Dennis Sullivan Course Notes
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Remember this point that even if you're right and I'm being stupid, it's still your fault because you didn't communicate to me. Does anybody have any homework to hand in? Let's see?

[Did you use the eraser?]

What we're doing here is putting a rubber band around some nails and then you turn the nails.

There's something magical about this because Thurston was doing this as a graduate student and he noticed that there was a limiting pattern, and he described all the limiting patterns on a given surface and he found a sphere now called the Thurston boundary of Tychmueller space. Right, count the number of strands going up, it's 0,1,1, then 1,2,1, then 1,3,2, and so on.

You get a foliation, this picture already tells you that you can do this approximately, you make a foliation with singular leaves and then a corner leaf, and this is a limiting object. If you study the holonomy of the foliation, follow transversal arcs until they come back, you get a transformation, you break the interval up and put this one over here, this one over here, it's a lot like doing irrational rotation on the circle. There's this topologically defined return map.

Very nice, is this all you have, put an A, I'd give you an A.

And you know, what's interesting, I'm talking about in the more elementary class, how can you put a topology on this set of pictures so that there will be a limit?

[Do you need a tangent bundle to talk about foliations?]

Partition the space so that it is locally like a bunch of leaves. There's a different way to think about this, if you don't like the singularities, you need singularities because the Euler characteristic of the disc is one.

[So what?]

You can't have a foliation of the disc without singularities. If you had a foliation of the disc, you can double it to get a foliation of the sphere. Then at every point on the sphere you'd have a line you could draw. Since the lines are not oriented, the Euler characteristic is two so you need four singularities.

Double this picture and you get a cross and then six isolated guys. This adds up to four. You can't orient a trivalent singularity. In the oriented theory, you travel around here, you have this vector, you travel this way, as I go around however I did it, well, this has degree -1 in the oriented theory. So this has index -2 if I go around twice. If you want to stay on the sphere and get rid of the singularities, there's another procedure, you leave air space and you pull it tight, make some kind of construction. Think of this on the plane minus three points. There's a geometric model of the plane minus three points and that's the sphere minus four points. Look at this hyperbolically so that everywhere it has negative curvature. Take a unique geodesic (in every class of closed curve) and it will have the minimal number of geodesics. In this space you see a lot of geodesics, all disjoint; they don't cross themselves. In the limit, because of the curvature, you get a Cantor set of geodesic. These are called laminations, geodesic laminations. This was one of Thurston's inventions in the late seventies.

He started with an abstract convergence. Given a curve γ on a surface, there is a function on other closed curves, that is the minimal number of transversal intersection.

This foliation is the limit under the weak topology of these simpler curves.

The theorem is that the foliation exists and is unique.

Let me start what I was going to talk about, this isn't an official discussion.

[Where do I get a measure?]

It comes from the fact that the lines are part of a single closed curve.

There's a familiar example from ordinary math, here's a torus, if you put an irrational in to a flat torus as slope of a line climbing around it, you cover densely. The limiting measure is the product measure of the curve.

Given an arc there is a transversal measure $\lim_{p/q} \to \sqrt{2}\#p/q$ intersection points for R divided by the length.

What I was going to talk about seems boring now. I'm going to stop the class now. Today there's a dynamical system seminar I wanted to go to.

I changed my mind. We're having class.

Travis' question was, what are the possible homology groups of a closed oriented manifold? Because of Poincaré and Pontryagin duality you get pairings of the groups. These are the conditions due to duality. There are four kinds of dimension. In $4i+2$, let me organize it. Then if you have $\mathbb{Z}$, and then down in $2i+1$ you have a possible free group here. Then the k and the $n-k$, this would be $2i+1$ as well. So maybe you're self-dual. The reason you have this duality is because of a pairing. This picture follows because the free parts in k and

$n-k$ have an intersection pairing, a perfect pairing $F_k \otimes F_{n-k} \to \mathbb{Z}$, similarly to the torsion in k and $n-k-1$ into $\mathbb{Q}/\mathbb{Z}$.

[Dennis answers the phone.]

Okay, okay, I'm teaching a class right now, I'll call you at 3:45 if I can, okay, okay. Teenager wants money.

So this is a skew-symmetric pairing (because the dimension is odd). In dimension $4i$ it will be a symmetric pairing. So in $4i+2$ the rank is even which answers the question to some degree. The middle one has to be of even rank. You can take the connected sum of $S^{2i+1} \times S^{2i+1}$, which will give you whatever you want in this middle degree. All the other ones can also be realized. We've completely analyzed $4i+2$. There's no information if the dual pairing is not a self-pairing. There will be lots of invariants in the other cases.

Now in the case of $4i$ you get a symmetric pairing, this is a big theory, symmetric matrices with integer coefficients up to similarity. We'll discuss that, it's very interesting, to realize these in dimension four, any can be realized by a topological manifold, and then there are constraints for smooth manifolds. Let's leave that because we're not going, well, the mathematics gets, here we're in Walmart, the next one is a little jewelry store on a Navajo reservation, but in $4i$ we're on Wall Street. The others are little trinkets in comparison. Some names are Donaldson, Segal, you can't have too much fun with that without some training first. In the $4i+1$ and then $4i-1$ dimensions, you get torsion groups either in $2i$ or in $2i-1$; these will be skew symmetric or symmetric respectively.

