# Infinite Dimensional Lie Algebras <br> April 27, 2005 

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So we have quite a bit of material to cover today. Are people familiar with modular forms? Have you heard of them? Let me then remind you where I stopped last time. We were discussing the space of functions on the Cartan subalgebra, namely we showed that the characters, and recall we can consider characters on the module as functions on the Cartan, ch $V(H)=\sum \operatorname{dim} V[\Lambda] e^{2 \pi i\langle\Lambda, H\rangle}$ for $H=h+u k-\tau d$

This is defined and holomorphic for $\operatorname{Im} \tau>0$. If $\Theta_{k}$ is the space of functions $f$ on the usual finite dimensional Cartan such that $f\left(h+\alpha^{\mathfrak{}}\right)=f(h)$ and

$$
\left.\left.f\left(h+\tau \alpha^{\mathfrak{V}}\right)=f(h) e^{-2 \pi i k(\langle h, \alpha \mathfrak{}}\right\rangle+\frac{(\alpha \sqrt{ }, \alpha \mathfrak{}}{2} \tau\right) .
$$

If I fix $\tau$ then every such character is in the span of the basis ch $L_{\lambda, k}$ that is, $L_{\Lambda}, \Lambda=\lambda+k \Lambda_{0}$. This is a basis in the Weyl-group invariant $\Theta_{k}^{W}$, for the normal Weyl group which acts obviously.

We have a basis $\theta_{\lambda}, \lambda \in k^{-1} P / Q^{\sqrt{ }}$ in $\Theta_{k}$.
These are invariant under $\mathfrak{h} / Q^{\sqrt{ }}+\tau Q^{\vee}$. This is roughly where we stopped last time.
Let me also say that what I did not really explain is how you relate these two bases. I'll say a little more today, but basically the character formula gives us a way to relate them, just as the way that in the finite case, you can write character formulas in terms of $e^{\lambda}$.

However, there is one important thing we have not discussed. This gives this if you fix $\tau$. However, we have not discussed what happens when you vary $\tau$. Another way to say this basis statement is that if you consider functions $f(h ; \tau), h \in \mathfrak{h}, \operatorname{Im} \tau>0$, then $c h L_{\lambda, k}$ form a basis of $\Theta_{k}$ over functions in $\tau$.

This brings us to the notion of modular forms. Let me just remind you some basic stuff about $S L_{2}(\mathbb{Z})$ and so on.

### 0.1 Modular forms etc.

Let me forget for some time about Lie algebras, Cartan subalgebras and all of that. Consider $E_{\tau}=\mathbb{C} / \mathbb{Z}+\tau \mathbb{Z}$. Let me write $\{\operatorname{Im} \tau>0\}$ as $\mathscr{H} \subset \mathbb{C} /$

The first observation is that this torus depends not on $\tau$ but on the lattice. So you take the lattice. If $V$ is a 1 -dimensional complex vector space and $L \subset V$ is a lattice, then you can choose a basis $\left\{\omega_{1}, \omega_{2}\right\}$ for the lattice, and then identify $V$ with $\mathbb{C}$ such that $\omega_{1} \rightarrow 1$. Then $\omega_{2}$ can be chosen with positive real part. So any lattice in $V$ can be identified with $\mathbb{Z}+\tau \mathbb{Z} \subset \mathbb{C}$ so the quotient $V / L \cong E_{\tau}$.

Choosing a different basis will not change $V / L$. For example, we can choose $1, \tau+1$, so the quotient will be the same. It's not very difficult to check when two tori are equivalent, it is when they are related by an element of $S L_{2}(\mathbb{Z})$.

Theorem $1 E_{\tau} \cong E_{\tau^{\prime}}$ if and only if $\tau^{\prime}=\frac{a \tau+b}{c \tau+d}$ for some $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in S L_{2}(\mathbb{Z})$. The best source I know for this is Serre, Course of arithmetic.

The second observation is that $S L_{2}(\mathbb{Z})$ is generated by two elements $T=\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ and $S=\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right)$. These correspond to $\tau \rightarrow \tau+1$ and $\tau \rightarrow-1 / \tau$.

Here is a generalization. Let $E_{\tau} \rightarrow E_{\tau}$ by $z \rightarrow \frac{z}{c \tau+d}$ then $\tau^{\prime}=\frac{a \tau+b}{c \tau+d}$.
Now suppose we have a function. We want to study the action of $S L_{2}(\mathbb{Z})$ on functions $f(z ; \tau)$ given by $f \rightarrow f\left(\frac{z}{c \tau+d} ; \frac{a \tau+b}{c \tau+d}\right)$.

