Dennis Sullivan Seminar
March 8, 2005
Paul Seidel Talk:
Khovanov Homology and Symplectic Topology (joint work with Ivan Smith)

Gabriel C. Drummond-Cole

March 18, 2005

Oh good, I've ordered food for fifteen, for dinner. 5:30, 6:00. Oh, I'm glad you came, Gabriel, how did you know to come, it didn't say Khovanov. Could you try to find [unintelligible]on the tapes, how important is it to have?

[Ah, forget it.]

Really? There's going to be a little delay because there's something that's not in the room.

Usually we go for an hour and a half, stop for tea, and then the die-hards go on for another session. Today we won't have tea, we'll maybe take a little break and then have an early dinner.

We're waiting on the tape. So if you have an unmentionable to say,

[I will have to cut out my sarcastic remarks about knot theorists.]

[(about an announcement on the board) Is ϵ a big or small number?]

Gromov uses ϵ to refer to a big number.

Ah, okay, are you ready? So. All right.

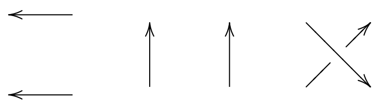
1 Main Talk

I'm very happy to be here. All the stuff I'm going to talk about is joint work with Ivan Smith, all based on work by Mike Khovanov.

[Next week, Eliashberg is coming to give a proof of the Mumford conjecture; I won't be here.]

Let me start by reminding you some stuff about the Jones polynomial. It starts with $K \subset \mathbb{R}^3$ an oriented link and associates with it a polynomial $V_K(q)$. You need an orientation because changing the orientation of a component changes the polynomial by a power of q depending on the linking number with the other components. I'm going to use a slightly nonstandard normalization, taking V of the unknot to be $q + q^{-1}$, which is usually taken to be 1.

Recall the Kauffman bracket relation. We have some diagrams which differ locally as



.

We will call these K_0, K_1, K, and you have $K_0 - qK_1 + K = 0$. This is easy to program, you get 2^n terms.

You can tell that I am not a knot theorist because I can only draw the trefoil as a braid closure. Any other attempt invariably results in the unknot. So if you smooth a crossing you get an unknot or a Hopf link; when you smooth the Hopf link you get two copies of the unknot. Then multiplying everything out you get $q^{-1} + q^{-3} + q^{-5} + q^{-7} - q^{-7} - q^{-9}$. Notice that there is cancellation here.

Now let's move on to Khovanov cohomology. Here we associate with K a bigraded Abelian group $Kh^{*,*}(K)$ with $Kh(unknot) = \mathbb{Z}_{0,-1} \oplus \mathbb{Z}_{0,1}$ and a long exact sequence $Kh^{i,j}(K) \to Kh^{i-w+1,j+3w-2}(K_0) \to Kh^{(i+1,j-1)}(K_1) \to Kh^{i+1,j}(K) \to$ where w is the partial writhe which counts the difference in orientation. When you do the Kauffman bracket calculus you get things coming out nicely, but not when you want an invariant.

The good thing to take here is $\sum(-1)^i q^j rank\ Kh^{i,j}(K)$ to make something like an Euler characteristic. So this is $V_K(q)$. So, right. What happens,

[Well, wait, why should there be this enhancement?]

There are different answers. One answer is that the coefficients are integers, so they are dimensions of a vector space. There is no obvious reason or it would have appeared ten years earlier. The relation to physics is still quite complicated.

For the trefoil you get $Kh^{i,j}(3_1) \to Kh^{i+3,j-8}(0) \to Kh^{i+1,j-1} \to \cdots$

[How'd you deduce that from this equation here?]

It's just the shift offset.

Anyway, let me give you the cohomology of the Hopf link, which you can use by looking at this exact sequence again. It is $\mathbb{Z}_{-3,5} \oplus \mathbb{Z}_{-3,7} \oplus \mathbb{Z}_{-1,1} \oplus \mathbb{Z}_{-1,-1}$, and you get a possibility for cancellation. So there is a possibility for the map to be nonzero and the Jones polynomial can't tell us.

