Dennis Sullivan Course Notes
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Scott O. Wilson, guest speaker

Gabriel C. Drummond-Cole

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[Travis: What's a clutching function?]

This doesn't go in the official record, right? What about this?

There's a simple way to characterize vector bundles over the sphere. You have a fiber, a total space, and a bundle

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

Part of the information is the local trivializations. Then the composition of transition functions correspond to a structure group G. Suppose you have a bundle over the n-sphere and you want to classify all bundles with fiber F and structure group G.

Dennis quickly said the words "clutching function, blah blah blah, and so this bundle is," but let's figure out how to work this.

Divide the sphere into the upper and lower hemispheres and the equator. Note that $E|_U$ and $U|_L$ are trivial, i.e., bundle isomorphic to the product bundle since U and L are contractible.

Let me outline a little proof for that. The claim is that any bundle over a contractible space is trivial, meaning it's equivalent to the product bundle. So part a is like a big theorem, but not really big, and it, one way to say it is if you have a bundle and two maps of another space into the base of another bundle. If the maps are homotopic then the pullback bundles f_1^*E is isomorphic (in some sense) to f_2^*E.

The pullback puts the fiber of $f(x)$ over x. You want to be able to say that the structure group is independent of coordinate transformations, but I dodged that. I have to be more precise about what the structure group is.

Another way to define the pullback is, [long pause], this is sufficient, isn't it?

This is a really fundamental result, that if you have homotopic maps the pullbacks are isomorphic. This immediately implies the claim since I can take the identity and the constant map. So the original and product bundles are isomorphic.

What we can look at, if I take some neighborhood containing the lower hemisphere and the equator and another containing the upper hemisphere and the equator, they overlap on a little neighborhood of the equator and are trivial. I want to say that the only possibility of something funny happening is here on the equator. So I want to restrict attention to the S^{n-1} now. You have these two trivializations right now, φ_1 and φ_2 and so I have a map $L \times F$ to S^1 to $U \times F$. So for every $x \in S^{n-1}$ I get a map from F to F. I mean from the fiber over $\varphi_2(x)$ to the fiber over $\varphi_1(x)$. This corresponds to an element of G, so we have a map $S^{n-1} \to G$, that is, an element of $\pi_{n-1}(G)$.

The claim is that bundles over S^n with connected group G and fiber F correspond bijectively to elements of $\pi_{n-1}(G)$. This homotopy group element is what Dennis called the clutching function. The bundle you construct from a map of $S^{n-1} \to G$ depends only on the homotopy type, and then the slick way to say it is with this bijection.

This doesn't make the problem much easier because homotopy groups are incomputable, sort of. That's all I wanted to say; I didn't prove much of anything, but it's an idea.

So I was just going to pick up where I left off last time, because Dennis just asked me to talk about simplicial complexes.

So, simplicial complexes are the future, and I mean that in the most literal sense. In some sense they're kind of hard to work with, because you have discrete data, but if you want to compute any of these things you need to be able to work with these discretizations.

They give an excellent approximation, as fine as you want, to differentiable and smooth things.

Some people even use the words simplicial and quantum in the same sentence; I just think there is a lot less known about these.

The j-simplex is the set of points $(x_1, \cdots, x_j)$ in $\mathbb{R}^j$ such that $\sum x_i \leq 1$ and $x_i \geq 0$.

These are simplices. If I label the vertices, every point x can be written as a linear combination of the vertices around it such that the sum of the coefficients is one.

One takes standard simplices and identifies faces under linear isomorphisms. A triangulation is just a homeomorphism from a simplicial complex.

Last semester we were interested in the boundary operator ∂, which might have

$$\partial[v_0, \ldots, v_n] = \sum (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n].$$

This occurs in the span over $\mathbb{R}$ of all of the simplices. Vall this $C_{\cdot}$ and grade it by dimension.

Now I want to talk about δ, the coboundary operator, which takes place in $C^{\cdot} = Hom(C_{\cdot}, \mathbb{R})$. So $\delta(a)(b) = a\partial(b)$.

Geometrically, if I have a triangulation, suppose I think of a cochain taking the value of 1 on a given edge, then its coboundary applied to a simplex is that cochain applied to the boundary of that simplex. So this is zero except on things with this edge as a boundary. I'm not going to put in signs, but the idea is that it's all the things of dimension one higher incident. I'm being loose but I'm allowed to as long as there are finitely many simplices.

You can "kind of" think of this as intersection in the correct dimension with the dual complex. The coboundary is the cell-dual of the boundary of the cell-dual.

So we have every simplex, which we can think of as cochains, and these are somehow built into the integrands. You have this definition of the coboundary operator, which in differential geometry you call Stokes' theorem (it's not a theorem in algebra, you call it a definition). You call it the Hom-dual extension or something. I don't know who Hom was, he was like the Stokes of algebra. You can write that.

Let me work top-down. Suppose K is a triangulation of a manifold M. This has differential forms. There is an embedding of the cochains (of K) into the $\mathscr{L}_2$ differential forms of M.

Say you have a zero cochain. This associates a number to every vertex. Now I need to associate a function on M. The simplest one would be the linear interpolation.

A 1-cochain would assign a number to each edge, and this would be an $\mathscr{L}_2$ function, but undefined on the vertices. The formula takes $[v_0, \ldots, v_k]$, which we should recall contains the points $p = \sum \mu_i v_i$, where μ_i are positive barycentric coordinates which sum to one. Then $W([v_0, \ldots, v_k]) = k! \sum_{i=0}^{k} (-1)^i \mu_i d\mu_0 \cdots \widehat{d\mu_i} \cdots d\mu_k$. These are directions and form a local coordinate system. This isn't a smooth or even a continuous form.

So what's the point? Here's a reason. There is a map in the other direction called integration $R : \wedge^{\cdot} \to C^{\cdot}$ A theorem is

Theorem 1 *1. $R \circ W = id$. That's the first statement; the second is that*

2. *R is a chain map and so is W. So $dW = W\delta$.*

   So what's the connection to cohomology? This is one way to prove that de Rham and algebraic cohomology give the same thing. There is a statement

3. *R, W are a cochain equivalence.*

   A cochain equivalence is a pair of maps

$$\begin{tikzcd} C^{\cdot} \arrow[r, leftrightarrow, bend left, "f\,g"] & D^{\cdot} \end{tikzcd}$$

   such that $f \circ g - Id = \delta A + A\delta$ and $g \circ f - Id = \delta B + B\delta$. Then this would mean $H^{\cdot}(C) = H^{\cdot}(D)$. So A and B are degree -1 maps.

There was no hope for $W \circ R$ to be the identity. Let me just say that the punchline is:

4. *If f is a sommoth j form on M then there exists a constant C depending on f such that $||WRf - f||_2 \leq C$ mesh(K). Once I describe the mesh, we'll be good to go.*

Let me suggest what mesh means. [Picture]. I don't have time to define it. I can take the supremum of the diameters (given a metric) over all simplices.

There's also a pointwise statement. This fourth is due to J. Dodziak. So this simulates fluid flow and stuff like that, it's easy to plug points into a computer.