Infinite Dimensional Lie Algebras
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Let me remind you what we know so far. Let me start with a harmonic oscillator, which we discussed before. There we had, classically, $(p, q) \in \mathbb{R}^2$, two numbers, position and momentum, and the Hamiltonian $H = 1/2(p^2 + \omega^2 q^2)$ where ω is frequency. If you do this quantum mechanically you get $[P, Q] = -i$, and then $a = \frac{1}{\sqrt{2}}(\omega Q + iP), a^* = \frac{1}{\sqrt{2}}(\omega Q - iP)$. Now $[a, a^*] = \omega$. Since now we don't have commutativity, we replace H with $a^*a + \omega/2 = aa^* - \omega/2$. Instead of a particle, now we have $\phi(z) = \sum \phi_n z^{-n}, \pi(z) = \sum \pi_n z^{-n}$ with $S^1 = \{|z| = 1\} = \mathbb{R}/2\pi\mathbb{Z}$. So instead of two coordinates now we have countably many coordinates, twice over.

The Hamiltonian we have chosen is $H = \frac{1}{2}\int \pi^2(z) + |\phi'(z)|^2$, where $\int$ is either $\frac{1}{2\pi i}\int dz/z$ or $\frac{1}{2\pi}\int_0^{2\pi} d\theta$. If you want to write things in coordinates, this is not so bad, it is $\frac{1}{2}(\sum \pi_n \pi_{-n} + n^2 \phi_n \phi_{-n})$. This is almost positive definite; ϕ_0 doesn't enter here so it's only semidefinite. This might create problems, but we will deal with them later.

To complete the picture we need analogues of the commutation relations. What will they be here? Let me answer, let me give you the answer. Suppose I quantize everything, but first let me describe the Poisson bracket. We don't need to know exactly what it is, here it is $\{\pi(z), \phi(w)\}$. So these are measuring your field at point w or the momentum of your field at point z. These are unrelated if $z \neq w$. So whatever this is, it involves a delta function $\delta(z/w)$ or $\delta(\theta - \theta')$ where $z = e^{i\theta}, w = e^{i\theta'}$. So then we write $[\pi(z), \phi(w)] = -i\delta(z/w)$. I still don't exactly understand what this means.

Let's do this in a straightforward and stupid way. This is $[\sum \pi_n z^{-n}, \sum \phi_n w^{-n}]$. Then can we write the delta function as a power series? It has integral one over the circle and is zero except at one point. The Fourier series of this is $\delta(z) = \sum \delta_n z^{-n}$ where $\delta_n = \int \delta(z) z^n = \int \delta(e^{i\theta})e^{in\theta}$, which will give you just the value of $e^{in\theta}$ at one, which is 1. So $\delta(z) = \sum z^n$.

Pretending that I am a physicist and do not care about convergence, I get that this is $-i\sum(z/w)^k$. This is very nice since I can write these two series as equal. If the power of z and power of w are not negatives of each other, the terms on the left vanish. So that gives $[\pi_n, \phi_m]$ equal to 0 if $n \neq -m$, and $-i$ if $n = -m$. So what do we have? What we have is the following. Forget about fields, integrals, and all that. Let me write the summary. The summary is that whatever the description, you should have a Hilbert space with operators π_n, ϕ_n, with commutation relations $[\pi_n, \phi_{-n}] = -i$, and everything else commutes. We also know that the Hamiltonian should be $\frac{1}{2}(\sum \pi_n \pi_{-n} + n^2 \phi_n \phi_{-n})$. So we want a representation of this. So this algebra effectively splits into a direct sum.

So you can write this as, you have a subalgebra generated by $\pi_{\pm 1}, \phi_{\pm 1}$, with $H_1 = \pi_1 \pi_{-1} + 1\phi_1 \phi_{-1}$, and then one generated by $\pi_{\pm 2}, \phi_{\pm 2}$, with $H_2 = \pi_2 \pi_{-2} + 4\phi_2 \phi_{-2}$, and so on. The Hamiltonian is the sum of these Hamiltonians.

This looks similarly to the original Hamiltonian we were looking at for the harmonic oscillator. A physicist would tell you that a free bosonic field is represented as a series of harmonic oscillators with positive integer frequencies. This is not exactly true because there are four operators, not two. But we will clear this up later.

To study representations of the whole thing. If you have a direct sum of Lie algebras, under reasonable assumptions the only representations of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ are $\oplus V_i' \otimes V_i''$, where these are representations of $\mathfrak{g}_1, \mathfrak{g}_2$, respectively. Semisimplicity of $\mathfrak{g}$ would be a sufficient reasonable assumption.

We still don't have exactly the harmonic oscillators, because we have four instead of two. So now, for every n, introduce $a_n = \frac{1}{\sqrt{2}}(n\phi_n + i\pi_n)$ and $a_n^* = \frac{1}{\sqrt{2}}(n\phi_n - i\phi_n)$.

Exercise 1 *If we write* $a(z) = \sum a_n z^{-n-1}$ *and* $a_n^*(z) = \sum a_n^* \bar{z}^{-n-1}$ *then*

$$\phi(z) = \frac{1}{\sqrt{2}}(\int a(z)dz + \int a^*(\bar{z})d\bar{z}).$$

This should be easy, since the Fourier series makes a like the derivative of ϕ.

[in response to a question]: We're going to have an adjoint condition that $\phi_n^\dagger = \phi_{-n}$, and likewise for π, so that $a_n^\dagger = a_{-n}^*$.

