Infinite Dimensional Lie Algebras
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Gabriel C. Drummond-Cole

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[Discussion of the enemy and $\otimes\mathbb{Q}$.]

I used to know a school in Moscow, and everywhere the students would go, they would leave a small statement saying something bad about one of their teachers that they didn't like. I have seen this not only in Moscow but also at MIT and in Paris. It must be completely cryptic to those not from Moscow.

We were talking about Fermions. We have ψ_i, ψ_i^*, with $i, j \in \frac{1}{2} + \mathbb{Z}$, with $\{\psi_i, \psi_j^*\} = \delta_{ij}$ and all other commutators zero. We say $\{a, b\} = ab + ba$.

So there is a representation $F = \bigwedge^{\infty/2}[\psi_i]$, so the idea is that there is an infinite strip of squares and eventually to the right and left the strip is full and empty. So $\psi_i\omega = \psi_i \wedge \omega$ and $\psi_i^*\omega = \frac{\delta}{\delta\psi_{-i}}\omega$, a contraction operator. This is uniquely specified by $\psi_i^*\psi_i \wedge \omega = \omega$.

We discussed this before and discussed that it has a vacuum vector $|0\rangle = \psi_{1/2} \wedge \cdots$ and $\psi_i|0\rangle = \psi_i^*|0\rangle = 0$ for $i > 0$. This is bigraded by deg $\psi_i =$ deg $\psi_i^* = i$ and deg $|0\rangle = 0$. There is another grading called charge which I will denote by c, with $c(\psi_i) = 1, c(\psi_i^*) = -1,\ c(|0\rangle) = 0$. And in fact sometimes it is more convenient to have more vectors than just the vacuum vector. So we define $|m\rangle = \psi_{1/2-m} \wedge \psi_{1/2-m+1} \wedge \cdots$ We want $c|m\rangle = m$. It is easy to check the degree of this. Do this for m positive. The difference between the two is that you have added m new ψ, namely the $\psi_{-1/2}, \psi_{-3/2}$, and so on. Now it's an exercise for I don't know which grade to compute the sum. So it is $-m^2/2$.

That's all very nice; it's not surprising that we get negative degree because it is always negative. I leave it to you to check that this formula works in general, for $-m$ as well. So let's see whether we can compute the dimension of this module. That doesn't make sense, but I want to check the dimension of the components of given degree.

So I will compute $F^{(0)} = \{\omega | c(\omega) = 0\}$. It must have the same number of monomials; you are moving them around.

My vacuum vector had three blocks for $1/2, 3/2, 5/2$; I move these, say, to $-5/2, -1/2, 3/2$.

Is there a natural combinatorial object that can tell you how to move blocks? The correct answer is that I'm trying to compute the dimension graded by degree. There is a way to naturally describe the basis consisting of such monomials, by partitions. Everyone knows what a partition is, right? I moved my first one three units, the second two units, and the third one one unit. Then I can say I moved the others by zero units. If you record this sequence of numbers, I claim that it's going to be a partition.

The only thing to check is that each one is smaller or equal to the last. It's pretty easy to see, it's a sequence of weakly decreasing numbers. It's pretty easy to see that it works the other way as well.

For the partition λ I write ω_λ; then the degree changes by one. So the degree of ω_λ is $-\sum \lambda_i = -|\lambda|$. Then the dimension of a space with given charge and degree is $-|\lambda|$. So the general formula is that $\dim\{\omega|c(\omega) = m, \deg(\omega) = -m^2/2 - k\} = \#\{\lambda||\lambda| = k\}$. So you start with $|m\rangle$.

Okay, so $\psi(z) = \sum_{n\in\frac{1}{2}+\mathbb{Z}} \psi_n z^{-n-1/2}$, $\psi^*(z) = \sum \psi_n^* z^{-n-1/2}$. So $a(z) = \psi(z)\psi^*(z)$, but this may diverge. Let me write this as $\sum a_k z^{-k-1}$ where $a_k = \sum_{i+j=k} \psi_i\psi_j^*$, where $k \in \mathbb{Z}$. Note that the $-1/2$ together give you a -1, and that i and j together as half-integers will give you an integer.

This is not well-defined. What is $a_0|0\rangle$? This is $\sum \psi_i\psi^*_{-i}|0\rangle$. Half of these give you zero, when the index is positive. So what if the index is negative? What if you have $\psi_{1/2}\psi^*_{-1/2}|0\rangle$. The first part kills $\psi_{1/2}$ so you are left with $|3/2\rangle$, and then the second one puts the same thing back again.

So what will we get if we do the same thing with $\psi_{3/2}$? We get the same thing, you can check that the sign works out the same. $\psi_n\psi^*_{-n}|0\rangle = |0\rangle$ for positive n. So then we have $\sum_{i>0}|0\rangle$, which is not finite.

