Dennis Sullivan Course Notes
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Gabriel C. Drummond-Cole

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Okay, this looks like the beginning of our course, right? I got them up to Riemann branched coverings, but they're very shaky, these undergraduates.

Let's discuss applications of this cohomological obstruction theory.

Say we have a $\mathbb{C}^n$ bundle over some space, and we're looking at the $2k$-skeleton. We want to find a nonzero section. We can normalize it to be unit length so it lives on S^{2n-1}. We know we can build sections up to but not including the $2n$-skeleton, because we'll get an obstruction on the $2n$-cell, this element of π_{2n-1}. If we're on the $2n-2$-skeleton, any two of these sections are homotopic. You're going to have a $2n-2$-cell, and sort of by induction, if you have two sections, if you try to build a homotopy between them, this is in $H^{\cdot}(X \times I, X \times \partial I; \pi_{\cdot-1}Y)$. But this is just the suspension of X, so the cohomology groups are just shifted. The first place you meet a problem is again the $2n$-skeleton. The homotopy on a $2n-2$ skeleton would be $2n-1$, so there's no obstruction.

Hello? Hi. Oh, wonderful. Oh.

Let's just remember that now. Consider the $2k$-skeleton where k is fixed, and now vary n, for $n > k$.

I want to define c_k of a $\mathbb{C}^n$ bundle. So c_n will be the Euler class in $H^{2n}(X, \mathbb{Z} = \pi_{2n-1}S^{2n-1} = \pi_{2n-1}(\mathbb{C}^n \backslash \{0\})$. This is done. Forget it. It's done. Forget it. Now I want to define c_k for $k < n$. The statement is, if $n > k$ then $2k \leq 2n-2$. Okay, so now, given this bundle E I can find a section on the $2k$ skeleton unique up to homotopy. I can look at the orthogonal complement. So I write E_n as the trivial bundle, direct sum with E_{n-1}. Now if this is still greater than k, I can find a new section and split it off. It's again unique up to homotopy. You keep doing this until you can't any more, and then you take the Euler class of that bundle. That's c_k.

This argument does not work as is for the real case, which would define Stiefel-Whitney classes. If you split off a complex line you lose two dimensions so you get this uniqueness. You have to use an associated frame bundle for that.

You need to write out the homework to understand, well, unless I explain it better, but that's no good either because the things you understand best are the things you learn yourself.

The Stiefel-Whitney classes are cool, but these ones appear all over the place, the Chern classes.

The mod two classes, some of them are really neat and all that, but somehow not as important for the gross national product.

[Do you get the same thing in other categories, smooth or holomorphic bundles?]

Not for holomorphic; holomorphic sections are another game, it uses sheaf theory so I'm not discussing that.

If you can build a continuous section, you can build a smooth section by averaging over a translation. The proof is something like $S\psi(x) = \in \psi(x+y)\phi(y)dy = \int \psi(y)\phi(y-x)dy$. You have to set up some kind of convexity. So let, you define the total Chern class to be $1 + c_1 + c_2 + \ldots$, working in the direct sum of the cohomology groups. If E is a $\mathbb{R}^n$ bundle, you define $P(E) = 1 + p_1 + p_2 + \ldots + p_i \in H^{4i}(X, \mathbb{Z})$ by $PE_{\mathbb{R}} = C(E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})$.

You should be asking questions. In words, the Pontryagin classes of a real bundle are defined to be the even Chern classes of the complexified bundle, so $p_i E = c_{2i}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(X, \mathbb{Z})$.

The vector space $E \otimes_{\mathbb{R}} \mathbb{C}$ has a property. You somehow only care about even terms. The odd ones have order two for some reason, the full classes are the Pontryagin classes and the Stiefel-Whitney classes $w_i \in H^i(X, \mathbb{Z}/2)$. Everything else is torsion, I don't remember. You can think of i as rotating either way; there's an isomorphism $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong E_{\mathbb{R}} \otimes_{\mathbb{R}} \bar{\mathbb{C}}$.

