Dennis Sullivan Course Notes
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Okay, you guys are getting short shrift from me because I can't think about more than one thing at a time. So I'm thinking about what a closed set is. Okay, so this is the second semester of a first year course in algebraic topology sort of slanted toward manifolds and manifolds with singularities rather than toward abstract homotopy theory. So homotopy theory will be a tool and not an object. Actually, I've been thinking, what exactly are the real numbers? Have any of you made your peace with what they are?

[Interlude]

There's this issue which continually comes up, which is, this builds the continuum upon which you build the geometric structures of mathematics, but this usual way of doing mathematics doesn't seem to be appropriate for a quantum field theory. Those are other mathematical theories, physicists get quantum effects and draw pictures of it. Taking the Cauchy completion of the rationals to get the reals may not be what is going on. Even though decimals aren't very canonical, thinking about cubes within cubes within cubes, maybe taking the Cauchy limit isn't right. Algebraic topology, putting algebra on the finite stages and doing some other kind of limit, might be another possibility. Finer and finer triangulations. When you press the physicists, "what is this path integral," you divide space up into cubes and then take a limit, and that doesn't commute with first taking the limit of the space, because that's not meaningful.

In the last lecture I just recalled the main thing I did last semester. When we defined homology of a cell complex, by the usual way of taking kernel mod image, and then made a geometric interpretation of a cycle, a linear combination of cells with boundary zero, these cells have to fit together so that their boundary is zero. If it's one or two dimensional you can think of it as a surface, but eventually it is not quite a manifold but a manifold with lower dimensional zippers in it. So a chain is then an $n+1$-manifold with some zippers in the middle. Then you can extend the notion of homology to any space, by considering mappings of these cellular cycles into the space.

So we have a geometric definition of homology, or an algebraic definition if you're divided into cells. The course this semester is going to be discussing cohomology; from the geometric side it is quite different than homology. This stuff appeared around 1900, cohomology appeared around 1930. Historically, cohomology classes arose as obstruction to doing various things, including deforming algebraic structures.

Okay, so let me discuss more structure on homology first. There is algebraic structure on the homology classes of X. Well, these are abelian groups; if you take your chain complexes to be vector spaces then they're actually vector spaces. Sometimes it's convenient to take your ground field to be $\mathbb{Q}$ so you get rational vector spaces.

So you have the diagonal map $\Delta : X \to X \times X$ which induces a map $HX \to H(X \times X)$, so you get an induced transformation in homology. Continuous maps can be approximated by cellular maps after subdividing the domain and then we get this induced map Δ. Sometimes this is essentially equal to $HX \otimes HX$, for example with field coefficients.

$$\begin{tikzcd} HX \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & H(X \times X) \arrow[d, "\cong"] \\ & HX \otimes HX \end{tikzcd}$$

Exercise 1 *Decide (if $HX = H(x, \mathbb{Z}/n)$), whether $H(X \times Y) \cong HX \otimes HY$.*

As a hint, every chain complex is isomorphic to the direct sum of simple ones. These are $0 \to Z \to 0$ and $0 \to Z \to^{\lambda} Z \to 0$. This is a good method. You don't need books. Universal coefficient theorem, tor, ext, if you're on a boat in the Greek islands. So half the point of this exercise is to make you independent of books. If you have your laptop with you you can just ask.

But any time the Kunneth theorem holds you get a map $V \to V \otimes V$, which is a coalgebra. The homology of a space "forms" a coalgebra. The quotes are here because you have to think about this carefully. There's always a map $HX \otimes HX \to H(X \times X)$, the question is whether this is an isomorphism. If you do cohomology, you get

$$\begin{tikzcd} HX & H(X \times X) \arrow[l] \\ & HX \otimes HX \arrow[ul] \arrow[u] \end{tikzcd}$$

So cohomology is always an algebra instead of only sometimes being a coalgebra, like homology. If you have a coalgebra, you can dualize this map and get

$$\begin{tikzcd} V^* & (V \otimes V)^* \arrow[l] \\ & V^* \otimes V^* \arrow[ul] \arrow[u] \end{tikzcd}$$

So the dual of a coalgebra is an algebra, but the dual of an algebra is not necessarily a coalgebra, because the vertical map is not an isomorphism for infinite dimensional spaces.

Sometimes late at night you almost prove the Poincaré conjecture, but it's wrong because of something stupid like that.

So one place where this is not obvious, if you take an algebraic variety, taking out a lower dimensional thing always generates a cycle. Then you can go into the diagonal map, and it's still unknown wherther these components are algebraic cycles. It's one of Grothiendieck's conjectures. Are the Kunneth components of an algebraic cycle algebraic?

