Dennis Sullivan Course Notes
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What is that, what is that triangle?

[It's a triangle.]

Is that pizza?

[Yeah.]

It's obvious that they should do that, isn't it? I've never seen it that way before. Hmm, you're making me jealous, I'm going to go get my lunch.

[Do you want to share?]

No, no, I have my own lunch. Uh, do we start at 2:10 or 2:20?

[2:20]

Okay.

Scott, would you go to the board and erase it, and write down any questions that people have, while I eat my lunch? Any questions about what I said last time? Any questions about geometric topology?

1. What the fuck is going on?
2. How do you get an algebra structure on homology?
3. If you have a Lie group, how are the manifold and Lie group multiplications related?
4. We talked about $\Delta(ab)$. Do we ever care about $a \to \sum a_1 a_2$, where $\Delta(a) = \sum a_1 \otimes a_2$.
5. What about transversal intersections?

For the last one, the codimension of the intersection is the sum of the codimensions of the other two.

That's even older than topology. They were doing that in algebraic geometry. They had cycles long before there were homology groups and so on. A lot of algebraic geometry is about the failure of equations being independent. The theory is so complicated that very few practitioners understand it. Serre understands it. It's there, though. The picture that I have is that the normal to this plane sticks out like that. Likewise for this plane. So the normal bundle for the intersection is the direct sum of the two.

The moving lemma in algebraic geometry lets you achieve transversality, and there's a whole book by Bill Fulton on it. You can't always do it. In geometry and topology it's relatively easy but still not simple.

First of all you have to be in a manifold. Two cycles needn't be transversal. The idea is that you can move one slightly and now you have a nice transversal intersection. But you can't work with manifolds in homology, and they're not always embedded.

But there's this trick due to Grothiendieck. $f : X \to Y$ is the composition of an embedding and a projection. Take $X \times Y$ and the graph of f, which is an embedded image of X. Then the projection takes you to f. Always think of the graph, as soon as you can. I talked about the wrongway map, which is taking a preimage. You cross z with X and then intersect that with the graph. You might get two or zero, but those are kind of homologous.

If both of these are manifolds, then the cartesian product is a manifold, and the cycle might be a manifold with singularities. We can think about intersecting subsets inside a manifold. So one way is like a half a semester course in itself, introducing manifolds with singularities, and the singularity set is generically a manifold and it has some sort of cone bundle structure. And at the bottom is something that's really a manifold and has some sort of structure. Then you get transversality by induction, starting from manifolds and working outward since transversality is an open concept.

We can work more easily. In Milnor's notes, his book "Topology from a Differentiable Viewpoint," he has a transversality lemma about maps between manifolds. You use partitions of unity. It only requires C^1. Smooth maps are already transversal over most of their preimages but that's a little more elaborate. It involves Sard's lemma. The main idea, one of the main ideas this way is that like if you, given some kind of map or embedding, you can make a small perturbation to make it transversal. Suppose you have a space and you do this, it's quite interesting. Map a sphere into a lower manifold and the preimage of a point will be a submanifold, so you create submanifolds, and all manifolds, with all kinds of conditions on them, can be created by transversal images like that. You start with a map and then you create manifolds which are invariants.

If you take another map and you get another one of these things, if these maps are homotopic, you approximate your homotopy smoothly, working in the C^1 category. The map will be transversal for a little while. Then it won't be transversal on the inside, but you can change that to make it transversal on the inside, and then you get a bordism, a manifold between these two manifolds. You can get a lot of different kinds of structure, complex, symplectic, almost complex, spin. You can extend it to manifolds with singularities by using this work, using what are called Whitney-Thom stratified sets.

Have any of you read anything written by Milnor? It's beautifully written. You can take it home and paste it on your bathroom wall and study it. Nice, beautiful.

