Infinite Dimensional Lie Algebras
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Let me get back to where we are. Today will be the last day about Bosons, Fermions, and so on. Then Kac-Moody algebras.

Okay, recall $[a_i, a_j] = i\delta_{i,-j}$ and $\mathscr{F}_m$ with $|m\rangle$ such that $a_i|m\rangle = 0$ for $i > 0$ and $a_0|m\rangle = m|m\rangle$

Let $V_{\mathbb{Z}} = \oplus \mathscr{F}_m$ and $e^{\lambda} : |m\rangle \mapsto |m + \lambda\rangle$ with deg $|m\rangle = -m^2/2$.

This was a while ago. For Fermions, $\{\psi_i, \psi_j^*\} = \delta_{i,-j}$ for $i, j \in 1/2 + \mathbb{Z}$. Now $F = \bigwedge^{\infty/2}[\psi_i]$ with $|0\rangle = \psi_{1/2} \wedge \psi_{3/2} \wedge \cdots$ and $|m\rangle = \psi_{1/2-m} \wedge \psi_{1/2-m+1} \wedge \cdots$ There is a bigrading by degree and charge; we denote by $F^{(m)}$ the vectors of charge m. Then $|m\rangle$ is the vector in $F^{(m)}$ of maximal degree, that being $-m^2/2$ as well. The dimension of this space is the partition number.

We can actually relate these two constructions. If I define the following operator in the Fermionic space F, namely $a(z) =: \psi(z)\psi^*(z) :$ or $a_k = \sum_{i+j=k} : \psi_i \psi_j^*$ then these satisfy the relation $[a_k, \psi_i] = \psi_{i+k}$ and similarly $[a_k, \psi_i^*] = -\psi_{i+k}^*$ and $[a_k, a_l] = k\delta_{k,-l}$. So $F^{(m)}$ becomes a module over the Heisenberg algebra.

Note that a contains ψ and ψ^*, one of each, so preserves charge. So we can break up by charge. Here is the answer.

Theorem 1 *$|m\rangle \mapsto |m\rangle$ gives an isomorphism $\mathscr{F}_m \to F^{(m)}$ and thus $V_{\mathbb{Z}} \cong F$ as modules over the Heisenberg algebra.*

This is known as, or is part of, the Boson-Fermion correspondence.

Let me prove it for $m = 0$; the other arguments are similar. Since the Foch module $\mathscr{F}_0$ is freely generated by a vector $|0\rangle$ with the above relations, I need first to check that $a_i|0\rangle = 0$ in F for $i > 0$. But this is easy. If i is positive one of $k + l$ is positive in $\sum_{k+l=0}$. So either I have ψ^* with positive index (checking for something with negative index) or ψ with positive index (wedging with something already in $|0\rangle$.

So the defining relation is satisfied.

Now we need to check that this map is an isomorphism. It's not immediately obvious. If we have $a_{-1}|0\rangle$, this maps to $\psi_{1/2}\psi^*_{-1/2}|0\rangle$ and $a_{-2}|0\rangle$ has two terms, and it's going to turn out to be hard to do this in a basis.

But it is injective since $\mathscr{F}_0$ is irreducible; it is surjective by comparison of dimensions, since $\dim \mathscr{F}_0[-n] = \dim\ F^{(0)}[-n]$, that is, the number of partitions of n.

This isomorphism also has other properties. It agrees with degree. The charge is exactly identified with the eigenvalue for a_0.

First of all notice that the degree, think in terms of degree on $V_{\mathbb{Z}}$. We can see that a_i shifts degree by i. Since we know that acting by as on this vector give you everything, so all degrees are specified by the ground vector and conditions on what a_i does to its degree.

You can do the charge part yourself. Next let me say something I'm not going to prove. On the Fermionic part we have a special basis, that given by monomials. On the Bosonic side you have the basis of the a_i. if I take a monomial basis $\omega_\lambda \in F^{(0)}$ in the Fermionic description then under the isomophism, if it is idenitfied with $S_\lambda(a_{-1}, a_{-2}\cdots)$

Under suitable change of basis this becomes the Schur polynomials

Let me do something different, something I can prove.

