Infinite Dimensional Lie Algebras
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Let me begin, I think I am done with Heisenberg and Fermions. Now I am going to study the nonAbelian analogue to the Heisenberg algebra. We started with a vector space with a form $(,\ )$ and the space generated by $[vt^n, wt^m] = n\delta_{n,-m}(v,w)$, and we generally worked with a one dimensional space. Now, instead we start with a finite dimensional Lie algebra $\mathfrak{g}$ and have here an invariant nondegenerate symmetric bilinear form $(,\ )$.

I assume that all of you are familiar with Lie algebras. Everything is over complex numbers, I'm not going to repeat it but everything is over complex numbers.

Examples, well, example number one is that you can take $\mathfrak{g}$ to be Abelian and then it is just a vector space, and the bilinear form gives you nothing.

Number two, which we will discuss most, is when $\mathfrak{g}$ is semisimple and $(,\ )$ is the (rescaled) Killing form. I trust you remember that $\mathfrak{g}$ has this kind of form; if $\mathfrak{g}$ is simple then it is the unique such form.

All of this is in Kac; it is not a good textbook but it is a very good reference. It is available in the university bookstore; I ordered it as the textbook for the class.

Definition 1 $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$,

so think of this as Laurent polynomials in one variable like xt^n or $x[n]$. This algebra looks like this. For every n there is one copy of $\mathfrak{g}$. You can think of this as functions in a punctured neighborhood of 0 with values in $\mathfrak{g}$. If I start in this point of view it is more natural to consider Laurent series, but let that come later. This is why this is frequently used. If you are thinking about a field theory on a Riemann surface, you can expand a function near a point in this way. This is just a motivation that we can leave aside for now.

The multiplication rule is as simple as it can possibly be: $[xf, yg] = [x, y]fg$. If you prefer, $[x[n], y[m]] = [x, y][n + m]$. As you can easily see this is a $\mathbb{Z}$-graded Lie algebra, called the loop algebra. It is called this because a punctured disc is like a circle. This is maps from a circle to your Lie algebra.

But this is a trivial thing. If I did this for $\mathfrak{g}$ Abelian I'd get a zero bracket. So I perturb it a little to get another term.

Definition 2 *$\tilde{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}k$ with $[k, \cdot] = 0$ and $[xf, yg] = [x, y]fg + Res_0(df)g(x, y)k$*

What is this explicitly? $(dt^n)t^m = Res_0(nt^{n-1+m})dt = n\delta_{n+m,0}$.

Equivalently you can write this as $[x[n], y[m]] = [x, y][n + m] + n\delta_{n,-m}(x, y)k$.

In the original example where the Lie algebra is Abelian I recover the Heisenberg algebra, which explains why this is called the nonAbelian version.

It is easy to check that this is a Lie algebra; the fact that it is is based on the fact that $Res(df)g + Res\ fdg = 0$ since $(df)g + fdg$ is exact. For Jacobi you need $d(fg)h = (df)gh + f(dg)h$, the Liebnitz rule. In this form it is completely obvious. Of course, you also need invariance of $(,)$.

For purely technical reasons I find it convenient to introduce one more variable. I want an element ad of which has n as eigenvalue on t^n.

Definition 3 *$\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}k \oplus \mathbb{C}d$ with $[d, x[n]] = nx[n]$ and $[d, xf] = xt\frac{df}{dt}$.*

The notation $\hat{\mathfrak{g}}$ and the term affine Lie algebra are interchangable between this and what I have called $\tilde{\mathfrak{g}}$. The analysis is very similar.

So what do we know about this Lie algebra?

1. It is $\mathbb{Z}$-graded. $\deg\ vt^n = n$, $\deg\ k = \deg\ d = 0$. You can then write $d[\hat{x}] = \deg(\hat{x})\hat{x}$ for homogeneous $\hat{x}$.
2. $\hat{g}$ has a natural symmetric invariant nondegenerate bilinear form which extends $(,\ )$. Namely $(x[n], y[m]) = \delta_{n,-m}(x, y)$, or $(xf, yg) = \frac{1}{2\pi i}\oint fgdt(x, y)$. Same thing.

   This is the most natural thing you can do. The fact that this is symmetric and nondegenerate is obvious.

   [What about k and d?]

   Well, I have saved this for last, you have $([\hat{x}, \hat{y}], k) = (\hat{x}, [\hat{y}, k]) = 0$ so $(k, [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]) = 0$, and k is in the commutant so you need $(k, d) \neq 0$. So then if you push it further you see (k, d) should be 1. Let me explain why.

   What is $([x[n], y[m]], d)$? According to the definition of invariant form this should be the same as $(y[m], [d, x[n]])$ and the commutator we now know so this is $n(x[n], y[m]) = n(x, y)\delta_{n,-m}$. On the other hand, this commutator is $[x, y][n+m]+n\delta_{n,-m}(x, y)k$. If you believe that it is reasonable to take commutator with things from the original algebra zero, then you get $n\delta_{n,-m}(x, y)(k, d)$.

   So we want $(k, k) = (d, d) = 0, (k, d) = 1$. So invariance is not hard to show, and this is natural in that it is forced by invariance.

If the Lie algebra is just general, this is about all I can say. But now let me consider the special case of when my Lie algebra is semisimple.