It turns out that this is multiplication by two so that $Kh^{*,*}(3_1)$ is

$$\mathbb{Z}_{-3,9} \oplus \mathbb{Z}_{-2,5} \oplus \mathbb{Z}/2_{-2,7} \oplus \mathbb{Z}_{0,1} \oplus \mathbb{Z}_{0,3}$$

So I hope I did this right. You have one copy of $\mathbb{Z}$, another three, and then one of $\mathbb{Z}/2$.

Now let me write a graded group where I collapse the two gradings diagonally. Then for me as a topologist I get $\bigoplus_{i+j=k} Kh^{i,j}(3_1) = H^{k-1}(S^2) \oplus H^{k-3}(\mathbb{RP}_3)$.

This is a bit arbitrary, since I've lumped together things that maybe don't go together according to grading. This is going to come back, keep it in mind.

Now I would like to go a little bit into the definition of Khovanov homology, which is less popular but shows you the ingredients you need.

1.1 Khovanov's arc categories H_m.

So we're going to talk about, well, we start with $A = H^*(S^2, \mathbb{Q}) = \mathbb{Q}[t]/t^2$, which is a commutative Frobenius algebra like the homology of any manifold, so it defines a $(1+1)$ dimensional TQFT. If you have a one-manifold with k circles you associate $A^{\otimes k}[k]$, shifted down by k to add symmetry. I shift to balance it, to put the middle at zero.

If you have a surface it gives a map from $A^{\otimes k}[k] \to A^{\otimes l}[l]$ of degree $-\chi(\Sigma)$.

If you didn't shift it, it would include the difference.

So you have multiplication, comultiplication, unit and counit. We will not use the unit and counit, do you use it, you don't actually need those.

Starting from the TQFT I will define a category which is not scary because it has finitely many objects. These are the arc categories H_m, $m \geq 1$ and the objects of this category are crossingless matchings in the upper half plane.

This means that you take the real line, take $2m$ points, connect them in pairs, with no crossings and staying in the upper half plane. With two points there are 1 possibility, with four two, and in general $\binom{2m}{m}/(m+1)$ (?). What are the morphisms? This is a linear graded category, so morphisms form a graded vector space.

I could do it over $\mathbb{Z}$, I guess. The morphisms, let me call an object μ or ν, say, with m fixed (defining the category), a matching gives a planar link $\bar{\mu}\nu$ by reflecting μ into the lower half plane and connecting. I will call the object X_μ.

So $Hom(X_\mu, X_\nu) = A^{\otimes k}[k-m]$ where k is the number of circles in $\bar{\mu}\nu$. I shifted it by k but then down by m for my personal pleasure.

The interesting thing is composition of morphisms. Assume you have μ, ν, σ, and you have the three diagrams. Then there is an obvious cobordism between the two circle sets. You move the two copies of ν toward one another dually and then let them join. This gives you a $TQFT$ map with degree m, which is why I shifted by m to give you degree zero.

Since there are only saddle points, all I need are multiplication and comultiplication. Which ones you need are dependent on the diagram.

This gives me a linear graded category. I haven't used that this is the two sphere. This is a linear graded category, but you can also think of it as an algebra $\oplus_{\mu,\nu} Hom(X_\mu, X_\nu)$. The product is zero when you can't multiply.

Okay, and now in particular, it makes sense to speak of H_m-modules, bimodules, and so on. Okay, just think of it as modules over the algebra. If you want, think of it as modules over the category.

For μ you get a projective module also called X_μ. Take the direct sum where μ is fixed. These are the elementary projectives, the summands of the free algebra.