Consider $tr_F(q^{-\deg} z^{charge})$ and also $tr_{\mathscr{F}}(q^{L_0} z^{a_0})$. You can replace q^{-L_0} with $q^{-\deg}$.

Anything in F can be gotten by applying some ψ and $\psi^* withnegativeindices$.

If I look at $F = \bigwedge[\psi_{-1/2}, \psi_{-3/2}, \psi^*_{-1/2}, \cdots]|0\rangle$ then I get the expansion of $(1+q^{1/2}z)(1+q^{3/2}z)(1+q^{1/2}z^{-1})$. So I have $\prod_{i>0, i\in 1/2+\mathbb{Z}}(1+q^i z)(1+q^i z^{-1})$.

If I want to take $tr_{\mathscr{F}_0} q^{L_0} z^{a_0}$, well, a_0 acts by 0 so z^{a_0} acts by 1 and gives you nothing.

So $\mathscr{F}_0 = \mathbb{C}[a_{-1}, \cdots]|0\rangle$, we get the option to apply things more than once. So I get $(1+q+q^2+\cdots)(1+q^2+q^4+\cdots)$. I'm using that polynomials in infinitely many variables is the tensor product of polynomials in each variable, so that I get a product on this side. Let me notice that these are $\frac{1}{1-q} \times \frac{1}{1-q^2} \times \cdots$ so this is $\prod_{n\geq 0} \frac{1}{1-q^n}$. What if we're working in $\mathscr{F}_m$? Then $tr_{\mathscr{F}_m} q^{L_0} z^{a_0} = (\prod \frac{1}{1-q^n} q^{m^2/2} z^m$ since $\mathscr{F}_m = \mathbb{C}[a_{-1}, \cdots]|m\rangle$.

So putting these together we see that the right hand side is $\sum_{m\in\mathbb{Z}}(q^{m^2/2}z^m)\prod_{n>0}\frac{1}{1-q^n}$. We write then

$$\sum_{m\in\mathbb{Z}} q^{m^2/2} z^m = \prod_{n>0}(1-q^n) \prod_{i>0, i\in 1/2+\mathbb{Z}} (1-q^i z)(1-q^i z^{-1}).$$

This is known as the Jacobi triple product identity. It's not very difficult to prove it directly by combinatorial arguments. Still, this way of deriving it is really rather nice. You compute the trace, the character in two realizations, and in comparing them you see you have an identity. There are many other such connections. Later I hope to give you more examples of the same sort.

The thing on the left is basically a θ function; this triple product gives you roots of it.

Jacobi did this, as far as I know, by pure combinatorial arguments. I don't remember how he proved it. You don't need vector spaces with a basis but vector spaces make it nicer. It's not easy to construct a correspondence between monomial bases. But trace is basis independent so you don't need to construct this.

[When you have the vectors that the Bosons and Fermions act on, you only have, the Fermions, how do they generate the same space since the Bosons have only the pairs.]

We know that we have an isomorphism between $V_{\mathbb{Z}} = \oplus \mathscr{F}_m$ and $F = \oplus F^{(m)}$ with a_i, e^λ on one side and ψ_i, ψ_i^* on the other. So how does ψ_i act on the Heisenberg algebra. So the question is, can we write ψ in terms of a_n, e^λ?

It's not an obvious question; let me tell you how to approach it. Recall that $[a_i, \psi_k] = \psi_{k+i}$. Recall that $\psi(z) = \sum \psi_k z^{-k-1/2}$. If you try to write this in terms of series, it gives you $[a_i, \psi(z)] = z^i \psi(z)$.

