Dennis Sullivan Course Notes
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Actually, you guys should prove this exercise.

Exercise 1 *Let $\Lambda = \{\lambda\}$ be any subset. Let L_λ be a straight line passing through λ and not vertical; suppose there exists an $\epsilon > 0$ such that for $\lambda_1 \neq \lambda_2$ then L_{λ_1} does not intersect L_{λ_2} for $-\epsilon < x < \lambda$. Then if $\bar{\Lambda}$ is the closure of Λ, the system of lines for Λ extends uniquely to a system of lines for $\bar{\Lambda}$ with the same ϵ-disjointness property.*

All right, so we have these four dimensions, $4i+2, 4i-1, 4i, 4i+1$. In the middle dimension we then have F with skew-symmetric $\langle\rangle$, T with symmetric skew-symmetric $\langle\rangle$, F with symmetric $\langle\rangle$, and T with skew symmetric $\langle\rangle$.

In the last case we have $A \oplus A \oplus \mathbb{Z}_2$ or $A \oplus A$. In the first case we have that F has even rank and things pair up canonically as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; in the third case we have a symmetric matrix; in the second we had things like $\begin{pmatrix} \epsilon_1 & 1 \\ 1 & \epsilon_2 \end{pmatrix}$ and in the fourth $\begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix}$ for order two t mod 2^n, and lines.

The realization theorem I think is the following. I'm 100% sure it's true, and I have 98% of a proof.

For $4i+2$ all are realizable by closed manifolds. All $4i-1$ are realizable as well. This is a little harder. For $4i$, this is pretty interesting. There's a natural partition in $4i-1$; type one means some a_{ii} is odd. The other is all a_{ii} are even. It's not so easy to write down a matrix of determinant one where all the diagonals are even.

$$
\begin{pmatrix}
2 & & & 1 & & & & \\
& 2 & 1 & & & & & \\
& 1 & 2 & & & & & \\
1 & & 1 & 2 & 1 & & & \\
& & & 1 & 2 & 1 & & \\
& & & & 1 & 2 & 1 & \\
& & & & & 1 & 2 & 1 \\
& & & & & & 1 & 2
\end{pmatrix}
$$

Anyone ever taken a determinant of an 8×8 matrix?

[I've done it on a computer.]

I've done a four by four. There's an eight dimensional manifold related to E_8 whose boundary is combinatorially equivalent to the sphere, but an exotic seven-sphere. It has this as its intersection matrix.

Show the determinant is one.

So over $4i$ type two is realizable in all dimensions. Type one is realizable only in dimensions four, eight, or sixteen.

For $4i+1$ let's call the extra $\mathbb{Z}/2$ type I and the lack of it type II. Then type one is realizable only for $i = 1$ in dimension five. I proved that a realization is diffeomorphic to the manifold $SU(3)/SO(3)$.

The rest later.

I'm not saying, for $4i$, that you can get these smoothly; I'm only saying it combinatorially. I'm only saying it for topological manifolds. People are still hammering away at dimension four. The K_3 surface is like the torus, a very simple object. It's a simply connected four-manifold with second Betti number 22. It has matrix two E_8 and three of the simple planes. There's a whole school of mathematics that tries to get rid of the plane blocks.

I'll just give you a construction that gives something in dimension four, but not a manifold. I think I gave the construction. Take a $4i$-ball and glue on $2i$-handles.

[lost]