

# Dennis Sullivan Course Notes

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All right, so Nathaniel, I've decided, I'll lead into a sort of introduction into these stable mod two operations so you don't need to talk about the rest of it.

All right, all right, so, I made some boo boos last time. When I tell my daughter I made a boo boo she runs and gets me a Band-Aid, but that's not what I mean. The whole thing is quite coherent, though.

I'm discussing which homology groups with a given Poincaré duality are satisfied by closed manifolds.

Let's consider the projective planes. We have the real projective plane, you can think of it as attaching a two cell by degree two onto a circle. That's the real projective plane. It's nonorientable, and it's really in the family, but let's do the complex projective plane. This is  $S^2 \cup_f e^4$ , where  $f : S^3 \rightarrow S^2$  is the Hopf map. The homology is  $\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}$ . It's a fibration where the fiber is the circle. The four ball has boundary a three sphere coiled around the two sphere. So in duality the center guy needs to have the one by one matrix  $(\pm 1)$ . If we have a chain complex. we take the dual space  $Hom(\cdot, \mathbb{Z})$ , and that's the cohomology. If we apply cohomology to  $X \rightarrow X \times X$  the diagonal map we get a ring structure on cohomology. Now if  $x$  generates the middle homology then top homology is  $x^2$ .

There's another projective plane over the quaternions, with  $S^4 \cup_g e^8, g : S^7 \rightarrow S^4$ , and one more over the Cayley numbers, with  $S^8 \cup_h e^{16}$  and  $h : S^{15} \rightarrow S^8$ . There are no more maps from a sphere to a sphere so that you can get an  $x, x^2$  like this.

These are bundles over spheres, and have trivializations over the two hemispheres. You get a map from  $S^k \rightarrow SO(k+1)$  for  $k = 1, 3, 7$ .

Now think of one of these spheres sitting inside  $\mathbb{R}^{k+1}$ . We can evaluate,  $SO(4)$  is the auto-

morphisms of the unit sphere in  $\mathbb{R}^4$ . So this is

$$\begin{array}{ccc} S^3 & \longrightarrow & SO(4) \\ & & \downarrow \\ & & S^3 \end{array}$$

The composition has degree one, meaning you have a frame bundle, so that the sphere is parallelizable. Only the 0, 1, 3, 7 spheres are parallelizable.

You choose a collection of spheres to realize  $a_{ij}$ . I had a collection of spheres with linking numbers given by  $a_{ij}$ . In order to lift this I need to be able to lift to a degree one map, i.e., extend a vector to a frame. So you can only realize a construction for  $a_{ij}$  with some  $a_{ii}$  odd when we're in the situation of  $S^1, S^3, S^7$ .

We can realize every even integer for all odd spheres, since we don't have the Hopf bundle but we always have the, in other words, if I take the even dimensional sphere and its tangent bundle, a bundle that always exists. When I look at the equator, I trivialize the bundle and get a map into, into  $SO(2k)$  and then evaluate this on  $S^{2k-1}$ , giving a map of degree two.

So you can realize any even integer.

Now it's actually a theorem that you cannot realize these odd matrices except in these three dimensions. Let me start stating that.

So let's look at cohomology mod two. This is  $H^*(, \mathbb{Z}/2) = Hom(H_*(, \mathbb{Z}/2), \mathbb{Z}/2)$ . There is a linear operation  $H^k \rightarrow H^{2k}$  by  $x \rightarrow x \cup x$ , which is linear modulo 2. Now, there's a kind of amazing fact, this is a natural operation, there's the suspension isomorphism

$$\begin{array}{ccc} H^k & \longrightarrow & H^{2k} \\ \downarrow & & \downarrow \\ H^{k+1} & \longrightarrow & H^{(k+1)+k} \\ \downarrow & & \downarrow \\ H^{k+2} & \longrightarrow & H^{(k+2)+k} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

Steenrod found the existence of maps like this, linear and natural which increase degree by  $k$ .

