Infinite Dimensional Lie Algebras
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Okay, let me begin because there are several things I want to do today. I want to show some things you can do with the operator product expansion and also I want to start with fermions. As a matter of fact, maybe I'll do them together.

For the last two or three weeks we studied the Heisenberg algebra which started with a simple harmonic oscillator with relation $[a, a^*] = 1$, and then we represented this as $\mathbb{C}[a^*]|0\rangle$ with $a|0\rangle = 0$. Then a^* acts by multiplication and a by $\frac{\delta}{\delta a^*}$. So we generalized it to $[a_n, a_m] = n\delta_{n-m}$. Then the Foch space we were talking about was $\mathscr{F} = \mathbb{C}[a_{-1}, a_{-2}, \cdots]|0\rangle$ with $a_i|0\rangle = 0$ for nonnegative i and $a_n = n\frac{\delta}{\delta a_{-n}}$ for $n > 0$. You can say $a(z) = \sum a_n z^{-n-1}$ and then $[a(z), a(w)] = z^{-1}\delta w\delta(w/z)$ but δ converges nowhere. So $a(z)a(w)$ is equal to some regular function plus a factor $\frac{1}{(z-w)^2}$ for $|z| > |w|$. You get the same thing for $a(w)a(z)$ so they are equal after analytic continuation.

So the commutator is essentially zero but not quite, which explains why it is nowhere convergent.

Oh yeah, we also had this series $L(z) = \frac{1}{2} : a^2(z) := \sum L_n z^{n-2}$. These generate the Virasoro algebra. So $L_0 = -\deg \geq 0$ and you can write commutation relations between L and a several ways, but perhaps easiest as $[L_n, a_k] = -ka_{n+k}$. So the product is regular everywhere except the diagonal.

Okay, now let me, this is the Heisenberg algebra, also called the algebra of bosons. Now let me talk about fermions. So first of all, what does the name mean? If you make 1 very small then the Heisenberg algebra is basically commuting. But for fermions we want them to anticommute. So you want $\psi\psi^* + \psi^*\psi = 1$, where 1 is small. So they almost anticommute.

Now let me show you the analog of this representation. Let me assume that I have a vector space V and the dual space V^*, and then I can consider ψv for $v \in V$ and $\psi^* v^*$ for $v^* \in V^*$ and then the relation is $\psi v\psi^* v^* + \psi^* v^*\psi v = \langle v, v^*\rangle$. So for those of you who know, this is the Clifford algeba of $V \oplus V^*$. Now in the Heisenberg case the natural representation was an algebra of polynomials.

So first of all let's try to guess. Start with a vector $|0\rangle$ and let me impose that I want $\psi_v|0\rangle = 0$. So what does this generate? Oh, I forgot to mention, the ϕ anticommute as do the ϕ^*. In particular, then $\phi_v^2 = 0$. So you will definitely not get an algebra of polynomials. What involves anticommuting variables? These are differential forms, or exterior algebras. You try to apply to $|0\rangle$ an arbitrary product of ψ and ψ^*. Since ψ acts by zero and from commutation we get that we only need to care about the actions of ψ^* on $|0\rangle$. Pick a basis ψ_i^* and so you get expressions like $\psi_1^*|0\rangle$ and $\psi_1^*\psi_2^*$, which anticommute. So you get the exterior algebra $\wedge V^*|0\rangle$. Every element acts by wedge product.

Now, how does ψ_v act. It acts by zero on the vacuum vector. Let's do an example. $\psi_1\psi_1^*|0\rangle = -\psi_1^*\psi_1|0\rangle + |0\rangle = |0\rangle$. This is the same kind of idea as before. Can anyone now tell what is the general formula, what is the action of the ψ?

Let me write a more general thing. $\psi_v\psi_v^*|0\rangle = \langle v, v^*\rangle|0\rangle$. This is an operator which is applied to forms of degree one to give a number. This is what happens when you plug a vector into an n-form. These are called contraction operators. In more generality, this satisfies a kind of Liebnitz rule. $\psi_v(\omega_1 \wedge \omega_2) = \psi_v(\omega_1) \wedge \omega_2 + (-1)^{\deg\ \omega_1}\omega_1 \wedge \psi_v(\omega_2)$. In general when you move something of degree 1 past something of degree k you get a sign of $(-1)^k$. So this is like what we had before but instead of a polynomial algebra, we have an exterior algebra; instead of derivations you have contractions, signed derivations.

So what's the difference? The module you generate is finite dimensional, unlike the case for the Heisenberg algebra if you start with a finite dimensional V. Second is the following interesting result. There is a stronger result which I will write but won't prove. It's actually a very easy result.

Theorem 1 *Call this algebra Cl for Clifford.*

1. $\psi_{v^*}^*\omega = v^* \wedge \omega$,
   $\psi_v\omega$ *as above define an action of* Cl *on* $\wedge V^*$.
2. $Cl \cong End(\wedge V^*)$.

This is a very easy exercise to prove this. Let me say, you need to be able to go from any form to any other form. Start with your form and kill all the terms to get one. Then multiply back out by what you need. You can make it a little smarter by only killing those you don't need.

As a remarkable thing you might say that ψ and ψ_* have broken symmetry because now we have $End(\wedge V^*)$. It is known that this thing has only one representation. Then if $End(\wedge V^*) \cong End(\wedge V)$ we get $\wedge V^* \cong \wedge V$ as Cl-modules. Then $|0\rangle$ is identified (noncanonically) with ω_0, a top degree differential form.

