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There are a lot of simple closed curves around two of three punctures in the plane which are not isotopic. This is not true for two punctures.

Isotopy is a family of homeomorphisms, while homotopy is a family of continuous maps.

At Berkeley, there was a confrontation, they wanted to paint murals on the wall and the trustees wouldn't let them. Thurston and I spent all day painting one of these on a wall. He came up to me and asked if I thought this was interesting. I said yeah and so he asked if I'd help. Five years later this was a basis for one of his theories. If you taffee back and forth he saw wha the limiting picture was. Start with two isotopies of the plane interchanging punctures one and two and two and three, and a circle around two and three. Then apply these one after the other. He saw the limit. Do four of these for your homework.

Deform the region slightly. You have a triangle and three teardrops of some sort. All the rest is foliated. So there's a continuous foliation with four singularities. If you look at the holonomy of the pattern, you get the golden number. This is invariant under the taffee transformation. The length increases exponentially fast, with rate the golden number, and, um, the limiting geometric object is a kind of foliation which is invariant under this transformation. This is Thurston's theory of [something], they have these invariant transformations.

There are actually two pictures which are transverse, depending on which transformation you end with.

So I wanted to, I wanted to give a lecture I meant to give Tuesday, I've wasted half my time already. That first year who wanted to know about T_3, was that you? He's not here, okay, let me discuss Poincaré duality a little more.

So $H_i(M,\mathbb{Z}) = \mathbb{Z}^k + \mathbb{Z}/n_1 \oplus \mathbb{Z}/n_2 + \cdots$ There is a map $0 \to T \to H_i(M,\mathbb{Z}) \to F \to 0$, and since this is free you can split it. So M is closed and oriented, we have a pairing $F_i \otimes F_{n-i} \to \mathbb{Z}$, a perfect pairing, where this is the intersection number. It's a perfect pairing that every linear functional arises uniquely as pairing with a fixed element.

Then two, $T_i \otimes T_{n-i-1} \to \mathbb{Q}/\mathbb{Z}$ is a perfect pairing. Okay?

And now, what's the picture? The picture of the first one is two cycles in complementary dimension, and there's an intersection number of two. If you move a cycle by a homology, you get a homology of the intersection numbers.

For two let's be in dimension three for concreteness. So we have $T_1 \otimes T_1 \to \mathbb{Q}/\mathbb{Z}$. So if you take λ copies of a cycle, it bounds something, then let y, we move this transversal to the picture so that it has some number of intersection numbers with this homology to zero, and then we divide by λ and reduce mod one to get into $\mathbb{Q}/\mathbb{Z}$.

If you chose a different homology, you'd have to take orientation into effect. The difference between the two numbers is zero, because the total intersection number, this is a one-cycle intersecting a two-cycle.

Oh, I forgot to say that the intersection number factors, it's, the torsion doesn't map to $\mathbb{Z}$ so $T \otimes H$ the map is zero. So you have $(T_i + F_i) \otimes (T_{n-i} + F_{n-i}) \to \mathbb{Z}$. But with torsion you get zero.

So the numerator is independent of the choice of homology. Oh yeah, right, why do you have to divide by λ and reduce mod one? Take something with λ sheets. If I want a cycle disjoint from that one, maybe I have something zero. I move across the boundary and I pick up λ intersection points. So the number changes by λ so dividing by λ it changes by one. So I get a well-defined answer in $\mathbb{Q}/\mathbb{Z}$.

So this F equation sort of has all the information of integral Poincaré duality. There are some algebraic consequences of this. One is if you look at M^{2k}, then we'll look at M^{2k+1}. So when k is odd, for M^{2k}, we get that β_k is even, since $F_k \otimes F_k \to \mathbb{Z}$, since k is odd you introduce a sign when you switch these, so it's skew symmetric and nondegenerate, with full so even rank.

If k is even you get a signature invariant, so $F_k \otimes F_k \to \mathbb{Z}$ is a unimodular symmetric form. There's a whole slew of invariants of such things, but the signature is in a way the most prominent one. This is the number of plusses minus the number of minuses in the quadratic form. There's a lot of this in dimension four. The Euler characteristic and signature control a lot of behavior. In general it's an important invariant.