Now I want to organize the problem, take a uniform description of this. In the other cases we can say a lot. We'll see the algebraic possibilities. I want not just the homology, but the homology with duality. In the time allotted I can say something nice about the first one. Actually, uh, what's true in the $4i+2$ is that every skew-symmetric form is equivalent to the direct sum of planes, two dimensional subspaces. When you have a form on the plane like this one, it's called a hyperbolic plane, so this is a direct sum of hyperbolic planes.

I'll make a table of results here. Now we're talking about torsion. When you're talking about quadratic forms, the theory in the prime two is always more delicate than in odd primes, because of this formula, $(x+y)^2 = x^2 + 2xy + y^2$. Every torsion group can be written as the direct sum of the odd and the two primary parts. So these completely fracture into the different primes. I'll put the primes together. Now I want to talk about these dualities. We have to understand $\langle , \rangle$.

	2-primary	odd
$4l+1$	sum of planes and $\mathbb{Z}/2$ lines	sum of hyperbolic planes.
$4l+2$	sum of planes $\begin{pmatrix} n & x \\ x & n' \end{pmatrix}$ where x are units, n are not, and lines	$\langle , \rangle$ is a sum of lines

There's a nice argument to get you started. An example of a line is $\langle x, x \rangle = \lambda$ where x is a generator and λ is a unit.

Take a subgroup. Split the form onto the subgroup. Start with a space with a perfect pairing into R, so that any linear function to R is taken by taking the inner product with something.

Say we have a subgroup on which the form is a perfect pairing. Now take the quotient by this subgroup, we choose a lift and look at the inner product between the lift and the subgroup; this is given by an inner product with something in the subgroup. Subtract that off and you're left with an orthogonal piece. If these were orthogonal and the new one wasn't a perfect pairing then the whole thing wouldn't be.

Now take a skew-symmetric pairing on a free Abelian group over $\mathbb{Z}$. We have $(x, x) = 0$. Get y so that $(x, y) = 1$. Then on the subspace spanned by x, y you get a perfect pairing and you can split this off.

Now let's prove some of these other ones. We have a symmetric pairing on an odd torsion group. Write this as $Z/n_1 \oplus \cdots Z/n_2 \oplus \cdots$ with $n_1 \leftarrow n_2 \leftarrow n_3 \leftarrow$ It's like the skyline of Manhattan, you take out the tallest building and then the second tallest building, and so on.

All of the calues have to be in Z/n_1, so you have to be in the n_1 roots of unity. So $T \otimes T \to Z/n_1$. In the symmetric case I need a unit in here for some x. So $(x+y)^2 = x^2 + 2xy + y^2$. If the squares are all nonunits, here suppose all of these are divisible by, say 3. Then $2xy$ might not be divisible by 3. Because of the perfect pairing, xy can be anything you want and then 2 is a unit.

You're going to have to break this up into primary parts. The primes are independent so that's true.

Then if (x, x) is a unit, the line generated by x gives, well, x is a unit, a generator. Then the form restricted there is a unit. Take the complement and repeat the argument. A symmetric pairing, odd torsion, is a sum of lines. I'm supposed to stop now.

So now, I see, now this one, the odd $4l + 1$ case. This argument goes just like the free argument. You have hyperbolic planes, these are skew, the linking is a skew form. So this goes exactly like the F. The odd case is easy, it's like the free case. The 2-primary part are the two hard ones. We can analyze the odd ones geometrically, there's no problem. A sum of planes and $\mathbb{Z}_2$ lines. We have that $\rangle x, x\langle$ has order 2. We have some $Z/8$, some $Z/4$, some $Z/2$. So if you take any x and choose y so that $\langle x, y\rangle$ is a unit, look at the matrix of this part. Say x is of order 8 and y another one so that the inner product is a unit in $\mathbb{Z}/8$. For the intersection matrix I get a 1 in the upper right and a -1 in the lower left, and then elements of order 2 in the diagonal. So in $\mathbb{Z}_8 \oplus \mathbb{Z}_8$ I can split these off. It turns out that there might be an even or odd number of $\mathbb{Z}_2$. So T is either $A \oplus A$ or $A \oplus A \oplus \mathbb{Z}/2$. Its order is a square or twice a square. We'll show that when a $4l + 1$ manifold is a boundary, we have no extra $\mathbb{Z}_2$. Then the next thing is that SU_3/SO_3 has $\mathbb{Z}_2$ in the 2 level and 0 everywhere else in the middle. A corollary of this discussion is that this M^5 is not the boundary of a 6-fold.

We'll pursue this more next time.

I want to give an exercise.

Exercise 1 *Write out proofs of the right hand sides, odd torsion.*