Infinite Dimensional Lie Algebras February 14, 2005

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Okay, let's see what's going on. We have $\mathscr{H} = \langle a_n, n \in \mathbb{Z}\rangle$, subject to the commutation relations $[a_n, a_m] = n\delta_{n,-m}$ and then $a(z) = \sum a_n z^{-n-1}$ with relations $[a(z), a(w)] = z^{-1}\delta_w \delta(w/z)$.

[something about vector fields on S^1]

Then there are L_n with $[L_n, a_k] = -ka_{n+k}$ and $L(z) = \sum L_n z^{-n-2} = \frac{1}{2} : a^2(z) :$ and then we have $[L(z), a(w) = z^{-1}a(w)\delta_w\delta(w/z)$.

We want to be able to think of $a(z), L(z)$ as operator valued functions, i.e., $a(z)$ as a function on $\mathbb{C}^\times$ with values in $End(\mathscr{F})$.

So $a(z)|0\rangle = \sum a_n|0\rangle z^{-n-1} = \sum_{n<0} a_n|0\rangle z^{-n-1}$. This is bad because it's not in $\mathscr{F}$. We have $\mathscr{F} = \mathbb{C}[a_{-1}, a_{-2}, \cdots]|0\rangle$. We want to work with $\bar{\mathscr{F}} = \mathbb{C}[[a_{-1}, a_{-2}, \cdots]]|0\rangle$, that is, infinite linear combinations. So this is $\sum v_k$ where the degree of v_k goes to negative infinity. So $a(z)$ is a function on $\mathbb{C}^\times$ with values in $Hom(\mathscr{F}, \bar{\mathscr{F}})$.

There is a problem, as usual. Such operators cannot be multiplied. E.g., can $a(z)a(w)|0\rangle \in \bar{\mathscr{F}}$ be defined? So $a(z)a(w)|0\rangle = a(z)\sum a_{-n}w^{n-1} = \sum_{n>0} nz^{-n-1}w^{n-1}|0\rangle$ plus other terms. This last part can be written $z^{-1}w^{-1}\sum n(\frac{w}{z})^n$. Recall that $a_n a_{-n}|0\rangle = n|0\rangle$ for $n > 0$.

Theorem 1 *1. For $|w| < |z|$ $a(z)a(w) =: a(z)a(w) : +\frac{1}{(z-w)^2}$ where the fraction is expanded in a power series in w/z.*

2. $: a(z)a(w) :$ is a well-defined operator $\mathscr{F} \to mat\bar{h}scrF$ for any $z, w \in \mathbb{C}^*$.

Corollary 1 *$a(z)a(w)$ can be defined as a function of $z, w \in \mathbb{C}^*$ for $|w| < |z|$ with values in $Hom(\mathscr{F}, \bar{\mathscr{F}})$.*

So for the proof of the theorem, for the first part we have

$$a(z)a(w) = \sum a_n a_m z^{-n-1}w^{-m-1};$$

$$:a(z)a(w) := \sum : a_n a_m : z^{-n-1}w^{-m-1}.$$

The only places these differ are when the first has $a_n a_{-n} z^{-n-1}w^{n-1}$ and the second $a_{-n}a_n z^{-n-1}w^{n-1}$. So

$$a(z)a(w) - :a(z)a(w) := \sum_{n>0} n z^{-n-1}w^{n-1}$$

$$= z^{-1}\sum_{n\geq 0}\frac{nw^{n-1}}{z^n} = z^{-1}\delta_w(\sum_{n\geq 0}(\frac{w}{z})^n)$$

$$= z^{-1}\delta_w(\frac{1}{1-\frac{w}{z}}) = \delta_w\frac{1}{z-w} = \frac{1}{(z-w)^2}.$$

Corollary 2 *Matrix coefficients of $a(z)a(w)$ can be analytically continued to $(\mathbb{C}^*)^2\backslash$diagonal and they will have a second order pole at $z=w$. Physicists write this as $a(z)a(w) \sim \frac{1}{(z-w)^2}$.*

So this is $a(w)a(z) =: a(w)a(z) : + \frac{1}{(z-w)^2}$ for $|z|<|w|$ and $a(z)a(w) =: a(z)a(w) : + \frac{1}{(z-w)^2}$ for $|z|>|w|$. Now, this means that $a(z)a(w) - a(w)a(z)$ is nowhere defined. As formal power series this would be the expansion of $\frac{1}{(z-w)^2}$ for $|z|<|w|$ less the expansion of the same function for $|w|>|z|$.

So without analytic continuation $[a(z),a(w)]$ is nowhere convergent; with analytic continuation $[a(z),a(w)]=0$. The moral is that all information about $[a_n,a_k]$ can be obtained from the polar part of $a(z)a(w) \sim \frac{1}{(z-w)^2}$.

Example 1 $a(z)L(w) =: a(z)L(w) : + \frac{a(w)}{(z-w)^2}$ *for* $|w|<|z|$.

Definition 1 $: a_n L_k := a_n L_k$ *for* $n<0$, $L_k a_n$ *for* $n\geq 0$.

So

$$a(z)L(w) - :a(z)L(w) := \sum_{n\geq 0, k\in\mathbb{Z}} [a_n, L_k] z^{-n-1}w^{-k-2}$$

$$= \sum n a_{n+k} z^{-n-1}w^{-k-2} = \sum n z^{-n-1}w^{n-1}a_{n+k}w^{-n-k-1}$$

$$= (z^{-1}\sum_{n\geq 0}\frac{nw^{n-1}}{z^n})(\sum a_l w^{-l-1}) = \frac{1}{(z-w)^2}a(w).$$

So $a(z)L(w) =: a(z)L(w) : + \frac{a(w)}{(z-w)^2}$ for $|w|<|z|$ and $L(w)a(z) =: a(z)L(w) : \frac{a(w)}{(z-w)^2}$ for $|w|>|z|$. Then $a(z)L(w) \sim \frac{a(w)}{(z-w)^2}$.