$sq^k(x) = x^2$  if  $x = k$ ,  $Sq^k(x)$  is interesting if  $\dim x > k$ , and  $Sq^k(x) = 0$  if  $x < k$ .

So these are natural stable (commute with suspension) modulo two operations extending  $x \rightarrow x^2$ . These are called Steenrod squares.

Now we can consider any natural stable operation which raises dimension; lowering makes you canonically zero while keeping equal makes you zero or the identity. So this is the Steenrod algebra, and these squares actually generate that algebra. Even  $Sq^j$  generate the algebra.

Unfortunately, Travis' question is tied intimately to the structure of these things at the prime two.

Take any of the three examples and suspend it. What you get is a two cell complex, sphere union cell,  $S^N \cup_{\Sigma f} e_{N+2}$  and so on. I claim  $Sq^2$  connects the two of these.

Here's a big fact that everyone, Milnor, Adams, were fighting to prove. I think Adams got it first, then Milnor, and finally Atiyah could do it on the back of a postcard. It turns out that if you have any sphere with a cell attached by a map  $S^N \cup_F e^{N+k}$ , then  $k$  is one, two, four, or eight.

There's an algebra of stable operations,  $x \rightarrow x^p, \text{ mod } p$ . If you like algebra, it's some kind of central structure, and it's fairly deep. You could just discuss, whatever, whatever the previous outline yesterday, you could just start learning about Steenrod squares. They correspond to something about the diagonal. See, Poincaré duality is expressing, well, it's giving Steenrod squares.

If we're trying to realize  $F\langle a_{ij} \rangle$  in middle homology, then what's this space going to look like? It will have a generator for each element in  $f$  and then a  $4l$ -cell. If some  $a_{ii}$  is odd, then there will be a class with  $x_i^2 = a_{ii} \text{ odd}$ . Collapsing lower cells to a point up to the  $i$  cell, I'll get the two-cell complex and I'll need and  $x^2$  going through the two. You can only realize for  $l = 2, 4, 8$ . My proof works for those cases.

Now let's go to  $4l + 1$ . Recall that I have  $T, \langle, \rangle$ . We know that  $T$  is  $A \oplus A$  or  $A \oplus A \oplus \mathbb{Z}/2$ . Let me concentrate on this, suppose I could realize this; let me just talk about the  $\mathbb{Z}_2$ . If I could just realize  $\mathbb{Z}/2$  in the middle, that looks like a sphere  $S^{2l}$  so I need an  $e^{2l+1} \cup e^{4l+1}$ . This is what it looks like homologically. If I call the sphere  $x$  I get  $xSq^1(x) \neq 0$ . But I claim that  $Sq^{2l}$  has to connect the latter two. If you look at cohomology, they give no boundary.

The reason is this. There are two interesting facts. If you have any manifold, and you look at the cohomology with  $\mathbb{Z}/2$  coefficients, and you have  $v_i \in H^i(M, \mathbb{Z}/2)$  and  $v_i \cup x = Sq^i x$  for all  $x \in H^{n-i} M_1 \mathcal{H}_2$ .

If you define  $V = 1 + v_1 + \dots$ ,  $Sq = Id + Sq^1 + Sq^2 + \dots$ , then  $SqV = 1 + w_1 + \dots = W$ .

If your manifold has a boundary, then any evaluation of them on  $M$  is zero.

Thom proved the converse, if something is not a boundary, everything is zero. A manifold is a boundary if and only if some of the  $V$  numbers are nonzero.

Anyway, we've got the conclusion that if a manifold is not a boundary then a square hits the top dimension. There has to be a square going from the middle to the top.

Then by the big result I get that  $2l$  is either 2, 4, or 8. I gave you  $SU(3)/SO(3)$ . I was guessing that maybe there's a restriction. In fact, you can use the planes to make these examples.

You take the three planes and suspend them once.