So now, what do I have with this change of variables, so what?

Theorem 1 *So defined, a_n, a_n^* satisfy:*

- $[a_n, a_m^*] = 0$
- $[a_n, a_m] = n\delta_{n-m} = [a_n^*, a_m^*]$.

Let's check the first property. $[a_n, a_k^*]$ is clearly zero unless $k = -n$, because subscript n commutes with everything except subscript $-n$. So that's the only case to check. Then

$$[a_n, a_{-n}^*] = \frac{1}{2}([n\phi_n, -i\pi_{-n}] + [i\pi_n, -n\phi_{-n}])$$

$$= \frac{1}{2}([i\pi_{-n}, n\phi_n] - [i\phi_n, n\phi_{-n}])$$

$$= \frac{in}{2}([\pi_{-n}, \phi_n] - [\phi_n, \phi_{-n}]) = 0$$

So to do the ones for a, you do the same thing.

$$[a_n, a_{-n}] = \frac{1}{2}([n\phi_n, i\pi_{-n}] + [i\pi_n, -n\phi_{-n}])$$

$$= \frac{1}{2}(-[i\pi_{-n}, n\phi_n] - [i\phi_n, n\phi_{-n}])$$

$$= \frac{in}{2}(-[\pi_{-n}, \phi_n] - [\phi_n, \phi_{-n}]) = -2i\frac{in}{2} = n.$$

At this point we say goodbye to physics and go to mathematics.

Definition 1 *The Heisenberg algebra $\mathscr{H}$ is generated by $a_n, n \in \mathbb{Z}$, subject to the relations $[a_n, a_m] = n\delta_{n-m}$.*

If we are talking about a free massless bosonic field, such a thing is described by a pair of commuting Heisenberg algebras, one denoted by a_n, the other by a_n^*. If I forget about adjointness, all you need to do to study the representations of such a pair is to study representations of one, and then look at tensor products. The Hilbert space of the theory will be $\oplus V_i \otimes \bar{V}_i$. I'm hiding a lot, but I'm not giving a course on quantum field theory.

So from now on, I'm interested in representations of one Heisenberg algebra. As before, this Heisenberg algebra has a single part a_0, and then pairs a_1, a_{-1} and a_2, a_{-2}, and so on. After suitable rescaling this is the direct sum of harmonic oscillators. The easiest and only way of constructing irreducible representations of $\mathscr{H}$ is to tensor irreducible representations of each of these together. And subject to a positive energy condition there is a unique representation. So for a_1, a_{-1} you take v with $a_1 v = 0$, so that the representation is $\mathbb{C}[a_{-1}]$. For the next it is $\mathbb{C}[a_{-2}]$.

So for the whole thing $\mathscr{F}$, you generate it by v_0, called the vacuum vector, such that $a_i v_0 = 0$ for $i > 0$. How do negative ones act? As a vector space, this is $\mathbb{C}[a_{-1}, a_{-2}, \cdots]v_0$, with all of these variables commuting.

Definition 2 *$\mathscr{F}$ is called the Fock module over the Heisenberg algebra.*

It is described in a very simple way. Its vector space structure is clear, and we see how a_i with negative indices act on it, by multiplication. How about the action of a_i with positive indices?

Lemma 1

1. *$a_n = n\frac{\delta}{\delta a_{-n}}$ for $n > 0$.*
2. *$\mathscr{F}$ is irreducible.*
3. *$\mathscr{F}$ is graded, that is, $\mathscr{F} = \oplus_{n \leq 0}\mathscr{F}_n$, where $\deg\ v_0 = 0$ and $\deg\ a_i = i$. So $a_{-2}^3 a_{-4} v_0$ has degree $3(-2) + (-4) = -10$.*

I should check that this is consistent. This uniquely determines the degree of all vectors in the space; now the degree of the positive ones comes from seeing how they change the grading. If I define degree in this way, then the degree of a_i will be i for all i. I leave this as an exercise for you.

I was not entirely careful because of a_0. The only irreducible representation of a one-dimensional algebra like that of a_0 is a constant representation. So this acts by a constant, but there is no restriction, the constant can be anything. So $a_0 v = \lambda v$ for some fixed λ.

Theorem 2 *Any graded irreducible $\mathscr{H}$-module with the following positivity condition, $V_n = 0$ for $n >> 0$, where V_n is the space with grading n, is isomorphic to one of the Fock modules $\mathscr{F}_\lambda$.*

Let me make you a picture of this Fock module. On top we have v_0.

0	v_0	1
-1	a_{-1}	1
-2	a_{-2}, a_{-1}^2	2
-3	$a_{-3}, a_{-2}a_{-1}, a_{-1}^3$	3

I should stop around here. I didn't prove the theorem, but it is similar to the other one. Take the highest degree vector, then it must be killed by a_n for positive n, and so on. Okay, we still have minor problems, we don't really know, we have the choice of eigenvalue for a_0, which is related to the fact that the others have pairs. Next time I'll try to tell you how you can fix that. The operator a_0 forgot about some of the structure. We had $a_n = n\phi_n + i\pi_n$, which is bad for $n = 0$ because ϕ_0 enters with coefficient 0.

We might try to fix this, which we will do next time.

[What happens if you drop the positivity condition?]

You can interchange them; you could have a vector killed by positive odd and negative even i. Next time I will define the energy, the Hamiltonian, and then instead we can say, positive energy.

I will see you on Wednesday.