Assume $\mathfrak{g}$ is semisimple. The basic example I'll always consider is when $\mathfrak{g} = \mathfrak{sl}(2)$. Recall

1. $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra.
2. $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, the root subspace; $[h, x] = \langle \alpha, h \rangle x$ for $h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha$.
3. $\mathfrak{g}_\alpha$ is one dimensional. Given a properly normalized pair e_α, f_α with $(e_\alpha, f_\alpha) = \frac{2}{(\alpha,\alpha)}$ then the triple $e_\alpha, f_\alpha, h_\alpha = [e_\alpha, f_\alpha]$ satisfy the relations of $\mathfrak{sl}(2)$.

Let's concentrate on this case. On the whole I will consider simple Lie algebras. In this case there is one bilinear form up to a constant. The construction of $\hat{\mathfrak{g}}$ depends on the bilinear form so I want to fix how I do so.

For $\mathfrak{g}$ simple we fix $(\,,\,)$ by requiring that $(\alpha, \alpha) = 2$ for long root α. Should I remind you what the word long root means? No matter how you choose your bilinear form there are two possible lengths for roots; sometimes there is only one. Either you are in the "simply laced" case with one length or you have two lengths.

Our goal would be to have an analog of this for $\hat{\mathfrak{g}}$.

The first thing is to find an analog of Cartan. Hereon we consider $\mathfrak{g}$ as a subalgebra of $\hat{\mathfrak{g}}$ with t^0.

So $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}k \oplus \mathbb{C}d \subset \mathfrak{g}[0]$.

Before going on in full generality let's consider $\mathfrak{sl}(2)$. There we have e, f, h so now we have et^n, ft^n, ht^n, and then k, d.

I want a picture.

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
 & & \\
f[1] & h[1] & e[1] \\
 & & \\
f & h \oplus \mathbb{C}k \oplus \mathbb{C}d & e \\
 & & \\
f[-1] & h[-1] & e[-1] \\
 & & \\
\vdots & \vdots & \vdots
\end{array}
$$

This turns out to be general.

Theorem 1 $\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \bigoplus_{\hat{\alpha}\in\hat{R}} \mathfrak{g}_{\hat{\alpha}}$ *where* $\hat{R} \subset \hat{\mathfrak{h}}^*$, $[\hat{h}, \hat{x}] = \langle\hat{\alpha}, \hat{h}\rangle\hat{x}$ *for* $\hat{h} \in \hat{\mathfrak{h}}$ *and* $\hat{x} \in \mathfrak{g}_{\hat{\alpha}}$.

If we denote by $\delta \in \hat{\mathfrak{h}}^*$ an element such that $\langle\delta, h\rangle = 0$ for $h \in \mathfrak{h}$, $\langle\delta, d\rangle = 1$, and $\langle\delta, k\rangle = 0$ then an element xt^n with $x \in \mathfrak{g}_\alpha$ is in $\mathfrak{g}_{\alpha_n\delta}$.

$[\hat{h}, xt^n] = \langle\hat{h}, \alpha + n\delta\rangle xt^n$ is equivalent to $[h, xt^n] = \langle h, \alpha\rangle xt^n$ along with $[k, xt^n] = 0$ and $[d, xt^n] = nxt^n$.

This is almost the whole affine Lie algebra. Why do I say almost? I should also describe elements of the form Cartan times something. Let me now write the second part of the theorem.

Theorem 2 $\hat{R} = \hat{R}_{re} \cup \hat{R}_{im}$ *where* $\hat{R}_{re} = \{\alpha + n\delta | \alpha \in R, n \in \mathbb{Z}\}$ *and* $\hat{R}_{im} = \{n\delta | n \in \mathbb{Z}\backslash 0\}$.

Why do I want to split them? There are several reasons. One is that the dimensions of $\mathfrak{g}_\alpha + n\delta = 1$ but the dimension of $\mathfrak{g}_{n\delta}$ is dim $\mathfrak{h} = rank(\mathfrak{g})$.

Checking these theorems is essentially straightforward. We basically did it.

That is the end of the story. Now can we do the same things as in the finite dimensional case? Can you split into positive and negative roots, find the simple ones, a Weyl group and so on?

That's what I'm going to do but it's trickier. The Weyl group, among other things, can be infinite. In the finite dimensional case the bilinear invariant form was positive definite, but that doesn't happen here.

Let me describe $(\,,\,)$ on $\hat{\mathfrak{h}}^*$; This can be written as $\mathfrak{h}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$ where Λ_0 is dual to k.

Lemma 1 *The inner product on* $\hat{\mathfrak{h}}^*$ *is given by the the same thing as before on* $\mathfrak{h}^*$*. So it is positive definite on the real span of the roots.*

On the other two elements, it should not come as a surprise that $(\delta, \mathfrak{h}^*) = (\Lambda_0, \mathfrak{h}^*) = 0 = (\delta, \delta) = (\Lambda_0, \Lambda_0)$ *but* $(\delta, \Lambda_0) = 1$.

If you take the $\mathbb{R}$-span of all the roots, you get:

Corollary 1 *The real span of the affine roots is* $\mathfrak{h}^*_{\mathbb{R}} \oplus \mathbb{R}\delta$ *and* $(\,,\,)$ *is not positive definite because* $(\delta, \delta) = 0$.

So we have lost the geometry. What is a reflection in a space like this? So you see that in this space you have a span of roots which does not fully generate dual Cartan but a codimension one subspace. Also the form is here semidefinite, with kernel of one dimension. So you expect the Weyl group and so on to be funny. And that's how it turns out.

If I check that for ever root which is "real," i.e., those corresponding to roots in R, we have $(\hat{\alpha}, \hat{\alpha}) = 0$ while for imaginary ones you get zero.

So there are different kinds of roots which have to be treated seperately. Next time I'll go on with this. That's it for today, and I'll see you on Monday. On Friday Gabriel will talk about Khovanov homology.