Infinite Dimensional Lie Algebras
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Maybe I can write something from what I did last time. Recall

1.
   1. For finite dimensional $\mathfrak{g}$, every character $ch\ V$ defines a function on $\mathfrak{h}/Q^{\vee}$ via $e^{\lambda}(h) = e^{2\pi i\langle\lambda,h\rangle}$.
   2. $ch\ L_{\lambda},\ \lambda \in P_+$ form a basis in $(\mathbb{C}[\mathfrak{h}/Q^{\vee}])^W$ (Weyl group invariant functions on the complex torus). Note: $\mathfrak{h}/Q^{\vee} \cong (\mathbb{C}^*)^r$

      This is the finite dimensional case. Functions here are basically Laurent polynomials; for affine Lie algebras they will be more like θ functions.

Aside: θ-functions Let $E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau, \tau > 0$. Topologically this is just a torus, but it has a complex structure. This is a 1-dimensional complex manifold called an elliptic curve. Any holomorphic function on E_τ is constant, so $\mathscr{O}(E_\tau) = \mathbb{C}$. But I want to study meromorphic functions on it; in the normal case you can write a meromorphic function as the ratio of holomorphic functions but here I don't have enough. This is the next best thing.

Definition 1 $\Theta(E_\tau) = \{f : \mathbb{C} \to \mathbb{C} | f(z + 1) = f(z), f(z + \tau) = f(z)e^{-2\pi i(z+\tau/2)}\}$.

Modular forms will come later. You can argue that this is the simplest factor you could have here. That's the definition. Let me make a small note, that this condition on the second kind of periodicity, can be rewritten $f(z + n\tau) = f(z)e^{-2\pi i(nz+n^2\tau/2)}$. Equivalently, define $T_n f = f$, where $T_n f(z) = f(z + n\tau)e^{2\pi i(nz+\frac{n^2}{\tau})}$. This gives $T_n T_m = T_{n+m}$. This is not a trivial group action. We want our functions invariant under the group action.

Lemma 1 *$\Theta(E_\tau) = \mathbb{C}\theta(z, t)$ where $\theta(z, t) = \sum_{n\in\mathbb{Z}} e^{2\pi i(nz+n^2\tau/2)} = \sum e^{2\pi inz}q^{n^2/2}$, where $q = e^{2\pi i\tau}, |q| < 1$.*

It is a trivial exercise that this converges. Next it satisfies the desired condition. This is seen by saying $\theta = \sum T_n 1$, so it is invariant under the action of the T.

I have not yet proved that up to a factor this is the only solution to these equations. I'm not going to do that now; it's in Mumford's book "Tata lectures on theta."

These are to be seen as sections of a line bundle on the torus, so we have shown that the space of sections is one dimensional. This has one zero at $1/2 + \tau/2$. You can use these to construct all meromorphic functions on E_τ as $\prod \frac{\theta(z-a_i)}{\theta(z-b_i)}$. You can arrange things so that the product is invariant under τ. It is not hard to check the conditions on a, b. Then you can check that every meromorphic function on τ can be written in this way.

Those of you familiar with algebraic geometry might wonder about a slight generalization.

Definition 2 $\Theta_k = \{f | f(z+1) = f(z), f(z+\tau = e^{-2\pi i k(z+\tau/2)} f(z)\}, k \in \mathbb{Z}$.

In the language of algebraic geometry, this is the k tensor power of the old line bundle Θ, so it is a line bundle. You can ignore that.

So one way to get sections of this is to take the k tensor power of sections of Θ, but that isn't all of them.

Theorem 1 *1. for $k < 0$ there are no global sections.*

2. for $k = 0$, $\Theta_0 = \mathbb{C}$, these are constant.

3. for $k > 0$ this space is k-dimensional, and a basis is given by $\theta_a = \sum e^{2\pi i k(nz+n^2\tau/2)}$, which is not a difficult thing to guess, except, sum over what, and here is the answer: $n \in \mathbb{Z} + a/k$. This is $\sum T_n e^{2\pi i k(az/k + a^2\tau/k^2)}$ Of course, $0 \le a < k$.

This is not a very difficult theorem to prove. Write a fourier series for this function. This is not very difficult, rather a trivial exercise, like at the level of the comps. So I'm not going to do this.

So there is a nontrivial line bundle where sections of the power can be written like this, then meromorphic functions can be ratios of such sections.

What if, instead of $\mathbb{C}$ mod this, I have some vector space and a lattice in it.

Let V be a finite dimensional $\mathbb{C}$ vector space with $(\, , \,)$ which is positive definite on $\mathbb{R}M$ and lattice M such that $(M, M) \in \mathbb{Z}$. Then what is the analog of this torus? I take $V/M + \tau M$. What kinds of functions are there on this? The first question is, what is this topologically? It is isomorphic as a complex variety to $(E_\tau)^r$ where r is the dimension of V.