Now there's one more thing, which is not a formal thing, namely, if you consider, if you put $2m$ points on one line and $2m-2$ on another, with a planar tangle, $i-1$ lines and then a cup and then the rest of the lines, this gives you an H_{m-1}, H_m bimodule. Reverse the diagram and you reverse the order of H_m and H_{m-1}. This is not quite free, it's left and right projective. This is the categorification of the Tempoley-Lieb algebra.

You have many more planar tangles than these, but this is the basic one.

So the claim is basically that in order to make Khovanov homology all you need is the H_m and these planar bimodules. To be more precise I need more homological algebra. As soon as you leave the planar, you leave modules and enter chain complexes of such modules.

So $\mathscr{K}_m$ will be the homotopy category of bounded complexes of projective H_m modules.

So if you like, you can take $\mathscr{K}_m \subset D^b(H_m - mod)$, with this a full subcategory, but this is overkill because all of these are projective.

Okay, so if you have an A, B bimodule, you can tensor it with an A-module and get a B-module. So the bimodules we've constructed give functors $\mathscr{K}_{m-1} \iff {}^{\cup_i}_{\cap_i}\mathscr{K}_k$, which are biadjoint.

By the way, these functors act on the chain complex in the trivial way, you do this on each piece seperately. So now I want $\mathscr{K}_m \to^{T_i} \mathscr{K}_m$, so this is $Cone(\cup_i \circ \cap_i \to id)$, so that geometrically you need the unit and counit to make disjoint, well not for this one but for the next one.

So the cone I mean the mapping cone, the cone of bimodules here.

The main facts about these, now these facts will start to rely on S^2, are

- T_i is an automorphism (self-equivalence)
- $T_i T_j \cong T_j T_i$ for $|i-j| \geq 2$
- $T_i T_{i+1} T_i \cong T_{i+1} T_i T_{i+1}$.

so you get a (weak) action of Br_{2m} on $\mathscr{K}_m$.

Okay, so what do you do, you take a link, write it as a braid closure $K = \bar{\beta}, \beta \in Br_m$, with two canonical crossingless matchings $\bar{\mu}$ and μ. We have the $2m$-stranded braid $\beta \in Br_m \times 1_m$ and so we can take $Hom_{\mathbb{Z}}^{*,*}((\beta \times 1_m)(X_\mu), X_\mu) = Kh^{*+?,*+?}(K)$

If you think in terms of extended TQFT's, you have the lower and upper parts, which are like the duality of Hom. So why is this bigraded? Because I forgot something. H_m-modules are graded because of the degree of A. So the morphism groups are bigraded.

So this is one possible definition of Khovanov homology. This is a Hom rather than an Ext because these are all projective.

This is not the definition of Khovanov homology you usually see, which is more combinatorial. Up til now, I made the claims about the braid relations and didn't prove them. Well, if you do $\cap_i \circ \cup_i$, this comes with maps to and from the identity, with a shift, and this gives an exact sequence of bimodules. You can write down everything you need and from that you get that it is well-defined as a link invariant. In the simplest case you get that your algebra sits in between k and k. You get the same theory over another sphere with the grading screwed up. There's some sense in putting in other spheres? The theories you get from even spheres are all equivalent. There is another sign from the odd one. Khovanov's theory is the even one, and the odd one is Khovanov's evil twin. The sign conventions in the odd theory are very complicated in the combinatorial world.

The crossing arises because you have the cone on the 0 smoothing into the 1 smoothing.

This is as much algebra as is going to happen. The Tempoley-Lieb algebra is given by looking at these tangles and quotienting out by certain things. This lives above it in a very precise sense. TL_m is the Grothiendieck group of this category $K_0(\mathscr{K}_m)$. This is not true. This is the representation of the algebra; you get the algebra itself by looking at the category of bimodules. Then you get the TL-algebra. You shouldn't really trust me on this, because I haven't really checked, thought about these [unintelligible].