So I get cycles and homologies out of some sort of transversal intersection. There you can work with cell decompositions. You can move a chain or cycle into a given triangulation or cell decomposition. Then you have the dual cell decomposition (using the fact that it's a manifold) and every chain in the dual is transversal to every chain in the original decomposition. So move one into one subdivision and the other into the dual subdivision. So you need the cellular approximation stuff, that you can homotope anything into the right skeleton, and see, when you count homology classes only the top cells count so it's like linear algebra. The intersection, see, is in the third subdivision, which is the common refinement of the two. You have to be energetic to do this. But then you get something well-defined. Or read Scott's thesis. How are you going to do it?

[I have to rethink set theory first.]

Oh, well, we can go into, if some of you want to go into this, at office hours. It's kind of nontrivial. They stopped thinking about it because, it's very important to have a geometric idea, and then you have to figure out what to do with it when you write your paper.

Whoops, what are, what the Hell's going on, what was this one?

[What about other algebraic relations?]

Oh, right, Gabriel's. Well, uh,

There's another way, using the cup product, as a technical tool, if you have cocycles.

Let's say you can intersect cycles and get a cycle and if you move cycles by a homology, you can intersect the homology with the cycle, so at this level the intersection is zero, here two points, and here a homology from two points to zero.

So take the complex projective plane $\mathbb{CP}^2$, lines in three-space, and look in it at H, lines in a two space in three space. This is a two-cycle in $\mathbb{CP}^2$, and if you move this plane in three-space to another plane, then you're moving these cycles and they're homologous, in fact homotopic. Now $H \cap H'$ is a single line, so the two cycles intersect in a point. So $h = [H] = [H']$ but $h \cdot h \neq 0$, since a point is not homologous to zero. So h is not zero. This actually shows, you can put a $\mathbb{CP}^n$ here and H a hyperplane section, and H' and so on, and if you did one more, anyway, you can see that you can keep doing this until you get down to a line, which is not zero, and so you deduce that $h^3, h^3, \cdots, h^n \neq 0$. Another argument is that the homology groups are cyclic in each even dimension. The Betti numbers are one in the even dimensions and zero in the odd. But this intersection argument gives you the ring structure.

Okay, so, uh, so let me, uh, so, go over some of those examples last time, and fill in some of the missing steps. Suppose we have the ability to move things to make them transversal. So let M be an oriented d-manifold.

1. We intersect (finite) cycles to get a ring structure on homology.

Does this ring have a unit? If you intersect with the whole manifold, this isn't going to be a finite cycle. Let's define an infinite cycle. It's just a locally finite but possibly infinite family of cells that satisfies the appropriate condition. So for example if you take the infinite cylinder, then, well, let's keep them seperate. We have finite cycles and ordinary homology. You only do this in geometric contexts and it helps you understand cohomology. You also have infinite cycles and homologies, and you call it the something homology, like the support homology or the Borel Moer homology. Now the whole manifold is a cycle in the infinite homology, and then you land in the finite case. The intersection of an infinite cycle with a compact cycle is finite, moving it a bit to be transversal. Then this equation now makes sense. So the unit is outside the ring. This gives a geometric example of adjoining a unit.

There's a duality because of this pairing in noncompact manifolds between finite and infinite cycles. And the infinite cycles give you a picture of ordinary cohomology, while finite cycles give you a picture of cohomology with compact support. So just write that down and remember it for later. So when you talk about a de Rham theorem and forms vanishing at infinity that will define a cohomology with compact support, which will be dual to the finite cycles.

I mention this for a couple of reasons, there's a unit, it gives a picture of cohomology for a manifold, a finite dimensional manifold. Even for an infinite dimensional manifold, you can still talk about submanifolds of finite codimension, closed submanifolds of finite codimension. That's a picture of ordinary cohomology, like a function space of maps, and a subspace of that has the property that you can intersect finite cycles with subspaces with finite codimension.

There will be a pairing of these objects against cycles. Integrating is like computing intersections. In a closed manifold, the picture of cohomology reduces to the picture of homology. If you run into one of these things it determines a cohomology class, a cycle running off to infinity. In infinite dimensions it's something closed of finite codimension.

