Dennis Sullivan Course Notes
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This damn question of Travis' is driving me nuts. I lie awake at night thinking about it. Thank you, Travis.

I wrote this out on my blackboard and didn't transfer it to my notebook; I hope I remember all of it.

Last time I didn't give a good answer about skew-symmetry. Let me, uh, I'll catch up. We really haven't properly pursued Poincaré duality, we're still pursuing what are the consequences of the statement.

To study quadratic forms you have to split up into odd and 2-torsion. Distributing over the tensor product you get the product of T_2 and T_{odd} as zero. The same argument shows that $\mathbb{Q} \otimes \mathbb{Q}/\mathbb{Z} = 0$.

Can we take a digression to show how the first Hilbert problem was solved? It involved the invention of the tensor product functor, the tensor product of $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$. That's a wild thing.

So the different primes don't communicate with each other across the tensor product. So the map from $T \otimes T$ to $\mathbb{Q}/\mathbb{Z}$ splits and $T_2 \otimes T_2$ maps to two torsion while the odd stuff maps to odd torsion.

A perfect pairing means that taking an element of order n there is a homology whose boundary is n times the element. Taking another element look at the intersection number with the homology. Divide by n and reduce modulo 1. When it moves across the boundary, λ intersection points disappear.

Later we'll see why it's a perfect pairing. To see if it's symmetric or skew-symmetric span the second element by something, and look at the intersection of the two homologies. That's a one-dimensional object; take the boundary of it. The sign in the Liebnitz rule is positive if the homology has even dimension, then it's skew-symmetric.

I've got to get the signs right for this. Everything comes from interesection theory is all I was trying to show. For intersection theory you have to work with the codimension. Then we have to see why it's a dual pairing; then let's analyze the form. The argument for $4l+1$ odd is the same as the free argument for $4l+2$.

	T_2	T_{odd}
4l+1		$\langle , \rangle$ is a sum of hyperbolic pl
4l-1	$\langle x+y, x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$. So there exists x with	

There are some invariants in $4l-1$ odd. Look at $\mathbb{Z}/p^k$. Then λ is only defined up to squares. In $\mathbb{Z}/p$ the units are $\mathbb{Z}/(p-1)$; in $\mathbb{Z}/p^k$ it's $\mathbb{Z}/(p-1) \oplus \mathbb{Z}/p^{k-1}$. Over 2 this is not true. So there is one $\mathbb{Z}/2$ invariant, whether this is a unit.

Now look at the 2-torsion. I'm going to break it up into more cases. We have a perfect pairing and I want to split off into free modules over 2^k. Say I only have $\mathbb{Z}/2$ and $\mathbb{Z}/4$. Then $Hom(G, \mathbb{Z}/4)$ is isomorphic to G. So the homomorphism of order 2 are $Hom(G, \mathbb{Z}/2)$.

I've got to sit down for this. You can't just draw a single picture for these algebra arguments. You walk into the hotel, you walk into your room, you walk into the bathroom, it's so many steps. I'm in the bedroom right now.

Any guy from a $\mathbb{Z}/4$ is two times something so if you tensor it with $\mathbb{Z}/2$ you get 0; So the elements that detect $\mathbb{Z}/2$ are in the $\mathbb{Z}/2$. Keep the same argument with higher 2^k. Repeat the same argument next with the lowest 2^k. I've been assuming this all along. So we can deal with the homogeneous cases.

Now there is an interesting invariant called the Wu class. In the $\mathbb{Z}/2$ case, $x \mapsto \langle x, x\rangle$ is a homomorphism. There is a canonical element ν called the Wu class with $\langle \nu, x\rangle = \langle x, x\rangle$. If it's nonzero on a generator, consider the one dimensional line generated by it. You split off $\mathbb{Z}/2$ lines intil ν is identically zero. Then you can split off hyperbolic planes. So the space is lines and hyperbolic planes.

Exercise 1 *Show that as quadratic forms,* $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Over $\mathbb{Z}/4$ in the skew symmetric case you still have a Wu class because self-pairing is an isomorphism.

Exercise 2 *Over* $\mathbb{Z}/4$*, find the isomorphism classes of the four* 2×2 *matrices* $\begin{pmatrix} 0/2 & 1 \\ -1 & 0/2 \end{pmatrix}$. *Choose a basis so it looks like this. How many isomorphism classes? Let me sort of guess what I think it will be. It's* 3*, I think.*

Everybody is a direct sum of these.

Let's discuss $\mathbb{Z}/8$ a little bit, oh, we don't need to. Okay, symmetric. Is there an element $\langle x, x\rangle$ which is not divisible by two? If so, split it off. Then, until, $\langle x, x\rangle$ is always even.

Exercise 3 *Start with* $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$; *Look at* $rx+sy$. *You can't change this one, I don't think.*

What are the isomorphism classes? There are three possibilities. I think that's a correct argument.

	skew symmetric	symmetric
free $\mathbb{Z}/2$-modules	lines and hyperbolic planes.	same discussion as skew.
free $\mathbb{Z}/4$-modules	$\langle , \rangle \cong \oplus \begin{pmatrix} t_i & 1 \\ -1 & t_i' \end{pmatrix}$ where t_i is torsion of order 2.	
free $\mathbb{Z}/8$-modules	This works the same as above.	

For the final free case, let's move down one by one.