So, um, if now you're so familiar with gauge theory and so on, you will notice similarities to, say, the Floer exact triangle, the Oszvath Szabo triangle. So can you give an explanation of Khovanov homology in terms of this gauge theory. Khovanov told me, "this is the manifold you want to look at." One nice thing about this seminar is that I can say why it is you want to look at it.

1.2 Springer Varieties

The reference is Chriss, Ginzburg, Representation theory and Complex Geometry.

I could say somehting about the gauge theory but it's not defined yet. I'll be talking about symplectic topology. I will be looking at $\mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{C})$. We have $G = SL_{2n}(\mathbb{C})$ which acts on $\mathfrak{g}$ by conjugation. Then we have $\chi : \mathfrak{g} \to \mathfrak{g}/G \to \mathfrak{h}/W = \mathbb{C}^{2n-1}$. There are finitely many possibilities given by the Jordan normal forms. The particular piece I want to look at is the "cone" $N \subset \mathfrak{g}$ of nilpotent matrices. So take $n^+ \in N$ which is a nilpotent matrix with two $n \times n$ Jordan blocks. So you have n^+ and the conjugates Gn^+. What is the tangent space? It is $\{[x, n^+] | x \in \mathfrak{g}\}$. I want to make a transfer slice, a complementary space intersecting this. I want to make this because you like to see what happens transversal to the group action.

It turns out that there is a nice construction how to make such slices, by Jacobson-Morozov, there exist $h, n^- \in \mathfrak{g}$ unique up to conjugatation with $[h, n^+] = n^+, [h, n^-] = n_-$, and $[n_+, n_-] = h$. These are the structure relations of $\mathfrak{sl}_2(\mathbb{C})$ so we get $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{C})$. So we know about the structure of $\mathfrak{sl}_2$-modules. So from $\mathfrak{sl}_2$ representation theory we get $\mathscr{S}_m = n^+ + \{x | [n^-, x] = 0\}$ is a transverse slice.

This seems a lot of work to get a transverse linear space, but now we can use conjugation by h to contract this to n^+. There is a natural $\mathbb{C}^*$ action contracting $\mathscr{S}_m$ to n^+.

Now what I want to look at, for $\mathfrak{g} = \mathfrak{sl}_2$, with $n^+ = 0$ then $\mathscr{S}_m = \mathfrak{g}$, So $\mathscr{S}_m \cap N = N = \{a^2 + b^2 + c^2 = 0\}$. There is a canonical desingularization $X_m \to^{\pi} \mathscr{S}_m \cap N$ with $X_m = \{(x, F) | x \in \mathscr{S}_m \cap N, F \in Fl_{2m}(\mathbb{C})$ is a complete flag, $x(F^i) \subset F^i$ for all $i\}$. So why is this nonsingular? Let's look at this case. Let me write down $\mathscr{S}_m \cap N$. These are determinant zero trace free matrices. Either A is conjugate to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and you have to choose $\mathbb{C} \times 0$, or A is the zero matrix where F is argitrary. So we have a hyperboloid (?) and an S^2, the minimal resolution of the singularity.

The resolution is $\mathbb{C}^*$ equivariant so X_m retracts onto $\pi^{-1}(n_+)$ called the "compact arc."

Should we have a break?

[Is this a good stopping point?]

There's some think stuff coming up.

[Some people won't come back, so you shouldn't stop in mid-sentence.]

I will relate it to the arc categories. The question is, what does it look like? X_m is of complex dimension $2m$, and $\pi^{-1}(n^+)$ is a union of irreducible components of dimension n.

People have studied these, but here comes the first interesting fact, I think due to Khovanov.

Lemma 1 *(Khovanov)*
The irreducible components of $\pi^{-1}(n^+)$ correspond bijectively to crossingless matchings in

the upper half plane.

It's actually very geometric. Recall that π is just forgetting the flag, so this consists of $\{F \in Fl_m | n^+(F^i) \subset F^i\}$. Because of nilpotence it must map to F^{i-1}.