Exercise 1 *The $c_{2i+1}(e \otimes_{\mathbb{R}} \mathbb{C})$ have order two in $H^{4i+2}(X, \mathbb{Z})$.*

[What if you tensor with a nontrivial complex line bundle?]

That's a good point. Suppose you have a bundle with fiber $\mathbb{C}$, and then forget and say it's just an oriented $\mathbb{R}^2$-bundle. The first Chern class of a $\mathbb{C}$ bundle is the Euler class. But what is the Pontryagin class? You go to $\mathbb{C}^2 = \mathbb{R}^2 \cong \mathbb{C}$. Now we take c_1 and c_2 of this. The point is that $\mathbb{R}^2 \otimes \mathbb{C} \cong \mathbb{C} \oplus \bar{\mathbb{C}}$.

If you take a secretly complex real vector space and tensor it with $\mathbb{C}$, you have two complex structures, and the second is isomorphic to $V_{\mathbb{C}} \oplus \bar{V}_{\mathbb{C}}$.

A complex structure is the same as having an operator J which squares to the identity, which means you can find a basis where the operator looks like blocks of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So if you look at the conjugate complex bundle, $C(\bar{E}) = \overline{C(E)} = 1 - c_1 + c_2 - \ldots$ Then you can work hard and get the Whitney sum formula, $C(E_1 \oplus E_2) = C(E_1)C(E_2)$. If you apply that here, if you start from a $\mathbb{C}$-bundle and forget and then complexify, you get $E_{\mathbb{C}} \oplus \bar{\mathbb{C}}$, which has Chern class $(1 + c_1 + c_2 + \ldots)(1 - c_1 + c_2 - \ldots)$. When we multiply this out, we get $p_1(E_{\mathbb{C}}) = 2c_2 - c_1^2$, so p_1 of a complex line bundle is $-c_1^2$.

The first Chern class, the extra information is an orientation. $\pm c_1$ are going to have the same square. To take the square root you need the orientation.

There's a famous formula. If you have a four-manifold, the second homology, if you look at $H_2(M^4)$, simply connected maybe, and oriented. The homology is $\oplus^k \mathbb{Z}$ in the second place, and zero then $\mathbb{Z}$ below and above. Then you have the intersection matrix a_{ij}. The signature is the difference between the number of positive and negative eigenvalues. Now $sig\ M^4 = -\frac{(p_1(M^4), M^4)}{3}$. Now say this is secretly a complex manifold, e.g., $\mathbb{CP}^2$. This has $\mathbb{Z}$ in its middle homology and intersection matrix [1]. So $p_1/3 = 1$ up to a sign, so $p_1 = \pm 3$. Now, $c_2 \in H_4$ and $c_1 \in H_2$ of the (complex) tangent bundle is the Euler class, which, evaluated on the fundamental class, is the Euler characteristic, is 3. So $c_2 = 3$. But p_1 is going to be $2c_2 - c_1^2$. So we get that p_1 should have been -3 so that $c_1^2 = 9$ and $c_1 = \pm 3$.

Now, I never know whether it's ± 3; algebraic geometers have some sort of convention. You get an interesting condition that, suppose you have a four manifold, and you want to know if its real tangent bundle is complex. You have to be able to write $-3sig = 2\chi - c_1^2$, so you get $c_1^2 = 3sig + 2\chi$. This is an integer you can compute given a 4-manifold, and you have to be able to find a vector whose square (under the quadratic form) is this number. If you can do it, well, Witten, doing something in quantum field theory, came up with the expression $3\sigma + 2\chi$ doing some σ-model of string theory or something, I forget. If you can do this, it's a cohomology class. It has to have another property, $c_1 \in H^2(M, \mathbb{Z})$ so $c_1 \mod 2$ is in $H^2(M, \mathbb{Z}/2)$ so $c_1 \cdot x = x \cup x$. Remember the Wu class? c_1 has to restrict to this element and then its square has to be given by this formula. That's a necessary and sufficient condition.