Similarly, the cup product in cohomology gives you the same information. Kind of unobvious stuff.

So let's make a score card.

Structure on homology

space	structure
any space	"coalgebra"
(oriented) manifold	algebra of degree zero with codimension grading. Wrongway
group (Lie group, topological group, loop space)	algebra of degree zero with dimension gr
group bundle over a manifold	$\cdots$

So suppose X is an oriented d-manifold. Then we can take two cycles and look in $X \times X$ and form $z_x \times z_y$ if these are the two homology representatives. Now we have the diagonal in X, and since X is a manifold this has a normal bundle. Any cycle in here can be made tranversal to this diagonal by moving it slightly. Then the intersection will be a cycle in the diagonal, compact even if X is noncompact. So we get a map from $H_iX \otimes H_jX \to H_{i+j-d}X$.

If we put this all together we get a map $H \otimes H \to H$. So H is an algebra. I was pretty sloppy about that. If your normal bundle is oriented, and the cycle is oriented, then you can orient the intersection. That gives a well-defined operation here, and did I give the dimension? If d is even, this is graded commutative and associative. If d is odd then it doesn't, but there's a modification. The modification is $x \cdot y = (-1)^{|x|} x \cap y$. You grade by the codimension of the cycle, so $d - |x|$. If d is even these dimensions are equivalent. There's some subtleties here with signs. Again, Poincaré duality says that a cycle in a manifold is associated to a cocycle, the Thom class of a neighborhood, naturally graded by the codimension, and this is naturally isomorphic to the ring structure on cohomology. Again, cohomology simplifies this algebraic discussion. People expressed things in cohomology and then moved it across with Poincaré duality.

The subtlety is that when you interchange the two factors in odd dimension, you change the orientation of the normal bundle. If we're only worried about this thing up to sign, we can't say anything, but with correct signs it's a graded commutative associative algebra.

You can ask how the coalgebra structure $\Delta : HX \to HX \otimes HX$ is related to the algebra structure. Δ is a module map, a mapping of left and right modules. They usually say bimodules.

I'm going to prove that in a more general setting later. It's very easy to prove in a slightly more general setting.

There's another property you have, you have the wrong-way maps. If you have a proper map $M \to N$ between two manifolds (preimages of compact neighborhoods are compact) then you get a map $HN \to HM$ in homology by the transversal preimage.

[Scott: What does that mean?] You move it so that it is transveral. That means that the graph is transversal to N cross the cycle. This requires some work, but the codimension is preserved, so it's a degree 0 map; otherwise it's a shift by the difference in dimension.

Now let G be a group space like a Lie group or even a topological group. Any space with a continuous associative multiplication with identity. To a group space you have a map $G \times G \to G$. You can take the cartesian product of two cycles and then there is a multiplication again $H_i G \otimes H_j G \to H_{i+j} G$.

Now we can ask, other examples, if this is a Lie group we've got two algebra structures and the coalgebra structure. The group algebra structure doesn't require finite dimensionality. Generally it's noncommutative but it's associative. For example, if you take a bouquet of two two-spheres and take the loop space Γ on that, then you get the tensor algebra of two indeterminates of degree one. That's sort of like the statement that π_1 of a one-point union of two circles is a free group on two generators. It's kind of analogous to that statement.

So what is the relationship between this algebra structure, again we always have the coalgebra structure. It was a map of bimodules before, now Δ is a map of algebras.

There's a third property of a diagonal map, which, oh–

[Is this a Hopf algebra?]

It's a bialgebra because it's got an algebra structure and a coalgebra structure which is an algebra map. A bialgebra is a Hopf algebra if it has a canonical antiautomorphism which in this case is the inverse.

Hopf showed that this forces the homology of a Lie group to have a very rigid structures.

These are two of the three properties I know.

If you have an algebra, and a coalgebra, there are only three things I know how to ask, you can ask it to be a map of algebras, of bimodules, and be a derivation. Two of them have come up already.

[Can't you ask about their compositions?]

Yeah, then you become lost. You can get all kinds of graphs, with any number of inputs and generate operations like quantum field theory. You have identities of a Hopf algebra, but what is this graph quotient those identities, and what that is is not known.

Some are known well, like for operads. Even if you take a Lie algebra, well, if you take an associative commutative algebra, there's really only one $n \to 1$ composition. So you understand commutative associative algebras. If you're just associative, then there are $n!$ inputs. If you take something like a Lie algebra, how many operations are there? With n

entries you have $(n-1)!$ operations.