Take a crossingless matching with $2m$ endpoints. You put in the components of the flag between the endpoints, increasing flags. You start, of course, with 0. You start with F^7 (in the picture) and arise at F^3, with two arches underneath. I interpret this as $(n^+)^2 F^7 \subset F^3$.

This gives you a component $C_\mu \subset \pi^{-1}(n^+)$.

So all that needs to be shown is that every flag fits into this. So you get a nonunique diagram to which such a thing corresponds.

Lemma 2 *(Khovanov)*
Each C_μ is an iterated $\mathbb{P}^1$-bundle, which is topologically trivial so $C_\mu \sim (S^2)^m$.

So you just have to see what freedoms you have. If you take F^7, this has to contain the kernel of $(n^+)^3$, it contains this six dimensional vector space, so you have one more choice. This determines F^3 and F^1, as $F^7 \in \mathbb{P}(\mathbb{C}^4/\ker(n^+)^3) = \mathbb{P}_1$. So then the F^6 has to contain the preimage of F^3, which you've chosen already. So again F^6 is given by a line $F^6 \in \mathbb{P}(F^7/(n^+)^{-1}(F^3)) = \mathbb{P}^1$.

Lemma 3 *$C_\mu \cap C_\nu \sim (S^2)^k$, where k is the number of circles in $\bar{\mu}\nu$.*

This is very familiar from the arc algebra. The X_m of ours, for an arbitrary x, not in $\mathscr{S}_m \cap N$. That is, $X_m \subset \{(x, F) | x \in \mathfrak{g}, F \in Fl_{2m}(\mathbb{C}) \text{ with } x(F^i) \subset F^i\} \cong T^* Fl_{2m}$ so this carries a complex symplectic form Ω.

So $X_m, \omega = re(\Omega)$ is a real symplectic form. Then each C_μ is a Lagrangian submanifold. We look at $F(X_m)$ the Fukaya A_∞-category.

Conjecture 1 *The subcategory of $F(X_m)$ with objects C_μ is quasi-isomorphic to H_m.*

The object C_μ will be the same object as X_μ. You take the Floer homology. This is a cochain level thing.

[You can do this when they're not intersecting transversally?]

It's like Bott Morse theory. You have to check multiplication and so on, but the Fukaya category comes with higher order structures, and the conjecture is that this is formal, i.e., the higher order structures contain no more information.

This is formally formal in lower crossing. Also X_m is a hyperK ahler manifold. You would expect Hodge theory to happen here. For the moment this is a conjecture. The X is not compact, which is bad for formality.

This is different than the spectral sequence idea because that begins not with Khovanov but with the evil twin.

[Normally we have tea now but there's going to be food so you can have water, bread and no water. Let's start up again in fifteen minutes.]

2

There will be no more categories. Keep the connection a little bit alive. We're still going on with these manifolds, but I want to describe the, We had the Lie algebra $\mathfrak{g}$, $\chi : \mathfrak{g} \to$ [*unintelligible*] $= \mathbb{C}^{2n-1}$ and the slice $\mathscr{S}_m \subset \mathfrak{g}$, the nilpotents $N \subset \mathfrak{g}$ and X_m a resolution of $\mathscr{S}_m \cap N$ with $t \in Conf^0_{2m}(\mathbb{C})$, unordered configurations of points with center of mass zero. You can think of this as lying inside $\mathbb{C}^{2m-1}$. You let thses be roots of a monic polynomial, then look at the coefficients.

I need $Y_{m,t} = \mathscr{S}_m \cap \chi^{-1}(t)$. This is a smooth $2m$-dimensional smooth affine variety differentiably, independent of t. Things in the slice are n_+ plus something that commutes with n^-

Lemma 4 *$Y_{m,t} \cong X_m$ where this is diffeomorphism.*

This follows from the Grothendieck simultaneous resolution. Inside the family you can connect one to the other.

