Dennis Sullivan Course Notes
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I'm going to talk about cohomology operations, and give the sort of basic example of them, which is Steenrod squares, and then I'll find the Wu class and the Stiefel-Witten classes, and on Monday I'll talk about Thom; his Fields medal in 1958 was for showing that a mnanifold was a boundary basically only if these things all vanish.

1 cohomology operations

Definition 1 *A cohomology operation is a natural transformation of functors* $\Theta : H^m(\cdot, G) \to H^m(\cdot, G')$ *so that the appropriate diagram commutes:*

$$\begin{array}{ccc} H^m(X,G) & \xrightarrow{\Theta} & H^n(X,G') \\ \uparrow{\scriptstyle f^*} & & \\ H^m(Y,G) & \longrightarrow & H^n(Y,G') \end{array}.$$

There are a few obvious ones and they're actually the only obvious ones.

Example 1
- *a group homomorphism* $\phi : G \to G'$.
- $\delta : H^n \to H^{n+1}$
- $x \mapsto x \cup x$.

Note that as in the last case, these need not be ring homomorphisms.

Now I want to give a more concrete example. Those who took Dusa's class last spring will remember that we had this as the definition of cohomology.

$H^n(X,G)$ is set-isomorphic to $[X, K(G,n)]$ where $\pi_n(K(G,n)) = G$ and the other homotopy groups are zero. G is a finitely generated Abelian group. This space is unique up to homotopy type. The only example I'm interested in is $G = \mathbb{Z}/2$. Brackets denote homotopy classes of maps from the first to second argument. I'll give the correspondence but will not prove it to be a bijection.

We have the Hurewicz homomorphism $h : G = \pi_n(K(G,n)) \to H_n(K(G,n))$, which is an isomorphism because there is no other homotopy, i.e., it's $n-1$-connected. Since this an isomorphism its inverse $h^{-1} \in Hom(H_n(H(G,n),G)) \cong H^n(K(G,n),G)$. Now if I call $\omega = [h^{-1}]$ then f corresponds to $[f^*\omega]$.

So now I claim I don't have to define this operation for all spaces and all coefficients, I just have to define it for ω on $K(G,n)$. That is, there exists a bijection between cohomology operations and $H^n(K(G,n),G')$.

How do I want to prove this? If I have a cohomology operation Θ it acts on ω to give $\Theta(\omega)$. I claim that this is a bijection, and well-defined. Suppose $\alpha \in H^m(X,G)$, that's the same as having $f_\alpha : X \to K(G,m)$, and $\Theta_\omega \in H^n(K(G,m),G')$ so $\Theta_\omega : K(G,m) \to K(G',n)$. Composing these gives me the map I want.

The next claim is that every cohomology operation arises in this way, and that is because of naturality.

If I have a cohomology operation $H^m(\cdot,G) \to H^n(\cdot,G')$, and $\beta \in H^m(X,G)$ then that corresponds to $f_\beta : X \to K(G,m)$ and $\Theta(\beta) = \Theta(f_\beta^*\omega) = f_\beta^*(\Theta(\omega))$.

That characterization tells us something about cohomology operations that we didn't know. They can't lower dimension, and can only fix dimension if they are a group homomorphism because $H^m(K(G,m),G') = Hom(H_m(K(G,m)),G') = Hom(G,G')$.

Now I'll define Steenrod squares:
$Sq^i : H^n(X,\mathbb{Z}/2) \to H^{n+i}(X,\mathbb{Z}/2)$

1. naturality
2. stability: $\sigma : H^n(X) \to H^{n+1}(\Sigma X)$
3. ring homomorphism: $Sq^i(\alpha+\beta) = Sq^i(\alpha)+Sq^i(\beta)$; $Sq^i(\alpha\cup\beta) = \sum_j Sq^j(\alpha)\cup Sq^{i-j}(\beta)$

A few of these are easy. $Sq^i(x) = x$ if $i = 0$; $Sq^i(x) \sim \delta x$ if $i = 1$. There is a short exact sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ We can take Hom and get

$$0 \to Hom(\cdot,\mathbb{Z}_2) \to Hom(\cdot,\mathbb{Z}_4), \cdots$$

[argument ensues.]

It is $x^2 = x \cup x$ for $i = \dim\ x$. It's 0 for $i > \dim\ x$.

[Keep going, do whatever you want.]

The main thing that I wanted to say was to give some idea of where these things come from. I'll draw the picture that Dennis would draw. He's going to draw that picture.

[A circle with two hemispheres attached?]

[That's a crappy picture.]

I'll try to explain that a little bit better. I can think of $x \otimes x \in H^n(X) \otimes H^n(X)$ so $x \otimes x \in H^{2n}(X \times X)$.

Now $T : X \times X \to X \times X$ swaps factors; So $x \otimes x$ corresponds to a homotopy class of maps from $X \times X \to K(2n, \mathbb{Z}_2)$.

I claim that this is homotopic to $(x \otimes x) \circ T : X \times X \to K(2n, \mathbb{Z}_2)$. So then that homotopy is a map $I \times X \times X \to K(2n, \mathbb{Z}_2)$. This is a homotopy f and I can also look at $f \circ (id \times T)$ Pieces these together and get a map from $S^1 \times X \times X \to K(2n, \mathbb{Z}_2)$. This is contractible, so I can fill in the homotopy and extend to $D^2 \times X \times X \to K(2n, \mathbb{Z}_2)$. I use this to prescribe a map on "the other side" by $(s, x_1, x_2) \cong (-s, x_2, x_1)$. This gives me $S^2 \times X \times X \to K(2n, \mathbb{Z}_2)$. I continue and only run into trouble at $2n$. I claim I can get around it and get a map $S^\infty \times X \times X \to K(2n, \mathbb{Z}_2)$.

[How do you get around the obstruction?]

It's not obvious that you can. The homotopy is only $\mathbb{Z}/2$, so you can't miss it by a lot. The original map is injective.

[discussion.]

Now I'll say how to use such a map to get Steenrod squares. I defined this so that it respects antipodes so this will pass to the quotient $\mathbb{RP}^\infty$.

$$\begin{array}{ccc} S^\infty \times X & \qquad & S^\infty \times X \times X \longrightarrow K(2n, \mathbb{Z}_2) \\ \downarrow & & \\ \mathbb{RP}^\infty \times X > @[rru] & & \end{array}$$

So $H^*(\mathbb{RP}^\infty, \mathbb{Z}_2) \cong \mathbb{Z}_2[\tau],\ \tau$ of degree one.

So $\alpha \in H^{2n}(\mathbb{RP}^\infty \times X, \mathbb{Z}/2)$ we can write $f_{\alpha \otimes \alpha} = \sum_j \tau^{2n-j} \otimes a^j$; then $Sq^j(\alpha) = a^{n+j}$.

$\alpha \otimes \alpha$ is in the $2n$ cohomology of $X \times X$. Then the above diagram shows how to get the diagonal map of which I take a Kunneth decomposition.

I'm going to write down one thing and then we'll all leave.

I have the pairing on cohomology of a closed manifold, $H^{n-i} \times H^i \to H^n(M) \cong \mathbb{Z}_2$. So

$Sq^i \in Hom(H^{n-i}, \mathbb{Z}_2)$. Then we can say, there exists V_i such that $V_i \in H^i$ with $V_i \cup x = Sq^i(x)$ for all $x \in H^{n-i}(M)$.

Then $W_k = \sum_{i+j=k} Sq^i(V_j)$