Dennis Sullivan Course Notes
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I'm going to begin by making the comment that this is the second year in a row that I've given an algebraic topology talk on my birthday.

Let's start out with some algebraic objects associated with closed, finitely many component manifold M^n, most of them defined last time.

1. cohomology operations $Sq^i : H^k(M) \to H^{k+i}(M)$.
   These form an algebra, $A(2)$, the Steenrod algebra over $\mathbb{Z}_2$. Here's a fun fact, $H^*(K(\mathbb{Z}_2, n)) = A(2)$.

   From these guys I defined two characteristic classes, and I'll repeat those definitions.

2. $H^i(M) \times H^{n-i}(M) \xrightarrow{\cup} H^n(M) = \mathbb{Z}_2$.

   Now $Sq^i \in Hom(H^{n-i}(M), \mathbb{Z}_2)$ so is represented by some $V_i \in H^i(M)$ meaning $Sq^i(x) = V^i \cup x$. These are called the Wu classes.

3. The Stiefel-Whitney classes $W_k = \sum_{i+j=k} Sq^i(V_j) \in H^k(M)$.

   The point is that these last classes are used for the cobordism type of a manifold and show if it is a boundary.

4. Finally, I'll use the Stiefel-Whitney numbers, I will form degree n monomials of W_k. All such creatures will operate on the n homology of the manifold.

   So $\langle w_1^{r_1} \cdots w_n^{r_n}, u_M \rangle$ is a number, where u_M is the fundamental class, that is, the sum of representatives of the components.

   The surprising thing is that two manifolds are cobordant if and only if they have the same Stiefel-Whitney numbers, and in particular a manifold is a boundary if and only if all of its numbers are zero.

These numbers are part of a manifold, intrinsically.

Next I'm going to talk about the cobordism algebra.

I can define, well, let's make our manifolds smooth to embed them in Euclidean space. Say $M \sim N$ if there exists an $(k+1)$-manifold B such that δB is diffeomorphic to $M + N$, where $+$ is disjoint union.

This is an equivalence relation. Let's denote the resulting equivalence classes μ_k. Then I claim that $\{\mu_k\}$ is a group under $+$. The group is $\mathbb{Z}_2$ in dimensions $0, 4$, and 5. and is zero in dimensions $1, 2$, and $3, 6$, and 7. This last is one of the facts that Milnor used to work with exotic seven spheres. Thom's was the Fields medal right before him.

Everything is its own inverse. This means every element is of order two, so that $\mathbb{Z}_2$ acts on these in a stupid way, so it's a $\mathbb{Z}_2$-vector space. That is, $M + M$ is a boundary (of $M \times I$).

Let me note, it's not just a vector space but also an algebra, induced by cartesian product. So $\oplus \mu_k$ is a graded algebra with respect to $+, \times$.

You can do this in the oriented case, and you get $-(M, o) = (M, -o)$. Now this is not a $\mathbb{Z}_2$-vector space. They're only different when dimension is 0 mod four , and then the unoriented case is $\mathbb{Z}_2$ tensor the oriented case.

Theorem 1 *(Thom 1954) $M = \delta B$ iff all Stiefel-Whitney numbers are zero.*

Corollary 1 *$M \sim N$ iff all Stiefel-Whiteny numbers agree.*

One direction is easy to prove, if it's a boundary then all of the numbers are zero. The other direction is not easy even a little bit. I'll do the easy direction; you'll feel like you've proved this a bunch of times already.

Proof. Suppose $M = \delta B$.

- $\partial : H_{n+1}(B, M) \to H_n(M)$ maps u_B to u_M.
- $\delta : H^n(M) \to H^{n+1}(B, M)$.
- $\langle v, \partial a \rangle = \langle \delta v, a \rangle$.

Suppose $w = w_1^{r_1} \cdots w_n^{r_n} \in H^n(M)$. Then $i^* : H^n(B) \to H^n(M) \to H^{n+1}(B)$. So $w \in Im\ i^*$ where $i : \delta\ B \hookleftarrow B$. Then $\delta w = 0$, so that $\langle, w, u_M \rangle = \langle w, \partial u_B \rangle = \langle \delta W, u_B \rangle = 0$.

In order to make a statement to allow me to compute these guys in general I need to compute the Grassmanian.

Definition 1 *$G_n = \{n$-planes in $\mathbb{R}^\infty\}$, and over this we construct $\gamma^n = \{(n$-vector plane in $\mathbb{R}^\infty$, vector in that plane$)\}$, which is a subset of $G_n \times \mathbb{R}^\infty$ and is a bundle over G_n.*

This is

$$\begin{array}{c}\gamma(n)\\ \downarrow\\ G_n\end{array}$$

This is called the universal bundle because it is universal in the following sense: any $\mathbb{R}^n$-bundle ϵ over some paracompact B has a bundle embedding into this thing unique up to homotopy, so sometimes you define classes on this bundle so you can define them elsewhere with the pullback.

Here's another cheerful fact. $H^*(G_n, \mathbb{Z}_2) \cong \mathbb{Z}_2[w_i(\gamma^n), \cdots, w_n(\gamma^n)]$.

[Dispute.]

Definition 2 *The Thom space of a bundle E is the quotient of that bundle by all vectors of length greater than 1, the one point compactification of the total space=$E(\epsilon)/(b, v), |v| \geq 1$. So the quotient is t_0.*

Theorem 2 *(Thom)*
$\Pi_{n+k}(T(\gamma_n), t_0) = \mu_k$

You can do the same thing for the oriented case. I want to end by giving an idea of the correspondence. The idea is, say I have a map $S^{n+k} \to T(\gamma^n)$. You can show, and this is a lot of the work, a smooth map g that intersects the zero section in the range transversally, a map homotopic to f. It doesn't make sense to say g is globally smooth, but you ignore the compactified point. Then the correspondence is between g and $g^{-1}(G_n)$.

I've left out how these are determined by Stiefel-Whitney numbers.