Infinite Dimensional Lie Algebras
February 2, 2005

Gabriel C. Drummond-Cole

February 7, 2005

I talked to Takhtajan, and he said he could give a lecture, but it will be next week, not this week. Also, there will be a colloquium talk tomorrow which I highly recommend.

We were talking about representations of the Heisenberg algebra $\mathscr{H} = \langle a_n \rangle_{n\in\mathbb{Z}}$, with commutation relations $[a_n, a_m] = n\delta_{n-m}$.

If we are talking about positive energy, there is a unique irreducible module, the Fock module $\mathscr{F}_\lambda$. We called the highest weight vector v_0 last time; today we will call it $|\lambda\rangle$, which is the most common thing. So $a_i|\lambda\rangle = 0$ for $i > 0$ and $a_0|\lambda\rangle = \lambda|\lambda\rangle$. The dimension of the graded component $\mathscr{F}_0[-n] = p(n)$, the number of partitions of n. So the partition $\lambda_1 \geq \cdots \geq \lambda_k$ is associated with $a_{\lambda_1} \cdots a_{\lambda_k}|0\rangle$.

Degrees can be chosen for the highest weight arbitrarily, but for good reasons I will choose $\deg|\lambda\rangle = -\lambda^2/2$, which doesn't have to be an integer. Note that if λ is an integer then this is a half-integer; if λ is in $\sqrt{2}\mathbb{Z}$ then this is an integer. Usually λ will be in $\sqrt{N}\mathbb{Z}$ so we will be in one case or the other.

So why do we choose this degree? For a harmonic oscillator we didn't need degree at all. Let me tell you how this is related to the Hamiltonian. The naive definition, the classical formula for the energy operator with the functions replaced by operators, gives

$$H_{naive} = \frac{1}{2}\sum a_n a_{-n} = \frac{1}{2}a_0^2 + \frac{1}{2}\sum a_n a_{-n} a_{-n} a_n.$$

This seems to be a reasonable definition. What is the commutation relation of H with a_i? It's an easy question. $[H, a_i] = \frac{1}{2}[a_i a_{-i} + a_i a_{-i}, a_i]$, so by Liebnitz this is $\frac{-1}{2}(a_i[a_i, a_{-i}] + [a_i, a_{-i}]a_i) = -ia_i$. What does this mean? So a_i shifts the eigenvalue of H by exactly i; if $Hv = hv$ then $H(a_i v) = (h-i)a_i v$. In other words, every element shifts the eigenvalue of H by the negative of the degree of the element. So $Hv = -\deg v$ plus a constant. The easiest way is to compute for the vacuum vector $|0\rangle$. This is $H|0\rangle$. The a_0 part and the $a_{-n}a_n$ kill it. So this is $\frac{1}{2}\sum a_n a_{-n}|0\rangle = \frac{1}{2}\sum(n + a_{-n}a_n)|0\rangle = (\frac{1}{2}\sum_{n\geq 0} n)|0\rangle$.

This is why we call this a naive definition. This splits into infinitely many pieces. If you add the lowest energy levels for all the oscillators, you get this infinity. The physicists say "we don't need to know energy, just the difference of energy." So let's get rid of the infinite constant, and define H to be $-\deg\ v$. There is a more elegant, or at least one more, way to explain the same thing.

$$H = \frac{1}{2}a_0^2 + \frac{1}{2}\sum_{n>0} 2a_{-n}a_n.$$

We're removing the constant by brute force. This operation of reordering is quite useful and has a special name and notation: $\frac{1}{2}\sum : a_n a_{-n} :$. It is called normal ordering and is defined as $: a_n a_k :$ is $a_n a_k$ or $a_k a_n$, written so that a negative one of these is first. If they are both positive or both negative, then they commute so the order doesn't matter.

Now $: a_n a_k :$ may differ from $a_n a_k$ by a constant; the idea is to kill infinite constants. The idea is that with this definition everything will be fine.

Lemma 1 *1. $H = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n$ is well-defined in any $\mathscr{F}_\lambda$, in that only finitely many terms will be nonzero when applied to any vector.*

2. $Hv = -(\deg\ v)v$.

3. $[H, a_i] = -ia_i$.

Barry McCoy argues that physicists are not doing real physics in the sense of analysis, they'd rather do algebra, manipulations like this avoid analysis. I tend to agree but I don't see it as a bad thing.

Let me show you how you, for example, prove the first part of this lemma. To prove it, suppose, for simplicity, we're in the vacuum model, with a vector $v = a_{-n_1} \cdots a_{-n_k}|0\rangle$. Then in Hv, only finitely many terms will be nonzero. I think actually each of you can prove it on your own. If it is bigger than the sum of the indices here, then $a_n v$ is zero. There is a more elementary argument but I want to use this one. The degree of v is $-\sum n_i$, and then the degree of $a_n v$ is positive so this vector is zero. This works even with many nonzero commutators, not just in this special case. Then only finitely many terms are nonzero. I don't want to prove the other two. The third we proved before. Subtracting an infinite constant doesn't change the commutator, you can use the same reasoning. Then the second one is going to justify our choice of degree for $|\lambda\rangle$. That is, $H|\lambda\rangle = \frac{1}{2}a_0^2|\lambda\rangle = \frac{\lambda^2}{2}|\lambda\rangle$.