All the commutation relations look nice in terms of generating series. So forgetting for a second about everything else, if you think of taking commutator with a_i as being like a derivative, then this is like $\psi' = C\psi$ so we should look for an exponential. So if $[a, b] = \lambda$ then $[a, e^b] = \lambda e^b$. So if we want to write $\psi(z)$, it should be $exp(A(z))$ where $[a_i, A(z)] = z^i$. But this equation is actually very easy to solve. If I want to write $A(z)$ as a formal power series in z and I want a_i to give this, then the z_i coefficient should be a_{-i}. If you match the coefficient precisely you'll see that the coefficient is $A(z) = \sum a_n \frac{z^{-n}}{-n}$. This is essentially the only solution. The general solution is that the operators a_i, my space, one of these Foch spaces is irreducible, so the only thing that commutes with all of them is a constant. So effectively that's the only solution.

Symbollically write this sum as $\int a(z)dz$ since $a(z) = \sum a_n z^{-n-1}$.

Okay. So we would normally expect that, this is just an idea, but we would write $\psi(z) = \exp \int a(z)dz$. That's the idea, not more. Why are there problems with it? How do you compute exp of a sum. You have a term like $a_1 z^{-1}$, forget about numerical coefficients, and you have $-a_{-1}z$. If you want to take exp of something like this, it should be the sum of $\frac{(\quad)^n}{n!}$. You will have problems if you look at this, even calculating the constant term. All of them come from an infinite series. So the naive definition has problems.

Let me tell you how you fix it. There's one way, you probably know it by now.

Theorem 2

$$\psi(z) =: \exp(\int a(z)dz) := e^1 z^{a_0} \exp(\sum_{n<0} a_n \frac{z^{-n}}{-n}) \exp(\sum_{n>0} a_n \frac{z^{-n}}{-n});$$

$$\psi^*(z) =: \exp(-\int a(z)dz) :$$

Here e^1 is the shift operator. It looks stupid but it's too late, we've set notation. The z^{a_0} makes sense formally as integrating $a_0 z^{-1}$ and then exponentiating. As an exercise, write ψ^* explicitly.

For the proof,

1. So defined, ψ satisfies $[a_i, \psi(z)] = z^i \psi(z)$. This is not the only thing we want. We need to know how it agrees with the shift operator. Now, ψ is supposed to move from $\mathscr{F}_m$ to $\mathscr{F}_{m+1}$. To do this we also need $\psi(z)|m\rangle = z^m|m+1\rangle$ up to terms of lower degree.

   Both of these facts are rather obvious. Why? For the first case, you are using the argument I outlined above. To commute $\psi(z)$ with a_1, you only look to the exponential of the negative parts; it commutes with everything else. So you repeat the argument yourself. Instead of $\psi = \exp(z)$ I have $\exp(A_+(z))\exp(A_-(z))$.

   The second one is also quite obvious. $\psi(z)|m\rangle = e^1 z^{a_0} exp(\sum_{n<0} z_n \frac{z^{-n}}{-n}) = e^1 z^{a_0}(|m\rangle + lower\ degree) = e^1(z^m|m\rangle + lower\ degree) = z^m|m+1\rangle + \cdots$

   That's basically the end of the story. There are two more steps.

2. These conditions define $\psi(z)$ uniquely.

3. Finally, the actual $\psi(z)$ satisfies the same relations. If I combine all three steps I will see that $\psi(z)$ is actually given by this formula.

The hardest part is the second part. I don't really have time to do this step.

[What is $[\psi(z), e^\lambda]$? Don't you need that condition?]

I don't need that. It's not immediately obvious. It may have been, there are some complications there too. I can write $e^\lambda \psi(z) = z^{-\lambda}\psi(z)e^\lambda$ but I need to prove it.

So the moral is that I can describe ψ and ψ^* in terms of a as long as you have also the shift operator. An interesting exercise is to show that these satisfy the relations for ψ. You need to be able to do computations with power series, and it's a good test of your ability to work with this language.

Next time I'll be done with these free things. Next time we'll have Kac-Moody algebras and others, where you start with something anticommuting.

Now let me tell you something else.

Corollary 1