So I was going to mention some more about that, because there's a great research problem, let's talk about the existence, I'm going to do manifolds of dimension two, four, and six. When does the tangent bundle have a complex structure?

In two, it's if and only if the manifold is orientable. If it's orientable, let rotation by ninety degrees be the J-operator. This is a specialization of every orientable $\mathbb{R}^2$ bundle being complex. The splitting in higher dimensions has to respect the higher differential.

Now go to four, and suppose you have a four dimensional space, and you consider on $\mathbb{R}^4$ all possible Js, and you make a bundle over the manifold, over each point you put the set of J applied to the tangent space at that point. You can imagine that we have a metric and J is orthogonal. Fix a vector in $\mathbb{R}^4$, our J takes this into a point on S^2 in the complement. Then an orientation on the remaining two dimensions gives us a complex structure on the remainder. The set of all J is thus S^2.

So this is an S^2 bundle over M^4. So the first obstruction is in $H^3(M, \pi_2 S^2) = H^3(M, \mathbb{Z})$. If this is zero, you get a second obstruction in $H^4(M, \pi_3(S^2))$. The first Chern class is u, where $\delta c = 2u$.

You have to compute that this is the second obstruction. This is some work. The sixth case is a magic dimension and everything is easier. There are these Calabi Yaus and these other manifolds, which are seven dimensional, but seven is close to six. So what is the space of J? Fix a vector, and then the set of all J maps to S^4, and the orthogonal complement is $\mathbb{R}^4$.

So this space is a fibering of S^2 over S^4. So J_2 is a point, J_4 is S^2, and J_6 is a six manifold fibered over S^4. I don't have time to look at the homotopy groups of this, but as a small miracle, J_6 is diffeomorphic to $\mathbb{CP}^3$, complex lines in $\mathbb{C}^4$. So $\mathbb{C}^4$ is the quaternionic plane $\mathbb{H}^2$, so there's a map to quaternionic lines which are S^4, and that's the map.

That has to be proven, but each line cuts the unit sphere S^7 in a circle. So $S^1 \to S^7$ fibers over $\mathbb{CP}^3$. Looking at homotopy groups, S^1 has one $\mathbb{Z}$ and then is zero forever, and so$\mathbb{CP}^2$ has a $\mathbb{Z}$ in dimension two and then matches S^7 from there on up. The first obstruction is thus in $H^4(M^6, \mathbb{Z})$. Can you fine c_1 which reduces modulo two to w_2, the second Stiefel-Whitney class.

This is the first obstruction, an element of order two. It's the obstruction to lifting the second Stiefel-Whitney class to a $\mathbb{Z}$-class. So if $H_3(M^6)$ has no 2-torsion, we can go on. We get the next few for free, and because you go up a dimension you not only get existence, you get uniqueness. The only question is then the w_2 lifting question. So S^6 has an almost complex structure, this is a famous thing. There's a famous open question of whether it admits an actual complex structure. Everyone thinks it doesn't. Chern said he had a proof before he died, but it wasn't convincing to Griffiths, I think. When I was a graduate student this was presented as strange, but any orientable M^6 with $w_2(M^6)$ the reduction of a $\mathbb{Z}$ class has one. Almost all of them do.

Oh, I was a little wrong; you need a choice of components of almost complex structure corresponding to a choice of first Chern class.

Here's a whole bunch of problems here; are there any further topological conditions on this having a complex structure? Having a nondegenerate 2-form is [[unintelligible]] about having a symplectic structure, but there are more topological invariants.

You have almost complex and almost symplectic, which are the same. It's a nondegenerate two form. Whether this is symplectic is the linear question of whether this is closed; the complex question is $dJdJ + JdJd = 0$. If both are true and compatible, you have K ahler manifolds. You discuss mirror symmetry in this part.