Theorem 2 If $\theta=\theta(z, \tau)$ is a $\theta$ function, $\theta \in \Theta_{1}\left(E_{\tau}\right)$ then $\theta\left(\frac{z}{c \tau+d} ; \frac{a \tau+b}{c \tau+d} e^{-\pi i \frac{c}{c \tau+d} z^{2}} \in\right.$ $\Theta_{1}\left(E_{\frac{a \tau+b}{c \tau+d}}\right)$.

These spaces are one dimensional, so up to a factor there are only one element in each space.
So then $\theta\left(\frac{z}{c \tau+d} ; \frac{a \tau+b}{c \tau+d}\right) e^{-\pi i \frac{c}{c \tau+d} z^{2}}=\varphi(\tau) \theta(z ; \tau)$.
Then $\varphi=\zeta\left(c \tau_{d}\right)^{\frac{1 / 2}{2}}$ where $\zeta$ is an eighth root of unity which depends on $a, b, c$, and $d$ in a messy way.

Basically you say that both sides are $\theta$ functions, you can see that they are the same up to a constant (in terms of $z$ ). Then to find it, you compare Fourier coefficients. It's rather messy. The point is that not only do these have nice properties under elliptic transformations $z \rightarrow z+1, z \rightarrow z+\tau$ for $\tau$ fixed, but also under modular ones $z \rightarrow \frac{z}{c \tau+d}, \tau \rightarrow \frac{a \tau+b}{c \tau+d}$. That's the summary. I haven't proved it at all. If I started computing it would take me a really long time and I don't think I want that. The point is that you have both the elliptic transformations,
but also modular changes. Theta functions translate nicely under both of them. I cheated a little bit; there should be some conditions on oddness. This is the summary I want you to remember. I should also say that if you only care about elliptic transformations, then you can multiply by any function of $\tau$, such as $q=e^{2 \pi i \tau}$. But the modular transformations will not respect this.

This was all about the simplest possible case, $\theta$ functions in one variable $z$. Do you have a similar statement in our land of Lie algebras? How about our characters, do they translate nicely under modular transformations?

The answer is yes, and let me write for you a precise theorem. Actually, I will do a simpler theorem. I'll only explain what happens with the character as functions of $\tau$, so I'll check for $h=0$.

Theorem 3 Consider $\chi_{\lambda}=\left(\right.$ ch $\left.L_{\lambda+k \Lambda_{0}-\frac{\lambda, \lambda+2 \rho}{2(k+h \checkmark)}} \delta(0-\tau d)\right) q^{-c / 24}$ which is

Then, actually, I have to put here one more correcting term with the $q^{-c / 24}$, where $c=\frac{k \operatorname{dim} \mathfrak{g}}{k+h \checkmark}$. Then the space $\left\{\chi_{\lambda}\right\}_{\lambda \in P_{+}^{k}}$ is invariant under $S L_{2}(\mathbb{Z})$.

In fact, $\chi_{\lambda}(\tau+1)=\chi_{\lambda}(\tau) e^{2 \pi i \frac{\lambda, \lambda+2 \rho}{2(k+h \checkmark)}}$ and $\chi_{\lambda}(-1 / \tau)=\sum s_{\lambda \mu} \chi_{\mu}$, where this is called the $S$-matrix and there are explicit formulas for it. This is important in conformal field theory.

This action of $S L_{2}(\mathbb{Z})$ is unitary. This is by no means obvious for the $S$-matrix part of this. As a corollary, you get that the function $Z=\sum_{\lambda \in P_{+}^{k}} \chi_{\lambda}(\tau) \overline{\chi_{\lambda}(\tau)}$ is $S L_{2}(\mathbb{Z})$-invariant. I leave this to you as an easy exercise.

This is called a partition function, by the way.
The proof, as far as I know, is long and boring. First of all, you establish transformation properties for $\theta$-functions of some special form. That's not very difficult, but you have to find the coefficients that are the analogs of the $\varphi$ s. After you have that, you write your characters in terms of $\theta$ functions. Then you plug one into another and after two or three pages of manipulation you are done. It's a valid proof but not very illuminating. It is easy to explain why characters have nice transformation properties. These transformations, the elliptic ones, come from the affine Weyl group. The modular properties don't seem to make sense. There is no automorphism of the Lie algebra that would allow you to take $\tau$ to $-1 / \tau$. Definitely these modular transformations do not come from a transformation of the Lie algebra. It turns out that there is a way to shed light on why this is, but they require telling a little bit about conformal field theory. Not only can you define them as traces of operators in your representation, but in ways that use not $\tau$ but the torus which $\tau$ determines. That would make the properties more transparent, but it would also require more preparation.

I was sketchy, I gave only statements and not the explicit formulas. If you try to prove this
with affine Lie algebras, this is not transparent, it's long and messy. This is known as the Kac-Petersen formula, and you can find it in Kac, along with the proof. Next time I'll do something similar. I'm going to talk about somewhat easier modular functions. Let's keep things here to prevent confusion.