I only have one idea, transversal intersection.

I think when I was a graduate student, I had this psychological mathematical experience, people are talking about manifolds all the time, mysterious things, but then you could get one from a homotopy class, something you can get out of algebra, from a spectral sequence. It was somehow nontrivial.

Okay, so, so, this, uh, another thing to say, is that if the noncompact manifold were fairly trivially noncompact, if it's homotopy equivalent to a compact manifold with boundary, then the infinite homology is by excision just the relative homology of the manifold modulo the boundary. In this case it's natural to talk about the one-point compactification. Then it's the reduced homology modulo the base point. It's the reduced homology modulo the point, which is why a point is homotopic to zero, since the point is the base point. A point wouldn't be a locally contractible space otherwise.

The duality we're going to discuss is again, this duality here between finite and infinite homology has to do with the duality between relative and ordinary theory, so called Lefschetz duality, another remark about infinite homology.

It's nice to have other geometric interpretations of things. So this is an algebra, we have a ring structure. We also have a ring structure here, we can intersect two infinite things. So ordinary homology is an ideal inside support homology, the same way compact support cohomology is an ideal inside cohomology.

So then, um, so the homology is an algebra, and what about this map? Take the diagonal map $\Delta : M \to M \times M$. I can say this in two ways, algebraically or geometrically. So the ordinary homology is the tensor product if I take rational coefficients. Algebraically I get a structure here. I can define $a(a' \otimes a'')$ to be $aa' \otimes a''$ and $(a' \otimes a'')a = a' \otimes a''a$. So this acts on the left and on the right. So $\Delta(a_1 a_2) = a_1 \Delta a_2 = (\Delta a_1) a_2$. So Δ is an element in the tensor product so is something like $\sum a_1 a_2' \otimes a_2'' = \sum a_1' \otimes a_1'' a_2$. This is some kind of nontrivial identity which relates the things in the ring structure. It's not immediately obvious why this should hold.

An algebraic way of saying this is that Δ commutes with left and right multiplication. It's a bimodule map. The proof uses this idea that you were suggesting, it sneaks out of the ordinary homology, going out into the infinite homology a little bit even though it's an identity about ordinary homology.

A picture for the proof, something, this thing is pretty complicated already, the diagonal map, to understand. I'll try to draw a picture but I need more than just a line, so say this plane is the diagonal. Say here's a_1, and a_2 intersects it in this one point. Now what you do is cross with M, one of these cycles, and so how does $a_1 \times M$ intersect the diagonal? At $a_1 \times a_1$. This is a transversal intersection. Then you have b sitting here, or a_2, so you see that $(a_1 \times M) \cap a_2$ (really $a_2 \times a_2$) and this is the same as $a_1 \cap a_2$ sitting on the diagonal.

Δ is just letting your arm fall over, right? It's a homeomorphism onto its image. In the larger intersection theory the whole manifold is the unit, so you can think algebraically as $(a_1 \otimes 1) \times \Delta a_2$, where the $\times$ is just the product of the multiplication in a_2. This is at the level of chains. I'm saying more than the proof requires, saying this bimodule structure is really the ring structure applied to this product. Okay, I have five more minutes, this is the relation for open and closed manifolds.

Theorem 1 *If M is an oriented d-manifold, then $H.M$ with the intersection algebra and the diagonal coalgebra satisfy the bimodule (Frobenius) compatibility condition.*

Um, I see no reason why this isn't true for infinite homology as well.

Maybe we can give this as an exercise:

Exercise 1 *Verify this for closed support homology*

That's one of the names for infinite homology. I think of it as closed submanifolds that go off to infinity. It's isomorphic to ordinary cohomology.

Well, next time I'm going to relate this to TQFTs, and then discuss it in connection with group spaces that have a ring structure on their homology.

My office hours are 12:50 on Wednesday.