I don't know how we got to here. You have structure on homology. What's the time of this class? 2:20 to 3:40? Okay, so there's another one related to string topology, what's the name for this?

If you have a group bundle over a d-manifold, where the fiber is a group. You have a total space and a projection map with fiber the structure of a group. Locally it should be isomorphic to the group cross the local set and all the maps on the fibers are group isomorphisms. So a principal bundle is not one of these examples. Each space is a free translation set of the group in a principal bundle. The fiber is isomorphic to the group, but not canonically. You have a section to the identity, which is all the juice of a principal bundle.

[So what fiber, what structure do they have other than a group action?]

It's a copy of the group but not canonically isomorphic to the group. The Euclidean plane has no privileged point but a vector translates it. But then you can do $E \times Y/G$ which is the associated bundle to a principal bundle with fiber Y. Take a linear representation of $GL(n)$ and you can get the associated bundle. One example is to let this be $E \times G$ and let G act on G by inner automorphism. In a principal bundle the fiber is E/G, so this projects onto E/G with fiber Y. We'll go over bundles more later. If you have a principal bundle, for the associated adjoint bundle. Y is G with a conjugation action. For a nonabelian group this is interesting.

So there are lots of examples of these. Any action of G on a space gives a bundle, but I need it to act on a group. If Y is a group and G acts as automorphisms of Y then I can get it.

But anyway, suppose I have a group bundle. Then we have another algebra structure on homology, so that $H_iE \otimes H_jE \to H_{i+j-d}E$. It's combining these two examples, which has to be done at the geometric level. Take a couple of cycles upstairs and project them downstairs. You can still intersect them downstairs. So you take $z_x \times z_y$, sorry, take z_x, z_y, cycles representing x and y, look at the projection $\pi z_x \cap \pi z_y$ in $M \times M$, and intersect with the diagonal.

A point upstairs is a pair of points, which give me two points, where z_x and z_y are in the same fiber, and then I multiply them to give me a single thing upstairs. So take two cycles upstairs. Intersect them and their shadows downstairs. Then above that you can multiply the preimage points.

For a trivial bundle it's the tensor product of the two algebras. This is a third algebraic structure on homology in a kind of weird class of spaces. There are two interesting special cases.

Example 1 *Consider the free loop space of a manifold $L(M)$ so this is maps from $S^1 \to M$, and the projection is projection to the base point. The fiber is the based loop space.*

If you take an appropriate equivalence relation, something about a topological group, Milnor in 1957, but you get the ring structure on the homology of the loop space. This is a paper with Moira Chas, this is a graded commutative associative algebra. If you think of the principal bundle, all paths starting at a base point, that's contractible. Then if you evaluate at the endpoint you get a projection. Any two such paths differ by a loop so this is an affine space for the loop space. Then the associated bundle by the adjoint action is this here. It's called the string product and this is called string topology.

If the manifold is simply connected, there's an algebraic model for the loop space related to the Hochschild complex for the cochain algebra.

[Let's say that M is symplectic. Does this have to do with Floer homology?]

Work in the cotangent bundle, do something there and it's isomorphic to this thing. That's Mathias, and it's a hell of a lot of work. These things haven't been completely mastered. But this is a very simple thing using these very classical things from the thirties.

Another example is

Example 2 *Take G to be a classical Lie group, keep it vague for a second. Well, let it be the unitary group. Then we have the Grassmanian $G^{\mathbb{C}}_{n,N}$ of n-planes in N space and these are contained, one in the other, $G_{n,N} \subset G_{n,N+1}$, and the union is the classifying space for the Lie groups U_n, so this is often called B_{U_n}; it's an infinite dimensional space. Over each of these you have the tautological bundle, all isomorphisms to $\mathbb{C}^n$, well, unitary ones. Take bundles with unitary structure. Then isomorphisms of the fiber to the standard unitary space is an affine space of U_n, and then the union of the natural principal bundles is the natural principal bundle is the union; we'll discuss this again with obstruction theory. Then we can form the associated adjoint bundle $E_N \subset E_{N+1} \subset \cdots$, the associated adjoint bundles, and so I have all of these homologies, all their homologies $HE_N, HE_{N+1}, \cdots$ and each of these is a compact manifold. I can look at the obvious map but also at the wrong-way map. The codimension is the same as downstairs; the fiber is U_n, so the codimension, if I shift everybody by the codimension of the base then the wrong-way maps will be degree zero, and then when I shift by dimension of the base I also have a degree 0 multiplication from the group structure. So I get this system of algebras, where the adjoint bundle over the infinite Grassmanian has this extra structure in it.*

I'm going to stop. I think this is really good, but I don't know what it's good for.