We have not yet gone to the infinite dimensional case, where we have infinitely many of these, one in each degree. So what happens if you want this? Let me go back to the case where I had one ψ, ψ^* in each degree. Let dim $V = 1$ with Cl generated by ψ, ψ^*. Then the infinite version $\widehat{Cl}$ is generated by ψ_i, ψ_j^* where $i, j \in \frac{1}{2} + \mathbb{Z}$. Now the commutation relations are $\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{i,-j}$ where the ψ, ψ^* commute among themselves.

Before I go any further, let me note that this has two gradings.

1. One is by degree, i.e., the degree of ψ_i or ψ_i^* is i.
2. This corresponds to the degree of differential forms. Call the charge of ψ_i one and the charge of ψ_i^* negative one. Then you think of ψ as a one-form and ψ_* as contractions. You can do it in reverse equally well; the choice is yours.

This is a graded algebra as long as the grading of constants is zero. So how do we represent a module over this? Before I say anything about modules, let me say there is a way of writing these in terms of formal generating series $\psi(z) = \sum_{\frac{1}{2}+\mathbb{Z}} \psi_n z^{-n-\frac{1}{2}}$ and $\psi^*(z) = \sum_{\frac{1}{2}+\mathbb{Z}} \psi_n^* z^{-n-\frac{1}{2}}$. Then you can check that $\psi(z)\psi^*(w) \sim \frac{1}{z-w}$. So for $\{,\}$ the anticommutator $ab + ba$ we get $\{\psi(z), \psi^*(w)\} = z^{-1}\delta(z/w)$.

So what is the nice way of defining representations of this? We can repeat the same thing that we difd before. Now we have countably many copies of V. So now we can talk about $\wedge[\psi_i]$ or $\wedge[\psi_i^*]$. These are not isomorphic. There is now no top degree form because there are infinitely many variables. You can also argue that $\wedge[\psi_i^*]$ has a vector 1 such that $\psi_i 1 = 0$. Now, is there a form in $\wedge[\psi_i]$ killed by wedge product with every generator? No, not for infinitely many variables.

But neither of these are good. I want positive energy. This will be related to the degree by i. I want the degree to be bounded from above. So I want degree negative. So in $\wedge[\psi_i]$ the degree is unbounded in both directions. The same is true in the other case.

There are two solutions, or rather one, with two descriptions. Let me describe the more intuitive way. We will consider the semiinfinite forms $\wedge^{\infty/2} = F$. Let me take the following vector: $\psi_{1/2} \wedge \psi_{3/2} \wedge \cdots$, so you really have an infinite sequence. I don't know what the Hell this means, but I can tell you how ψ and ψ^* act on it. Call this $|0\rangle$. Then $\psi_{1/2}|0\rangle = 0$ and $\psi_{-1/2}^*|0\rangle = \psi_{-1/2}^* \psi_{1/2} \wedge \cdots$ So you take those which differ from this $|0\rangle$ in finitely many places. The best way to describe it is graphically. When I have a monomial in ψ I can put them in order. Let me make a strip, labelled by the integers plus $\frac{1}{2}$. So I color in those squares in the strip which represent an index present in a given monomial. So we can compare pictures to see whether things differ in finitely many places.

The formal definition takes some ingenuity to write, but the formal definition is that F is the span of $\psi_{i_1} \wedge \psi_{i_2} \wedge \cdots$ such that $i_{k+1} = i_k + 1$ for $k >> 0$. As far as the action goes, you can figure it out on its own. The ψ act by wedge product and the ψ_* act as contractions.

If you're unhappy with this definition then here's another definition of the same space. Equivalently, let $F = \widehat{Cl}|0\rangle$ subject to $\psi_i|0\rangle = \psi_i^*|0\rangle = 0$ for $i > 0$. If you take this module, then first of all I claim that this is the same representation. As a vector space it is pretty easy to describe. So the ψ with negative indices give you a half-infinite $\wedge[\psi_i, i < 0] \otimes \wedge[\psi_i^*, i < 0]|0\rangle$ as a vector space.

So that's our vector space. What is the degree? There are two. Let me start with charge. We have that the charge of ω is an infinite, but we care about the difference which is finite. So the difference of charge is given by the difference between the number of ψ_i and ψ_i^* applied. It is quite possible to define charge by two conditions: $|0\rangle$ has charge zero and then the charge $c(\psi) = 1, c(\psi^*) = -1$. I can write a precise formula for the charge of a monomial: $c(\psi_{i_1} \wedge \psi_{i_2} \wedge \cdots) = \#\{i \in I, i < 0\} - \#\{i \notin I, i > 0\}$. This gives us integral charge for any integer. How about degree? You define the degree of $|0\rangle$ to be zero.

So we will get $\deg(\psi_{i_1} \wedge \cdots \wedge \psi_{i_k} \wedge \cdots) = \sum_{i \in I, i<0} i - \sum_{i \notin I, i>0} i \leq 0$.

This kind of picture and idea goes back to Dirac describing positrons as the absence or lack of electrons. I will continue next time and will also tell you how this is related to the bosons. As your homework, try to find the dimension of a graded piece of given charge and given degree.