Infinite Dimensional Lie Algebras
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Last time: we say that a lattice is integral if $(L,L) \in \mathbb{Z}$. What is the order of the set $\{\lambda_0+L, (\lambda_0, L) \in \mathbb{Z}\}$? That was the question. So $L = \oplus \mathbb{Z} e_i$, $A = (e_i, e_j)$, and $|L^*/L| = \det A$.

For example, let $L = \langle e_1, e_2 \rangle$ be the root lattice for A_2. Then L^* is the weight lattice, $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $|L^*/L| = 3$.

Okay, so $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}$ subject to the relations $[xt^n, yt^m] = n(x,y)\delta_{n,-m}$. Here $xt^n = x[n]$. So when $\mathfrak{h} = \mathbb{C}$ then $a[n] = a_n$.

Now $W = Vect(S^1) \otimes \mathbb{C} = \langle d_i \rangle$. Here $d_n = -t^{n+1}\frac{d}{dt}$. The relations are $[d_n, d_m] = (n-m)d_{n+m}$.

So W acts on $\hat{\mathfrak{h}}$ by $d_n x[k] = -kx[k+n]$.

A question: can we define the action of W in representations of $\hat{\mathfrak{h}}$? So $d_n \to L_n \in End(\mathscr{F}_\lambda)$. Here $[L_n, x[k]] = d_n x[k] = -kx[k+n]$. For $h = \mathbb{C}$ we have $L_n = \frac{1}{2}\sum_{k+l=n} : a_k a_l :$. Oh, $L_0 = H$.

So why this formula? Well, acting by a decreases the number of a by one so it should be quadratic. By degree considerations, L_n must be of degree n. So $k+l = n$. So it must be $\sum c_{kl} : a_k a_l :$ and then solving for constants gives this formula.

Now
$$[L_n, L_m] = \frac{1}{2}[L_n, \sum : a_k a_l :] = \frac{1}{2}\sum([L_n, a_k]a_l + a_k[L_n, a_l])$$
$$= \frac{1}{2}\sum(-ka_{n+k}a_l - la_k a_{l+n}).$$
Call this *. Now, if $n - m \neq 0$ this becomes
$$\frac{1}{2}\sum -ka_{n+k}a_l + \frac{1}{2}\sum -la_k a_{l+n}$$
$$= \frac{1}{2}\sum_{i+j=m+n}(n-i)a_i a_j + \frac{1}{2}\sum(i-m)a_i a_k$$

$$= \frac{1}{2} \sum_{i+j=m+n} (n-m) a_i a_j = (n-m) L_{m+n}.$$

When $m+n=0$ * makes sense since for all but finitely many terms we get cancellation. There we have

$$\frac{1}{2} \sum ((n-i) a_i a_{-i} + i a_{i-n} a_{n-i})$$

which we turn into

$$= \frac{1}{2} \sum : ((n-i) a_i a_{-i} + i a_{i-n} a_{n-i}) : + \frac{1}{2} \sum (n-i) i x(i) + i(i-n) x(i-n)$$

$$= 2n L_0 + \frac{1}{2} \sum_{i=0}^{n} (n-i) i = 2n L_0 + \frac{1}{12} (n^3 - n).$$

Here $x(i)$ is zero for negative i and 1 otherwise so that $: a_i a_{-i} : + i x(i) = a_i a_{-i}$. So $[L_n, L_m] = (n-m) L_{n+m} + \delta_{n,-m} \frac{n^3-n}{12}$.

Definition 1 *$Vir = W \oplus \mathbb{C}c = \langle L_n, c \rangle$ with commutation relations $[L_n, L_m] = (n0m) L_{n+m} + \delta_{n,-m} \frac{n^3-n}{12} c$ and $[c, L_n] = 0$.*

Theorem 1 *The action of W on $\hat{\mathfrak{h}}$ extends to an action of Vir on $\mathscr{F}_\lambda$ such that c acts by* $\dim \mathfrak{h}$. *[i.e., a representation with (Virasoro) central charge* $\dim \mathfrak{h}$*]*

Instead of writing commutation relations between a_n, L_n, consider $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$, that is, what physicists call functions in z with operator values. Then

$$[a(z), a(w)] = \sum [a_n, a_m] z^{-n-1} w^{-m-1} = \sum_{n \in \mathbb{Z}} n z^{-n-1} w^{n-1} = z^{-1} \delta_w (\sum_{n \in \mathbb{Z}} (w/z)^n)$$

where this last sum is the Fourier series for the δ function at w/z. Then this is $z^{-1} \delta_w \delta(w/z)$, where $\delta(z) = \sum z^n$.

Exercise 1 *For P a Laurent polynomial in z and w, $P(z,w)\delta(z/w) = P(z,z)\delta(z/w) = P(w,w)\delta(z/w)$. This is similar to $f(x)\delta(y) = f(y)\delta(x)$.*

Now $L(z) = \sum L_n z^{-n-2} = \frac{1}{2} \sum_{k,l} : a_k a_l : z^{-(k+l)-2} = \frac{1}{2} : a^2(z) :$. The commutation relations are

$$[L(z), a(w)] = \sum_{n,k} [L_n, a_k] z^{-n-2} w^{-k-1}$$

$$= \sum -k a_{n+k} z^{-n-2} w^{-k-1} = \sum a_i (\sum_{n+k=i} -k z^{-n-2} w^{-k-1})$$

Try to figure out where to go from here.

Okay, now

$$\frac{1}{2}[: a^2(z) :, a(w)] = \frac{1}{2}[a^2(z), a_w)] = \frac{1}{2}([a(z), a(w)]a(z) + a(z)[a(z), a(w)])$$

$$= \frac{1}{2}(z^{-1}\delta_w\delta(w/z)a(z) + a(z)z^{-1}\delta_w\delta(w/z))$$

$$= z^{-1}a(z)\delta_w\delta(w/z).$$

As homework, try to write $[L(z), L(w)]$. Here is a reference: E. Frenkel and B. Zvi, "Vertex Algebras and Algebraic Curves," and V. Kac, "Vertex Algebras for Beginners."

The operators "almost" commute except when $w = z$.