Infinite Dimensional Lie Algebras
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Gabriel C. Drummond-Cole

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Lattice algebras:
We have $\hat{\mathfrak{h}}$, that is, $x[n]$ for $x \in \mathfrak{h}, n \in \mathbb{Z}$, e^λ, and $\lambda \in L$, an integer $(\lambda\mu) \in \mathbb{Z}$ lattice (isomorphic to $\mathbb{Z}^n \subset \mathbb{C}^n \cong \mathfrak{h}$ in $\mathfrak{h}$. These are lattices $\sqrt{N}\mathbb{Z}$.

Then the commutation relation will be of the form $x[0]e^\lambda = e^\lambda(x[0]+(\lambda, x))$ and $e^\lambda e^\mu = e^{\lambda+\mu}$.

Representations of this lattice algebra: you cannot take a single Foch space because all vectors have the same eigenvalue for the zero degree element. So they will be $V_L = \bigoplus_{\lambda \in L} \mathscr{F}_\lambda$, where $e^\lambda|\mu\rangle = |\lambda + \mu\rangle$.

This is not the only representation. Can anyone suggest another representation of it? How about $\bigoplus_{\lambda \in \lambda_0 + L} \mathscr{F}_\lambda$. The point is that you don't really care where you start as long as you have the spacing right. These will not be isomorphic because $x[0]$ will act differently in each of them. For the future we want to add the condition λ_0 such that $(\lambda_0, L) \in \mathbb{Z}$.

For example, if $L = \sqrt{2}\mathbb{Z}$, I can consider $\cdots \oplus \mathscr{F}_0 \oplus \mathscr{F}_{\sqrt{2}} \oplus \cdots$, but you can also consider starting at $\sqrt{2}2$. For $L = \sqrt{N}\mathbb{Z}$ we will have $\lambda_0 \in L/N = \mathbb{Z}^n/\sqrt{N}$.

As homework, how many are there for a given integer lattice?

Okay, so take $\hat{\mathfrak{h}} = \oplus_{n\in\mathbb{Z}}\mathfrak{h}[n] \oplus \mathbb{C}$ with $[x[n], y[m]] = n\delta_{n-m}(x, y)$; we can also take this as $\mathfrak{h} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}$. So instead of writing $x[1]$, we write xt. Which one you prefer is a matter of convenience.

So how do we write the commutation relations? Well, they now become the following: $[xt^n, yt^m] = n(x, y)\delta_{n-m} = Res_0((xt^n)'yt^n)$, the coefficient of t^{-1}. With $n = -m$ you get $nt^{n-1}t^m(x, y)$. The residue of this thing is 0 unless $n + m = 0$. Otherwise the residue is the coefficient, $n(x, y)$. If you don't like the word residue, I can write it as $\frac{1}{2\pi i}\oint(xt^n)'yt^m dt$ So we can write $\hat{\mathfrak{h}}$ as polynomial functions on S^1 with values in $\mathfrak{h}$ direct summed with $\mathbb{C}$. Then $[f(t), g(t)] = \frac{1}{2\pi i}\oint(f', g)dt = \frac{1}{2\pi i}\int df, g$.

So what happens if we try to make a change of coordinates? That's a vague question. How do we make it precise? We have an infinite dimensional group and we want to see how the group acts on our algebra; we might instead ask about the action of the Lie algebra; if we take the group of diffeomorphisms, what is the Lie algebra? Vector fields, of course. So rather, vector fields on S^1. Consider the Lie algebra $W = \{f(t)\frac{\delta}{\delta t}\}$. Here t runs over the unit circle, so I want tangent vector fields. For now let me forget reality conditions and then I get the complexification of the Lie algebra of vector fields on S^1. It has a very nice basis. If instead I consider Laurent polynomials, it has the basis $L_n = -t^{n+1}\delta_t$ for $n \in \mathbb{Z}$. Now, what is the commutator of two vector fields like this? First of all, the minus sign disappears. We have $[L_n, L_m] = -(n-m)t^{m+n+1}\delta_t$ by easy calculations. So $[L_n, L_m] = (n-m)L_{n+m}$. This acts on a polynomial by changing degree by n, which is why I have chosen it this way.

So the first thing to do is to ask how my Lie algebra acts on my algebra. We can easily see how L_n acts on a polynomial with values in a vector space. We have $L_n.(xt^k) = x(-k)t^{k+n}$. If you prefer, you can write it in terms of $x[n]$ as $L_n.x[k] = -kx[k+n]$.

So far nothing interesting. The interesting thing is as follows. Can we extend this action to representations for example to $\mathscr{F}_\lambda$. What do I mean by extend? If I have an algebra and a group action on it, and a representation of that algebra. What is the natural meaning of an extension? If G acts on A, and we have V, a representation of A, then can we find a representation of G on V with $(g.a)v = g(a(g^{-1}v))$. If we already know how the group acts on the vector space, then it is obvious how to act on operators: by conjugation. How do you go the other way?

For Lie algebras, we need something quite similar: Can we define an action of L_n on $\mathscr{F}_\lambda$ such that $L_n.x[k] = [L_n, x[k]]$? The answer is yes, and we have the Sugawara construction. Take $L_n = \frac{1}{2}\sum_{k+l=n} a^i[k]a^i[l]$ where a^i is an orthonormal basis in $\mathfrak{h}$. In the case where $\mathfrak{h} = \mathbb{C}$ this is $\frac{1}{2}\sum_{k+l=n} a_k a_l$.

Why does it work? We have a serious problem with this as written; the sum is infinite. So for L_0 we get $\frac{1}{2}\sum a^i[n]a^i[-n] = H$, the Hamiltonian You remember, we have a problem with that. So we put the normal ordering around everything, to get $L_n = \frac{1}{2}\sum_{k+l=n}\sum_i : a^i[k]a^i[l] :$. It is easy to show that this doesn't depend on choosing a basis, but it's easier to just choose one.

So let's check commutators. Okay, extending an action, let me give you a philosophical remark. We want to do this construction in a coordinate-independent way. For the algebra we did it; it's an algebra of polynomials in one variable. But instead we can think of functions on the complex plane with poles at 0 and infinity. Or functions in a neighborhood of zero with a finite pole there. Then these are basically meromorphic functions near the origin. You need only to know how to differentiate. But the construction of the Foch module, as originally defined, was using the explicit formulas in terms of $x[n] = xt^n$, so the definition was dependent on the parameter t. If we can extend it, this will mean we can give a coordinate free definition of the Foch module as well, which doesn't require picking t but rather would be canonical, so that instead of a pole at zero, maybe we can say "take a Riemann surface and do blah blah" instead of explicitly identifying a neighborhood with $\mathbb{C}$.

But for now let's return to the mundane checking of the commutator. I'll do it in the one-dimensional case. Does $[a_k, L_n]$ equal ka_{n+k}? So by the way we defined L_n, we have

$[a_k, L_n] = \frac{1}{2}\sum[a_k, : a_i a_{n-i} :]$. The only two possibly nonzero terms are when i is $-k$ or $n+k$. We don't need to care about normal ordering because that just corresponds to adding a constant, which we know is finite. The calculation gives

$$\frac{1}{2}([a_k, a_{-k}a_{n+k}] + [a_k, a_{n+k}a_{-k}]) = ka_{n+k}.$$

This answer even works for the special case $n = 2k$.

Let's stop here. Next time I'll tell you that this isn't the end of the story, there is one more complication.