Dennis Sullivan Course Notes
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I used to be at this place called IHES and physicists and mathematicians worked together. Once I saw them arguing with one another, for like twenty minutes. They didn't understand what a quotient vector space was. They had a Hilbert space, and they wanted a basis in terms of the original functions. They could have just passed to the quotient.

I remember this one student, this engineering student. She was smart, I remember the weekend that she learned what it was. Only mathematicians do that. This, I made that longer so he'd get out of the room. He's having a problem with having a concept that's sort of abstract. It's very simple.

Anyway, uh, so, uh, what did you want to ask me?

[Students aren't coming to my office hours. Make them.]

[How?]

[Throttle them.]

[Withdraw points.]

[Why are you looking for trouble?]

[I just want them to learn.]

Right.

[It's a big day in math when you realize that you can do whatever you want in math. Let whatever.]

I could suppose my way out of a paper bag.

So, you're my grader for this course but you're not here on Monday. So I need, I'm going to be away March 11, Friday, and the following Monday too. So, I want, I'm going to have a review and quiz for my undergraduate class. You can do the review but you can't do the quiz. You can't do the Monday but you can do the Friday.

[By definition of Friday?]

Right. You can tell them about your office hours. I need a volunteer to give the quiz on Monday.

This problem Travis gave me, I feel like I'm the graduate student and he's the aloof advisor. I need a volunteer to survey René Thom's cobordism theory on Friday and then Monday. It's great stuff.

All right. So I'm struggling with this problem here.

[Can you classify two by two symmetric matrices?]

Over what ring? I got into it, got to a Diophantine problem and then felt was it was like, and got tired of pursuing that line. I remembered the original question was, "which ones are realized by manifolds?" This question actually helps you understand the classification.

So you've got T and the pairing $\langle , \rangle$. So either we're in $4l+1$ and this is skew symmetric with $T = H_{2i}$ or it's $4l-1$ and it's symmetric with $T = H_{2i-1}$.

There's going to be a construction from L to T. It's a functor. Let's do the symmetric one first. Suppose I have a free abelian group of finite rank and some pairing $L \otimes L \to \mathbb{Z}$, symmetric but not necessarily perfect. So you have a symmetric matrix a_{ib} where $a_{ij} = \langle x_i, x_j \rangle$. So the determinant is nonzero but it need not be ± 1. A perfect pairing it would be a unit.

In the rationals, consider tensoring with the rationals, then it is a perfect pairing over the rationals. Take $V = L \otimes \mathbb{Q}$.

Then you define the dual lattice L^* as $\{all\ v \in V : \langle v, x \rangle \in \mathbb{Z} \text{ for all } x \in L\}$. Notice that $L \subset L^*$, but L^* is bigger precisely when L is not a perfect pairing.

We have this map $L \to Hom(L, \mathbb{Z})$. This is injective but not necessarily onto. Now we consider the vectors in the rational version. L^* is in fact $Hom(L, \mathbb{Z})$.

You then get a short exact sequence $0 \to L \to L^* \to T \to 0$, where $T = L^*/L$ is torsion, and $|T| = \det(a_{ij})$.

Okay, so now notice that we have, if we take $\langle L^*, L^* \rangle \to \mathbb{Q}$. This implies that $T \otimes T \to \mathbb{Q}/\mathbb{Z}$. Take two elements of T, lift to L^*, and take the inner product. Since this changes by an integer if you change by L, you get the quotient by the integers.

This torsion pairing is perfect. It arises geometrically. Suppose, I'll say that $(T, \langle , \rangle)$ is an algebraic boundary if there exists $L, \langle$ giving it.

Say you have a manifold W with boundary$M$ with torsion in the middle and zero elsewhere. Now you have $L = H_{2l}W,\ L^* = H_{2l}(W, M),\ T = H_{2l-1}M$.

You get a long exact sequence of homology associated to a short exact sequence of chain complexes.

Then there's a duality between H_i of the manifold and H_{n-i} of the relative homology. Let's pretend that there's no torsion. Modulo torsion it says the intersection pairing is a perfect pairing. Your cycles have intersection matrix, the interior homology is not affected by adding a disk; then the paiting is perfect. Suppose $H_{2l}(W, M)$ is free and $H_{2l}(W, M)$ is torsion. If the map $H_{2l}(W, M) \to H_{2l-1}M$ is α then $\alpha^* : H_{2l}M \to H_{2k}(W, M)$ is relative.

Now what? By dualizing and playing around we get a theory. So, so we deduce, this is going to be half the proof of a theorem.