Okay, so now we do have a nice description of the Hamiltonian. Now I can say that this is a unique irreducible positive energy module. Positive energy now means that the eigenvalues of H are bounded from below. Okay.

Second, there is one more loose end in what I discussed last time. One more loose end is an element a_0 in the center, so we have no unique Fock module, we have a family of them. Since we had ϕ and π gave us a_n as $\frac{1}{\sqrt{2}}(n\phi_n + i\pi_n)$ and $a_n^* = \frac{1}{\sqrt{2}}(n\phi_n - i\pi_n)$. For $n = 0$ this is not an invertible change of variables. We are forgetting about ϕ_0, so we need to add information to recapture ϕ_0. So we can add it by hand.

a_0 is central. To fix it, let's add one more operator which does not commute with a_0, something like adding back the ϕ_0 we lost. We could recall $aa^* = -i$ and add an operator to coomute with a_0 to get 1. Instead usually people add an operators e^λ which commute in the following way: $a_0 e^\lambda = e^\lambda(a_0 + \lambda)$. This should satisfy $e^\lambda e^\mu = e^{\lambda+\mu}$ but it is not the exponential.

Let me explain the model example. Suppose you have $\hat{x} : f(x) \to xf(x)$. Then $[\frac{d}{dx}, x] = 1$. Now $exp(\lambda ddx) : f \to f(x + \lambda)$. How does this commute with x? In one order you get $xf(x + \lambda)$; in the other $(x + \lambda)f(x + \lambda)$, and the relation you get looks something like what we have here.

Another way to put it is as follows. So, if $a_0 v = \mu v$, then $a_0 e^\lambda v = (\lambda + \mu)e^\lambda v$. So.

Definition 1 *For a fixed N, $\hat{\mathscr{H}}[\sqrt{N}\mathbb{Z}]$ is the algebra generated by $a_n, n \in \mathbb{Z}$ and e^λ, $\lambda \in \sqrt{N}\mathbb{Z}$.*

Commutation relations are as before, these shift operators commute with a_0 as above, and they commute with a_i for $i \neq 0$.

For example I can take $N = 2$ and get $e^{k\sqrt{2}}$.

This is roughly $\exp(\phi_0)$. I can tell you the physical meaning, and I won't describe how to get from the physical meaning to this, but just in case, this describes what appears in a description of a free field on S^1 with values in $\mathbb{R}/\sqrt{N}\mathbb{Z}$. Physicists would tell you this is a free boson in the circle.

So this is the Heisenberg algebra with some shift operators as well. What are representations of this algebra? They must be representations of a Heisenberg algebra. So restricted to the Heisenberg part it should be the direct sum of several Fock modules.

Theorem 1 *$\mathscr{H}[\sqrt{N}\mathbb{Z}]$ has a unique positive energy irreducible representation*

$$V_{\sqrt{N}\mathbb{Z}} = \oplus_{\lambda \in \sqrt{N}\mathbb{Z}} \mathscr{F}_\lambda$$

Where $e^\lambda|\mu\rangle = |\lambda + \mu\rangle$.

So the idea is that the a act up and down in the grading and the shift operators act sideways, not horizontally but along a parabola, since the degree of $|\lambda\rangle = -\lambda^2/2$. This thing ensures there is only a finite dimensional space at each degree.

This is the simplest case of a lattice algebra. So far I was talking about Heisenberg algebras with one generator for each n. I could have two; more generally let $\mathfrak{h}$ be a finite dimensional space with nondegenerate $(,)$. Then $\hat{\mathfrak{h}} = \langle h[n] \rangle$, where $h \in \mathfrak{h}, n \in \mathbb{Z}$, so I'm just making countably many copies of my original space. If my original space was one dimensional, I have something like what we got in the case of the Heisenberg algebra. Each generator gives me countably many generators.

And what are the commutation relations? $[x[n], y[m]] = n\delta_{n-m}(x, y)$. If the vector space is $\mathbb{C}a$ with $(a, a) = 1$ then we recover the old Heisenberg algebra. Most of the things we've done can be generalized for this multidimensional Heisenberg algebra. Let me define the Fock module, then I'll have to stop.

This is generated by $|\lambda\rangle$ again, here for $\lambda \in \mathfrak{h}$, with $x[n]|\lambda\rangle = 0, n > 0; \quad x[0]|\lambda\rangle = (x, \lambda)|\lambda\rangle$, and negative ones act freely. Then $\mathscr{F}_\lambda = S(\mathfrak{h}_-)|\lambda\rangle$, where $\mathfrak{h}_- = \oplus_{n<0}\mathfrak{h}[n]$.

See if you can figure out on your own what is the analog of such a lattice algebra; I will describe it next time.

When you try to naively extend things you know to this infinite dimensional setting, you sometimes run into infinite constants you have to kill with brute force.