Dennis Sullivan Course Notes
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I'm going to stop the class early because of a seminar.

Travis is the one that asked the question I've been struggling with. The note at the end is irrelevant because I asked Milnor at his surprise birthday party and he gave a perfect answer in five seconds.

The question we're studying is, "what are the conditions on the homology of a manifold." A complete statement would be a list of all the possible homology groups. So which graded groups are the integral homology groups of oriented connected closed manifolds?

If the dimension is n, then H_0 is $\mathbb{Z}$ and $H_n = \mathbb{Z}$. We're really interested in homology from 1 to $n-1$. Now there's a construction, which given two manifolds, you remove a disk from each one and run a tube between them. The connect sum has the direct sum of the homologies between 1 and $n-1$. So then we can break down over connected sum into atomic pieces, and then try to see which atomic pieces can be realized.

Let me just talk about this intermediate stuff. Here are some diatomic pieces:

You can have a $\mathbb{Z}$ in dimension $n-k$ and a $\mathbb{Z}$ in dimension k with the rest zeroes. You can have a $\mathbb{Z}/\lambda$ in dimension k and a $\mathbb{Z}/\lambda$ in dimension $n-k-1$. So Poincaré duality says that, in general homology looks like a bunch of $\mathbb{Z}$ plus torsion. The $\mathbb{Z}$ in complementary dimension and then the torsion in the complementary dimension less one are equal. These are diatomic so that if $k = n-k$ this is $\mathbb{Z} \oplus \mathbb{Z}$ in the middle; similarly if $k = n-k-1$ we mean $\mathbb{Z}/\lambda \oplus \mathbb{Z}/\lambda$ in the middle. The diatomic pieces have $\mathbb{Z}$ in the middle dimension or $\mathbb{Z}/\lambda$ one below the middle.

If I could realize every atomic and diatomic piece then I could get every manifold. So Poincaré duality says homology of any manifold of dimension n is a direct sum of diatomic and atomic pieces.

There's something kind of naturally implied here, that I'm talking about graded groups with duality as part of the structure. There is a perfect pairing so that the tensor product maps so that any linear product on one factor is given by multiplication with an element in the other. So in the diatomic middle dimension, I mean that the pair is dually paired so that the matrix is $\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$, with the sign depending on the parity of n. So that's a bit more precisely a definition.

These are perfectly fine candidates, but which ones can be realized?

Realizations and Nonrealizations:

The free diatomic ones can be realized by products of spheres, namely, $S^k \times S^{n-k}$. This even works if $n = 2k$, because the spheres don't link with themselves, but with the others.

Milnor solved this instantly by concentrating on the k level group. I'm going to show that the torsion diatoms are completely realized also.

So all of the diatoms ae realized. So what you do is you concentrate on the lower one, Gabriel, this is the wrong thing to do, you look at the other side. We want to get an n-manifold, and n here is greater than or equal to $2k+1$. In the extremal case $n = 2k+1$. So we can just embed the two-cell complex, you have $S^k \cup e^{k+1}$ with attaching map of degree λ. Then you embed this in $\mathbb{R}^{n+1}$ Remember that it worked in the worst case with $S^1 \cup_\lambda e^2 \subset \mathbb{R}^4$. So $n+1 \geq 2k+2$ in general. Then you just take the neighborhood of it in Euclidean space and then take the boundary of the neighborhood. The other homology group is living on the other side and they come together in the middle.

When you do this in the extreme dimension, it produces two $\mathbb{Z}/\lambda$, not just one. And so the closed manifold picks up a contribution from each one of them. You can also just do it with Meyer-Vietoris.

So the proof is to embed the 2-cell complex with homology just the lower $\mathbb{Z}/\lambda$ into S^{n+1}, and take the boundary of a regular neighborhood.

Exercise 1 *Show that the homology of the M^n is the diatomic $\mathbb{Z}/\lambda, \mathbb{Z}/\lambda$. Use Meyer-Vietoris.*

That's actually what I was trying to say the first day, and I concentrated on one of the $\mathbb{Z}/\lambda$. So then we take connected sum. So any homology groups split into two parts by Poincaré duality we can realize.