Definition 3 *Even though there are no globally defined functions, I can talk about line bundles. $\Theta_k(V/M + \tau M) = \{f : V \to \mathbb{C} | f(z+\alpha) = f(z)$ for $\alpha \in M$ and $f(z + \tau\alpha = f(z) e^{-2\pi i k((\alpha,z) + \frac{(\alpha,\alpha)}{2}\tau)}\}$.*

So what do you think is the generalization of the one dimensional statement? You can just cut and paste this theorem, I'll just say what the changes are.

Theorem 2 *1. for $k < 0$ there are no global sections.*

2. for $k = 0$, $\Theta_0 = \mathbb{C}$, these are constant.

3. for $k > 0$ this space is $|M^*/kM|$-dimensional, and a basis is given by

$$\theta_a = \sum_{\alpha \in M + \frac{a}{k}} e^{2\pi i k((\alpha,z) + \frac{(\alpha,\alpha)}{2}\tau)} = \sum_{\alpha \in M} T_\alpha e^{2\pi i k((z,a/k)+(a,a)2k^2\tau}$$

Here, let me write the formula, $a \in M^*/kM$, where $M^* = \{\lambda \in V | (\lambda, M) \in \mathbb{Z}\} \supset M$.

If $V = \mathbb{C}$ but $(\lambda, \mu) = 2\lambda\mu$, and $M = \mathbb{Z}$ then M^* is $\frac{1}{2}\mathbb{Z}$ so $M^*/M = 2$.

Since you are summing over this plus M, you don't care about M^* itself, but if you change by kM it will give you the same thing, which is why we mod out by kM.

So all of this was to come up with the proper analog of the space of polynomials. Let me repeat something from last time.

2. Let $\hat{\mathfrak{g}}$ be affine.

   1. For every $L_\Lambda, \Lambda \in \hat{P}_+$, $ch\ L_\Lambda$ defines a function on $X = \{H = \mathfrak{h} + uK - \tau d | Im\ \tau > 0\} \subset \hat{\mathfrak{h}}$. As usual let k be the level.
   $ch\ L_\Lambda(H)] = \sum_{\Omega \in wt(L_\Lambda)} e^{2\pi i \langle \Omega, H \rangle} \dim\ L_\Lambda[\Omega]$, that is, $e^{2\pi i k u}$ times something that does not depend on u.

   Theorem 3 *(a) $ch\ L_\Lambda(h - \tau d) \in \Theta_k(\mathfrak{h}/Q^\vee + \tau Q^\vee), \Lambda \in \hat{P}^k_+$.*
   (b) For fixed τ, $\{ch\ L_\Lambda(h - \tau d)\}_{\Lambda \in \hat{P}^k_+}$ is a basis in $(\Theta_k(\mathfrak{h}/Q^\vee + \tau Q^\vee))^W$

   Proof. (sketch) the first part follows from: $ch\ L_\Lambda$ is invariant under the action of $\hat{W} \supset Q^\vee$ and the condition of invariance under $t_{\alpha^\vee}$ for $\alpha^\vee \in Q^\vee$ is equivalent to the condition defining Θ_k.
   This is simple but you have to be very careful. Let me write for you just one formula, the key formula. $t_{\alpha^\vee}(h + uk + \tau\alpha^\vee) = (h + \tau\alpha^\vee) + (u + (\alpha^\vee, h) + \frac{(\alpha^\vee,\alpha^\vee)}{2}\tau)k - \tau d$.

The characters are thus kind of theta functions on this torus which are in addition Weyl group invariant. They are elliptic, meaning that they change nicely if you translate by either of two lattices. They are invariant under one and change nicely under the other.

Once you fix the level you have only finite dimensional theta functions. So in a way this is even better than the finite dimensional situation.

So $(\mathbb{C}^*)^r$ we now have powers of $(\mathfrak{h}/Q^\vee + \tau Q^\vee)$ and instead of polynomials on $(\mathbb{C}^*)^r$ we have θ-functions.

If you have never seen θ-functions, you should look at an appropriate book. A reference for elliptic functions is "Scrapbook of theory of complex curves" or something like that, by

Clemens. I trust that you know how to use search engines. It gives a very lively description of elliptic functions.

That's it for today. What I'll discuss next time or a little later the modular properties. For fixed τ you have an elliptic curve, but what happens if you change τ, a modular change. Do you know anything about modular transformations, $SL_2(\mathbb{Z})$, all of that? A little bit, okay. It will have to be a quick review. There are no classes Monday, so next class is Wednesday. There will also be learning seminar this Friday, when I'll give a talk on quantum groups.