Infinite Dimensional Lie Algebras
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Gabriel C. Drummond-Cole

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We will move to 5-127. Class will be MW, 10:25 to 11:35, and there will be a seminar. We'll meet Friday at 10:40 to determine a topic and a time for the seminar. Are there some things you've wanted to learn, and what are they?

Now let me remind you what we did. Recall:

1. The Heisenberg algebra is generated by $\{a, a^*\}$ with $[a, a^*] = 1$. We also have $H = aa^* - 1/2 = a^*a + 1/2$. This is an attempt to describe a harmonic oscillator in a quantum mechanical sense. As functions these just commute and it's just aa^*, so you split the difference; this is $\frac{aa^*+a^*a}{2}$. Okay. So this is the Heisenberg algebra. We had some relations there.

2. The statement is that to describe the harmonic oscillator, the state space is an irreducible representation of the Heisenberg algebra such that

   (a) it is positive energy, so that the eigenvalues of the Hamiltonian are nonnegative.

   (b) it is unitary, in that $a^\dagger = a^*$. This justifies the notation.

   I think it's better if I consider this with frequency ω, so I replace scalars with multiples of ω. At the moment I decided not to scale my variables.

Of course the question was how many representations are there? Can you recover the information from the state space?

Theorem 1 *Stone- von Neumann*

1. *The Heisenberg algebra has a unique positive energy irreducible representation V with basis $\psi_0, \psi_1, \cdots$ and the action is recovered uniquely from $a\psi_0 = 0$ and $\psi_n = \frac{(a^*)^n}{\sqrt{n!}}\psi_0$. In this representation, the action of a, a^* is given by $a^*\psi_n = \sqrt{(n+1)\omega}\psi_{n+1}$ and $a\psi_n = \sqrt{n\omega}\psi_{n-1}$, along with $H\psi_n = \omega(n+1/2)\psi_n$.*

2. *There is a unique positive definite inner product on this representation such that* $||\psi_0|| = 1$ *and* $a^\dagger = a^*$. *Dagger denotes the adjoint under the inner product. If I use this inner product, then moreover* $\{\psi_i\}$ *is an orthonormal basis. I should also notice that as a vector space, this space has a simple description as* $\mathbb{C}[a^*]$, *polynomials in* a^*. *Then the action of* a, *up to a constant, is* $\frac{d}{da^*}$, *which makes sense because* a *decreases the degree by one, so making the constants agree, that's what a derivative does.*

The proof is relatively simple. Before I continue, this is the algebraic version. I'm talking about vector spaces, not Hilbert spaces. There is an analogue in terms of Hilbert spaces but that is technical and more difficult, as we'll see in the examples.

This is closely modeled on the proof for representations of $\mathfrak{sl}(2)$.

So let $\psi_0 \in V$ have lowest energy for V an irreducible representation. For our purposes it suffices to choose E_0 such that $E_0 - \omega$ is not an eigenvalue.

We know that $[H, a] = -\omega a$. That means that $a\psi_0$ is an eigenvector for H with eigenvalue $E_0 - \omega$. But therefore $a\psi_0 = 0$, since E_0 is lowest in this sense.

Now define ψ_n by this formula, $\psi_n = \frac{(a^*)^n}{\sqrt{\omega^n n!}}\psi_0$. A very easy lemma by induction gives

Lemma 1 $a^*\psi_n = \sqrt{(n+1)\omega}\psi_{n+1}$.
$a\psi_n = \sqrt{n}\psi_{n-1}$.

For $n = 0$ this is exactly the formula we have. Now you can write $a\psi_1 = a(a^*/\sqrt{\omega}\psi_0) = \frac{a^*a+\omega}{\sqrt{\omega}}\psi_0 = \sqrt{\omega}\psi_0$. So the lemma is easy to prove by induction.