I should say slightly more is true. If you choose your t in the right way, the slice is hyperK ahler. The advantage is that you can see the sphere structure. From now on I will use the $Y_{m,t}$. The structure that I chose in X_m rotates to the K ahler structure in $Y_{m,t}$. There's no resolution involved. Choose $n^+ = \begin{pmatrix} 0_2 & I_2 & & \\ & 0_2 & I_2 & \\ & & \ddots & I_2 \\ & & & 0_2 \end{pmatrix}$. Then $\mathscr{S}_m =$ $\begin{pmatrix} (A_1)_2 & I_2 & & \\ (A_2)_2 & 0_2 & I_2 & \\ \vdots & & \ddots & I_2 \\ A_m & & & 0_2 \end{pmatrix}$ where A_1 has trace zero and these are all two by two matrices.

Then $\chi = \det(y^m I - y^{m-1}A_1 - \cdots - A_0)$.

I hope I didn't do any signs wrong. I'm not good at linear algebra. In other words, if you consider $Y_{m,t} = \mathscr{S}_m \cap \chi^{-1}(t) = \{A = \begin{pmatrix} P & S \\ R & Q \end{pmatrix} | P, Q$ are monic of degree m and R, S are polynomials of degree $m-1$ in y and $\det(A) = \prod(y - t_i)\}$.

Choose an appropriate K ahler form on $\mathscr{S}_m$ This is just a linear space. Now we'll see what's actually happening. For $t \in Conf^0_{2m}(\mathbb{C})$, $Y_{m,t}$ is smooth and we get a symplectic manifold. If we have a path $\gamma : [0,1] \to Conf^0_{2m}(\mathbb{C})$ we can use parallel transport to get $Y_{m,\gamma(0)} \to Y_{m,\gamma(1)}$ via $h\gamma$ a symplectic isomorphism. You have to be careful choosing the K ahler form because these are not compact so there are technical problems.

In particular, if you take a closed loop in configuration space, you get a symplectomorphism from $Y_{m,\gamma(0)}$ to itself. So the braid group $Br_{2m} = \pi_1(Conf^0_{2m}(\mathbb{C}), t_0) \to \pi_0(Symp(Y_m, t_0))$.

[questions about holonomy and their symplectic properties]

If you forget about the symplectic structure and just go into $\pi_0(Diff)$ you will lose the information, factoring through S_{2m}.

So we have this braid group thing acting. Remember the definition I gave before of Khovanov homology. I had the braid group acting on a category and then I took Hom of something acted on by a knot and with itself and that gave me the homology.

Suppose $t = (0, 0, t_3, \cdots, t_{2m})$; then $Y_{m,t}$ becomes singular. Then $\bar{t}$, the last $2m-2$ points become a point in the $2m-2$ configuration space.

Now what is $Sing(Y_m, t)$? You can compute it by hand or whatever and you get $\{(A_1, \cdots, A_m)\}$ with A_m zero. Then this is $Y_{m-1,\bar{t}}$. transversally to the singular set we have an ODP (ordinary double point) $a^2 + b^2 + c^2 = 0$.

What does it mean to me, I have a singular submanifold. I can perturb slighly and smooth it by replacing the singularity with S^2, i.e., by looking to $(\epsilon, -\epsilon, t_3, \cdots, t_{2m})$. On the other hand, you can take the singular set.

Suppose I have a Lagrangian submanifold. I can put it in the singular set and flow it out so that every point corresponds to S^2. So you can move from Lagrangian submanifolds in one to the other. You have a Lagrangian submanifold in $Y_{m-1,\bar{t}} \times Y_{n,t}$. This corresponds to the functors $\cup_i$.

Anyway, what matters is we can go from $m-1$ to m. They don't vary. You look at it closely and see that they don't.

[How did you get your Lagrangian manifold in the product?] It comes with a natural map to the singular Lagrangian manifold.