What about the atomic pieces? First of all, if k is odd, then, well, can I change $n = 2k-1$ in the statement to make a uniform statement? If k is odd, there are big problems with realization. If you have anything with a bunch of $\mathbb{Z}$ in the middle, for example, take $k = 1$. Is there a manifold like that, an orientable surface with Betti number one, it doesn't exist. It's a skew-symmetric duality and a skew symmetric form has even rank. So self-duality is skew-symmetric.

People talk about these 6-manifolds a lot, Calabi-Yaus, and the 3-homology has this skew symmetric form, with even rank.

So none of the free atoms is realized. If Z/λ is realized for k odd then λ is 2. The proof is: $\mathbb{Z}/\lambda \otimes \mathbb{Z}/\lambda \to \mathbb{Q}/\mathbb{Z}$ is the linking. Recall that cyclic groups are direct sums of the prime powers. Let's do examples. The tensor product will be $\mathbb{Z}/\lambda$. The target must satisfy the same $\lambda x = 1$ so must be into $\langle 1/\lambda \rangle \cong \mathbb{Z}/\lambda$.

Now we'll show that a nondegenerate skew-symmetric pairing on a cyclic group means the group is $\mathbb{Z}/2$. We have $T \otimes T \to \mathbb{Q}/\mathbb{Z}$. So $(x,x) = -(x,x)$. So $2(x,x) = 0$, so (x,x) has order 2. Look at the map $x \to (x,x)$, this lies in $\mathbb{Z}/2 \subset \mathbb{Q}/\mathbb{Z}$. And this is a homomorphism, since $(x+y, x+y) = (x,x) + (x,y) + (y,x) + (y,y)$, so you see that this is a linear function, this thing that doesn't look very linear, it's linear. I don't need that right now.

Now suppose T is cyclic and let x be a generator. Then $(x,y) = (x,sx) = s(x,x)$, an element of order two. The linear functionals on $\mathbb{Z}/4$ is a copy of $\mathbb{Z}/4$. The whole linear functional takes values in $\mathbb{Z}/2$, so it can't work. If T has odd order, then since there's no element of order two in the image, the whole pairing is 0. On a group of order the power of two, it's an order two thing so not enough to be a pairing thing.

Exercise 2 *Show that if a finite cyclic group T admits a perfect skew-symmetric pairing $T \otimes T \to \mathbb{Q}/\mathbb{Z}$, then $T \cong \mathbb{Z}/2$.*

In the skew-symmetric case, then, Poincaré duality says almost none of these are realizable. I gave the problem last time to show that $SU(3)/SO(3)$ is a realization. I don't know about the others.

If we can sort out which atoms we can realize, we can focus out on what's left. I would conjecture, well, maybe I won't conjecture that. I have some arguments in mind for crossing this example with S^2 and then doing a surgery to kill homology. When I was a graduate student I did surgery but I haven't done it since then, that was forty years ago. I could analyze this better forty years ago, definitely, probably. I would have been scared, because there were all these hard problems.

I think if we settle this atomic issue positively, I conjecture that we can answer everything because, see, for example, the thing in the middle has to be even rank and you can break them up into pairs where the quadratic form is a direct sum of diatomic quadratic forms. So the diatoms completely realize all the things in the middle and all the other diatoms realize all the rest, so we've solved the problem for $n = 2k$, k odd, with diatoms. That's one fourth of the dimensions, $4l+2$ manifolds. Torsion and free parts are split into diatoms. The other three mod four have to be discussed.

Now, for $4k+1$, I have hope for $4k+1$, so what I hope is that for $4l+1$, that $SU(3)/SO(2)$ generalizes to realize the other $Z/2$ atoms. Then these plus diatoms yield all possible examples. I remember the theorem that the torsion in dimension $2l$ looks like $A+A$, plus possibly $\mathbb{Z}/2$. This is what I hope. That will take care of $4l+1$. Let's go to the other two dimensions, $4l$ and $4l-1$. These always get more attention in topology because of the following fact: For $4l$, only three of the free middle atoms are realizable, when l is $1, 2$, or 4. More generally, with a bunch of $\mathbb{Z}$ in the middle, there's a symmetric quadratic form, and Gauss studied these in two variables. These have determinant one, he didn't have determinant one, and which ones of these forms can be realized for four-manifolds has gotten a lot of attention. If we're not worried about the quadratic form, then we can take $\mathbb{CP}^2$ and then we're done. But the natural question is to give the groups with the duality. In dimensions