1. Now, by the lemma, all of the ψ_n are nonzero. since $a^n\psi_n = \sqrt{n!\omega^n}\psi_0$.
2. Then they are all linearly independent because they have different eigenvalues for H.
3. It already means that if I take the subspace in my original representation spanned by these, it will be a subrepresentation. Then if my original one was supposed to be irreducible, this must be the basis. It only remains to show that this one is irreductible. You can get from any basis element to ψ_0 by applying a sufficiently many times.

So this is just like what you normally do for $\mathfrak{sl}(2)$. In addition to specifying that e kills the highest weight vector, you also need to specify the highest weight. But you don't need that here; H is in terms of a and a^* completely.

Has anyone thought of why this algebra has no finite dimensional representation. Consider the commutation relation $[a, a^*] = \omega$. If it has a finite dimensional representation, what is the trace of ω in that representation? Can someone complete the argument for me?

This is because the trace of a commutator must be zero, but here the commutator ω must be a constant.

Now that we know that, maybe it's a good idea to go back and compare, we know that there is exactly one irreducible unitary representation before you complete it to a Hilbert space, so you might expect an analogue, but you have to be careful because your operators are not everywhere defined. But there is another way ignoring this analysis, by guessing.

Let $\mathscr{H} = L^2(\mathbb{R})$, and I always mean complex-valued functions, so here square integrable complex valued functions on the reals.

Let $Q = \hat{a}$ and $P = -i\frac{d}{da}$. Then $a = \omega Q + iP = \omega\hat{a} + \frac{d}{da}$ and $a^* = \omega Q - iP = \omega\hat{a} - \frac{d}{da}$. It's reasonable to expect that if we define a Hilbert space with an action of this algebra, that this would be a representation. So the Hilbert space analog should identify this with the representation from the Stone-von Neumann theorem.

How can we identify $\mathscr{H} = \overline{\mathbb{C}[a^*]}$? What is the analogue of ψ_0? It should be a vector annihilated by a, $a\psi_0 = 0$. So the equation reads $(\omega\hat{a} + \frac{d}{da})\psi_0 = 0$, an ordinary differential equation. You can write this $\psi_0' + \omega q\psi_0 = 0$, where $'$ is derivative with respect to q. Then this is $\psi_0 = Ce^{-\omega q^2/2}$. Then you choose the constant so that the square integral is one. We need to check that this is in L^2, because if it were not we wouldn't know what to do. Here C, I believe, is $\frac{1}{\pi^{1/4}}$. There should be ω here, let me think for a second. For $\omega = 1$ that is the correct answer. So how do you change variables? You can do that as well as I can. It's $\frac{\sqrt{\omega}}{\pi^{1/4}}$? I'm not really worried. So since we know how to get ψ_n from ψ_0 from the theorem, so up to a constant this is $C_n(\omega\hat{a} - \frac{d}{da})^n e^{-\omega q^2/2}$. So this will give a polynomial multiplied by ψ_0, and will be $C_n H_n(q)e{-}\omega q^2/2$. Up to normalization these polynomials are very well known and are called Hermite polynomials.

We've identified inside L^2 a subspace like the one we want. What we have done is identified the irreducible representation V we were talking about before with the following thing: the space of polynomials in one variable multiplied with our ψ_0, namely $\mathbb{C}[q]e^{-\omega q^2/2} \subset L^2(\mathbb{R})$, which is everywhere dense in L^2, so that $L^2(\mathbb{R}) \cong V$. You need some analysis to show that these are everywhere dense. There is some technique, purely analytical, to show this. One way is to describe Dirac's delta in terms of them. Again, I'm not teaching analysis so I won't go deeply into that.

One way you can do it is forget about analysis. If the closure is not the whole space, then there is an orthogonal complement, which would also be a representation of the Heisenberg algebra, and then there would be a vacuum vector there to satisfy the same equation, but the solution is unique. There are still problems because the operators are not acting on L^2, but on an every dense subset.

[Why is the complement a representation?]

Think about a group. If you think of these as generators of a Lie algebra as translations for P and multiplication by e^{ix} for Q. So the action of the Lie group will be invariant under the inner product, so that the complement is as well.