We don't get a good answer, so what do we do?

[Subtract infinity.]

Say $a(z) =: \psi(z)\psi^*(z) :$

Then you put annihilation operators to the right. If $j > 0$ we say $: \psi_i\psi_j^* := \psi_i\psi_j$; otherwise it is $-\psi_j^*\psi_i$. You might see that it is symmetric, actually, if $i + j$ is not zero then these give you the same answer. If $i + j = 0$ then $i > 0$ in this case.

Theorem 1 *1. So defined, a_k is well-defined on F.*

2. Of course, $c(a_k) = 0, \deg(a_k) = k$

3. $[a_k, \psi_i] = \psi_{i+k}, [a_k, \psi_i^*] = -\psi^*_{i+k}$

This first part is pretty straightforward, we're used to this. Now, in a_k we have one ψ, one ψ^*, which has charge 0. Then for degree the order doesn't matter, and the degree of a summand of a_k is the sum of the degrees of its factors, $i+j=k$. So the only thing to check is the last condition. Ignoring normal ordering, there is only one place where the ψ don't commute, namely for $b=-i$ in the following.

$$a_k\psi_i = \sum :\psi_a\psi_b^*\psi_i = -\sum \psi_a\psi_i\psi_b^* + \psi_{k+i}$$

$$= \sum \psi_i\psi_a\psi_b^* + \psi_{k+i} = \psi_i a_k + \psi_{k+i}.$$

The last one should be easy to check, you can do it on your own. Okay, so we have some time, let's do some computation.

Example 1 $a_1(\psi_{1/2}\wedge\psi_{5/2}\wedge\cdots)$.

Note that normal ordering only matters when $k=0$.

So $\cdots\psi_{3/2}\psi_{-1/2}^*$ applied to this kills $\psi_{1/2}$ and adds in $\psi_{3/2}$. The others are all zero. From examples like this you see that normal ordering doesn't come into play.

So

$$k\neq 0\ a_k\psi_{i_1}\wedge\psi_{i_2}\wedge\cdots = psi_{i_1+k}\wedge\psi_{i_2}\wedge\cdots+\psi_{i_1}\wedge\psi_{i_2}\wedge\cdots+\cdots$$

Only finitely many terms survive because you are pushing your blocks down by k units, so in the middle of a filled block you don't have room.

For a_0 you can see that $a_0|0\rangle = 0$ and $[a_0,\psi_i]=\psi_i$, and similarly but with sign for ψ_i^* so it changes the eigenvalue by ±1. If you like, that's the number of factors minus an infinite constant.

I didn't compute the commutators for the a_k. Believe it or not, they are easy to compute:

Theorem 2 $[a_k,a_l]=k\delta_{k,-l}$

If we were talking about finite wedge products then operations like these commute. You might expect them to be always commuting, with a correction. So first let me calculate $[a_k,a_l]|0\rangle$. If one of k,l is zero, it's another consideration, and you can check. If both are nonzero, what is $a_l|0\rangle$? If $l>0$ then this is 0. When $l<0$ this is $\psi_{1/2-k}\wedge\psi_{3/2}\wedge\cdots+\psi_{1/2}\wedge\psi_{3/2}-l\wedge\cdots+\cdots$ So you have l terms, a completely filled block with one block moved down.

So what is $a_la_{-l}|0\rangle$? Each term gives back the vacuum vector and then you get $l|0\rangle$.

I don't want to do the last part. If k and l don't add up to zero then this is zero.

But the important term is where you move l to the left and then to the right.

This only explains what happens with the vacuum vector. How do you do it without getting bogged down?

Well, you say $[[a_k, a_l], \psi_i] = 0$ by Jacobi. This is

$$[a_k, [a_l, \psi_i]] + [[a_k, \psi_i], a_l] = [a_k, \psi_{l+i}] + [\psi_{k+i}, a_l] = \psi_{k+l+i} - \psi_{k+l+i}.$$

Similarly it commutes with ψ_i^*, and so you can derive its action from the action on the vacuum vector directly.

So F becomes a module over the Heisenberg algebra. We constructed Bosons out of two anticommuting variables. By gluing together two fermions we have constructed a boson. The mathematically obvious fact is that if your ψ almost commute then your as do as well.

So an interesting question is what F looks like as a module over the Heisenberg algebra. The action of a_k preserves charge, so $F^{(m)} = \{\omega | c(\omega) = m\}$ is a submodule.

Let me state the theorem to be proven next time.

$F^{(0)} \cong \mathscr{F}_0$ is the Fock module. The map from right to left is obvious, and its injectivity is esay too.