This is what ties the different $Y_{m,t}$ together in terms of the the different m for symplectic geometry.

Suppose that you have a crossing free matching μ not in the upper half-plane. This leads to a Lagrangian $L_\mu \subset Y_{m,t}$ where $L_\mu \sim (S^2)^m$.

Use parallel transport to move a point around to make two of your points close. Then I move it to the origin and proceed inductively.

Now fix a t_0 which is just $2m$ points, say, on the line in the plane, and μ the cap off, and consider a knot as a braid closure.

We have $\beta \times 1_m \in Br_{2m}$, and then I take $HF^{*+m+w}(h_\beta(L_\mu), L_\mu)$ and we call this $Kh^*_{symp}(K)$. I hope the formal parallelism is clear. We had $\mathscr{K}_m$ with a braid group action. Here we have a manifold. The conjecture, in its full form, includes the braid group action. We haven't got the braid group action yet.

The conjecture is that $Kh^*_{symp}(K) = \oplus_{i+j=k} Kh^{i,j}(K)$. This is well-defined (not up to an integer) because you have L_μ in both places. The conjecture is that this collapses the bi-grading. This is sad because you can't get anything but the evaluation at -1, which counts the number of components. The A_∞ operations break the grading. The formality gives you the extra grading. I'm almost not assuming you get there.

[This is evidence that the formality conjecture is true.]

It's geometric, but the reasoning makes it like the formality conjecture.

What is the evidence? It's kind of scant. It has the same long exact sequence. Let's look at the trefoil. This is a computation that takes place inside $Y_{2,t}$. Inside here there's an open subset U localizing this. It looks like, well, it fibers over, maybe I shouldn't, it contains $L_\mu, h_\beta(L_\mu)$ where $\beta = \sigma_1^3$ and then $L_\mu \cap h_\beta(L_\mu) = S^2 \sqcup \mathbb{RP}_3$ and $Kh^*_{symp}(trefoil)$ is $H^{*-1}(S^2) \oplus H^{*-3}(\mathbb{RP}_3)$, which is the original Khovanov homology of the trefoil. You somehow have to think of the original picture in some strange way because you have to take pieces of your $\mathbb{RP}^3$ in different grading.

Up to now we proved independently that the symplectic version is a link invariant. In the nontrivial markov move is that the three strands are very close together. It's almost asymptotically fibered over the one where you have two strands less. The problem is to relate Y_{m-1} and Y_m. Your two Lagrangians look like two lines intersecting at a point fibered over what you want. Alternately maybe you could do it with long exact sequences in Floer theory.

[The fact that some invariants of braids give link invariants should correspond to a structure constraint]

It's the fact that you can write one as a fiber bundle over the next one lower.

Before we all collapse, there's one thing that I wanted to say, the connection with Oszvath Szabo theory. Since we're geometers, we like to see everything geometrically. Then we hope to see more properties. There are many variants, and each one uses some or other trick. If you think about it, you have 2×2 matrices, so you can do [unintelligible]equivariant [unintelligible], and here is one thing you can see that the algebraists had not seen. Remember the form of $Y_{m,t}$ as matrices of polynomials with their determinant equal to $\prod y - t_i$. We can take the transpose action; then the fixed point set $Y^\iota_{m,t} = \{\begin{pmatrix} P & ;R \\ ;R & Q \end{pmatrix}\}$. Now let me take the double branched cover branched along this set t, $E = \{(x, y) \in \mathbb{C}^2 | x^2 = \prod(y - t_i)\}$.

So on each S^2 it's the antipodal involution. So you fix the equator and get circles instead of spheres.

Lemma 5 *(Mumford)*
$Y^\iota_{m,t}$ is a Zariski open subset of $Sym^m(E)$.