That's the end of the story for irreducible representations and we can find out everything about it. We can do this algebraically and have it be polynomials in one variable, or complete it and have it be L^2 in one variable. I prefer the polynomials so I'll ditch L^2 as soon as I can.

Let's see now what happens when we move to something really interesting. I won't have time to complete it today, but I can start. This will be as physical as I'm going to get, you can try to ignore the physics and at the end I will make a mathematical statment. We'll go from a harmonic oscillator to free bosons.

H.o.	*Free Field Theory on Circle*
$(q,p) \in \mathbb{R}^2$	$\phi(z), \lvert z\rvert = 1, z = e^{i\theta}, \pi(z),\ \phi = \sum \phi_n z^{-n}, \phi_n = \frac{1}{2\pi i}\int \phi(z) z^n \frac{dz}{z} = \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{i\theta}) e^{in\theta} d\theta, \pi$
$\{p,q\} = 1$	
$H = \frac{1}{2}(p^2 + \omega^2 q^2)$	$H(\phi,\pi) = \frac{1}{2}\frac{1}{2\pi}\int \lvert\pi(z)\rvert^2 + \lvert\psi'(z)\rvert^2 \frac{dz}{z} = \frac{1}{2}(\sum \pi_n \pi_{-n} + n^2 \phi_n \phi_{-n})$
$a = \frac{1}{\sqrt{2}}(\omega q + ip)$	
$a^* = \frac{1}{\sqrt{2}}(\omega q - ip)$	
$q = \frac{1}{\sqrt{2}\omega}(a + a^*)$	
$p = \frac{1}{\sqrt{2}}(a - a^*)$	
$\{a, a^*\} = i\omega, H = aa^*$	

Here we have the classical picture and this analogous picture. π is like a time derivative. The phase space is an infinite dimensional vector space with coordinates $\phi_n, \pi_n, n \in \mathbb{Z}$. So $\mathbb{R}^2$ becomes two copies of $\mathbb{R}^{\mathbb{Z}}$.

So for physicists writing the Hamiltonian or Lagrangian we will define the system. Let me start with the Hamiltonian. You have a field and you have the momentum π which says how this changes with time.

The $\phi'(z)$ is $z\frac{d}{dz}\phi$. Sometimes you will see a term with a ϕ^2; this is a special case called massless theory.

That's what physicists tell us the energy should be. How do we rewrite it in terms of ϕ_n and π_n? Well, $\pi^2(z) = \sum \pi_n z^{-n} \pi_m z^{-m} = \sum(\sum_{n+m=k} \pi_n \pi_m) z^{-k}$. If you integrate these over the circle, what happens? I get precisely $\sum \pi_n \pi_{-n}$. If you do the same thing for the other term, what you get is $\sum n^2 \phi_n \phi_{-n}$. It makes sense because differentiation of the n term will pick up n in front of it. You can do the math yourself, but you can see the Hamiltonian in a very reasonable form $H = \frac{1}{2}(\sum \pi_n \pi_{-n} + n^2 \phi_n \phi_{-n})$. It looks like the variables are separating. Nothing involves ϕ_4 and ϕ_5 together. So this will be like an infinite collection of harmonic oscillators with various frequencies, and that is what happens but I will have to complete it next time. I need to talk about the Poisson bracket and then quantize. This is as physical as I'm going to get. If you're uncomfortable you won't have to worry. From the fact that π is a real valued function, you should get $\overline{\pi_n} = \pi_{-n}$, and similarly for ϕ. So this operator is positive definite, which is what we want.

The goal is to show that this simplest example of field theory gives a nice generalization of the Heisenberg algebra. Let me remind that we are moving to $5-127$, effective next time. As for next time, I want to have the organizational meeting for the seminar. My idea is that everyone says what topics he wants to hear, and then we say who will talk on what topic. I will send an email to the department. We continue on Monday with the Heisenberg algebra.