I'll do it with the first one, and we'll see how it looks. Some of you did the  $SU(3)$  in their homework, and had fun.

So when you suspend  $f$  you get an element of order two. Then  $S^3 \rightarrow S^3$  by degree two, then you get a sequence.

[a bunch of nonsense about (co)fibrations. Note to self: review Gitler, Alomar, whoever.]

Anyway, these are realizable. In  $4l, 4l + 1$  if the  $\mathbb{Z}/2$  is present, you can only realize these examples in two, four, and eight. and the analogous statement for the free case. You can only realize a type I (some  $a_{ii}$  is odd).

Then you analyzed the symmetric forms, but not in high generality. All of the skew in  $4l + 2$  were realizable. In  $4l$  even type II was realizable from tangent spaces to spheres.

Let's discuss when we can find the forms. We build the construction, and if it's above dimension four with determinant one and you can realize the linking with closed spheres, you get a manifold with boundary. In the dimensions above four, the Poincaré conjecture in high dimensions makes it a sphere and you can cone it off.

This is how Milnor discovered exotic spheres, some of these have exotic structures. If you want a smooth manifold, you have more obstructions.

Then down in dimension four, you're doing these knots. It's hard to figure out the fundamental group. It gets into the modern quantum world, Floer homology and all that, but if you just want a topological manifold, you get an independent construction due to Friedman which tells you that any form is realizable. Four has its own story which is quite interesting and complicated. People like Claude LeBrun work on precise questions about realizing things in dimension four using varieties.

Another fact is that, well, uh, ah, we'll return to dimension four. It's an interesting, more elaborate story. Oh.

[Checks his watch.]

So, uh, the situation is sort of in control for the quadratic forms and when there's a  $\mathbb{Z}/2$ . The two-by-twos are not very much in control. I've changed the original question. The original questions are, given a string of groups, are they the group of a manifold. You see that the duality gives you extra structure. Since duality imposes some odd constraints. Which duality are we talking about? This adds a big extra burden in the middle dimension.

Still, you go in there and you don't get to the bottom because of the  $2 \times 2$  that were in the last homework. Talk about a torsion group being a quotient of lattices, and another boo boo I made, is it's not quite true you can use the exact sequence of a manifold with boundary to do this. There was some use I made which was not quite, I make that confession to you. That's leading us in the right direction, however.

Let's change the question one more time. This is very much in the spirit of everything I talked about. Which homology groups are the groups of the manifold? The next level was to add "with duality." But duality comes from something at the chain level, namely that when you turn the arrows around by taking  $\text{Hom}(\cdot, \mathbb{Z})$ , and then taking it and turning it over physically, and starting both at dimension  $n$ , there is actually a chain equivalence. This is Poincaré duality. So the new form of the question is, which of these things can be realized by manifolds? Move to the chain level. Make that an object with a notion of equivalence.

Take a cell decomposition and then a dual one, and the boundary on one is like the coboundary on the other. This is now the right way to answer the other. It will take care of the  $2 \times 2$  matrices, and you'll have the answer. It gives you exactly this, what made me think of this was  $L \rightarrow L^* \rightarrow T$ , you have cycles mod boundaries.

This is the third version of the problem, and this problem, which chain complexes with duality are chain equivalent to chains on a manifold. In fact, the Wu classes and everything are built in.

This is very much in the spirit of this course. The Steenrod squares come from taking the chain complex, and the diagonal, and trying to organize this. So these come at the chain level. You see how they influence the question, but at the level of homology it's weird.

Poincaré duality is something deep, and even though there's been a shitload of work done on it, it's not understood now. The modern appearance of it in string theory and Frobenius algebras is not included in the past work. Now I'm refereeing two papers about Poincaré duality but both authors are getting it wrong because the chain level hasn't been explained yet. In our setting we can see that duality is not unique, but they can't see that yet in their complicated setting.

That's my incomprehensible ramblings.