Take $P, Q, R \in T^2_{m,t}$ and let $(y_1, \cdots, y_m)$ be the zeroes of p (this being monic of degree m and $x_k = R(y_k)$. If you look at this, if y is a point where P vanishes, you get $R(y)^2 = \prod(y - t_i)$ so $(x_1, y_1), \cdots, (x_m, y_m) \in Sym^m(E)$. It's a complement of explicitly given divisors.

Okay, so everything in us now cries out Oszvath Szabo theory. See $L^iota_\mu \subset Y^\iota_{m,t}$. Then $L^\iota_\mu \sim (S^1)^m$. Then

Lemma 6 *(Manolescu, Pemtz)*
L^ι_μ is isotopic to the Ozsvath-Szabo torus $\mathbb{T}_\mu$.

We really want symplectic, but they didn't define this like that.

He gave a description of the whole $Y_{m,t}$ using Hilbert schemes.

Where does this leave us, right? We have $Kh^*_{symp}(K) = HF^*(h_\beta(L_\mu), L_\mu)$. This has a $\mathbb{Z}/2$-equivariant version $HF^*_{\mathbb{Z}/2}(h_\beta(L_\mu), L_\mu)$, and everything here will be $\mathbb{Z}/2$ coefficients. We want to go from here to $HF^*(h_\beta(L^\iota_\mu), L^\iota_\mu)$.

Now, $Y_{m,t}$ isn't quite the symmetric product but it just misses out some divisors.

This makes $HF^*_{sym^d(E)}(\mathbb{T}_{\beta(\mu)}, \mathbb{T}_\mu) = \widehat{HF}^*(M_K \# S^2 \times S^1)$ where $M_K \to S^3$ is the double branched cover branched over K.

So this is a spectral sequence, E_2 is Kh^*_{symp}. You get from one to another by the localization theorem of Floer homology. It's approximately an isomorphism. To get from here to the extra divisors , there's a standard spectral sequence and we hope it to be trivial. I'm not ready to state it as a theorem.

In particular, a corollary, $rank\ \widehat{HF}(M_k \# S^2 \times S^1) = 2\ rank \widehat{HF}(M_k) \leq rank\ Kh^*_{symp}(K)$.

The localization was a pain in terms of Floer homology. Morse theory is allergic to restriction maps unless you have a minimum, which you don't because things arercyt unbounded below and above in Floer homology.

Then the localization map provides a map $HF^*_{\mathbb{Z}/2}(h_\beta(L_\mu), L_\mu) \to HF^*(h_\beta(L^\iota_\mu), L^\iota_\mu) \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[q]$.

This becomes an isomorphism mod torsion, but that's not what I want. This is a torsion module. You can get extra numerical invariants $\chi(cok(\lambda)), \chi(cok(\lambda)), \cdots$ and this recovers the Jones polynomial. The idea is what Dennis basically said is that in the fixed point theory you have Khovanov's evil twin, or something like it. You can see the original bigrading. I'm no longer sure what to say about general K. The advantage is that you can construct, the disadvantage is that you can't compute. The $HF^*_{\mathbb{Z}/2}(h_\beta(L_\mu), L_\mu)$ has not been done algebraically.

That's it.

Originally I hoped that this would be the evil twin. We don't have a geometric viewpoint of it. Suddenly tons of things occur when you write it this way. It's all unexplored.

[I have a question about formality. Where was this issue?]

When we were trying to identify this with Khovanov. You don't prove it for each link seperately. You try to prove it going back to the arc algebra. It boils down to the Fukaya category being formal. It has all these higher order structures and you hope for them to be formal.

The second step is to look at the action of the braid group on both sides. You want to show that they agree.

[Why do you want them to be the same? Maybe yours is different?]

My money is on them being the same.

[What about for alternating knots?]

It should be all on one row? In the alternating case we know it is the same from the spectral sequence, but it's a trick, not satisfactory.

[What is the philosophy?]

Through working in other directions on hyperK ahler manifolds. It's mirror symmetry stretched or something.