In the other case, the odd case, you have $T_k \otimes T_k \to \mathbb{Q}/\mathbb{Z}$.

Notice mod two that a skew-symmetric form doesn't have to have even rank. If you looked only to $\mathbb{Z}_2$ coefficients you wouldn't get this here. You can ask about mod p coefficients for p odd; then would the Betti number be even?

So what can you say about torsion groups. When k is even, this k, then this pairing is skew symmetric and I think there's an algebraic lemma which says that $T_k = A \oplus A (+\mathbb{Z}_2)$ possibly. So tensored with odd coefficients it has even rank again. The $\mathbb{Z}_2$ may or may not be there.

In the last case you get a, so this is actually $T_{2l-1} \otimes T_{2l-1} \to \mathbb{Q}/\mathbb{Z}$, and this is a symmetric pairing. It's already interseting even with a cyclic group. If you take $L(p,q) = S^3/z_1 z_2 \sim \omega z_1, \omega^q z_2$, where ω is $e^{\frac{2\pi i}{p}}$ and $(p,q) = 1$. So $x \otimes x \to q/p$.

I don't know the proof of the lemma or even the statement for M^{2k-1}. We can go further if we're not worried about missing anything. The uniform statement, I think I've said this before, is you can state Poincaré duality, if you want to go further with memorable statements, I haven't prove these but the dual cell complex proves anything, but you can take $H_i(M, \mathbb{Z}/\lambda) \otimes H_{n-i}(M, \mathbb{Z}/\lambda) \to \mathbb{Z}/\lambda$ so when you take $\mathbb{Z}/\lambda$ coefficients, you get two things. So you get everything being a cycle. The intersection duality with these coefficients is a reformulation of the linking duality.

Now one can, you know, take $\mathbb{Z}/p$ coefficients. So for an even dimensional manifold, you'd get a little further for $4k + 2$.

So you get pairings not just from i to $n - i$ but also to $n - i - 1$, for Z/λ as H_i over $\mathbb{Z}$.

If anybody's interested in this structure, there's a wonderful paper by Kerveaire and Milnor, "Groups of Homotopy Spheres I" in the Annals 1963, they take a manifold and do surgery to kill its homology, and also some stuff about characteristic classes and so on. I read every line and understood it, and that was my training. II doesn't exist because there was a problem they were trying to solve, they couldn't solve. There's a table in the front, a string of numbers, even numbers, and what they're counting is the number of diffeomorphism types of manifolds that are homeomorphic to the sphere. Up to combinatorial equivalence, actually, this is what we're looking at. It's just that combinatorial and homeomorphic are the same for spheres except in dimension four. You weren't supposed to know that. It's period four in this sequence. There's a big group every fourth dimension. The paper is computing the groups in terms of known Bernouli number computations.

Maybe that's next, to show that if two manifolds are cobordant then their signatures are the same. They're cobordant if they have the same characteristic numbers modulo torsion, for a manifold, it's $\frac{7p_2 - p_1^2}{45}$. [unintelligible][unintelligible]Anyway.

Call this group Σ_i and you mod out by the big cyclic group whose order is the numerator of $B_n/4n$. Also a little $Z/2$ sometimes. Take these and mod out by a known quantity and then you get Π_i. meaning $\lim_{N\to\infty} \pi_{i+N}(S^N)$. So a lot of hardcore homotopy theorists work on that. The orthogonal group has a $\mathbb{Z}$ in every fourth dimension. This is where you mod out by the denominator. The other side is something else. This is related to the skew-symmetric form and the Arf invariant. So they proved this up to a slight uncertainty with $\mathbb{Z}/2$ in certain dimensions. So that's what they prove in there. They call it homotopy spheres. So not in dimensions four or three but elsewhere it's combinatorially equivalent to the sphere. To get it in low dimension you have to say it the right way. So this is remarkable, here's a homotopy group, here's a geometric group, here's the Bernouli number, and then you get this relation.

The function $n \to B_n/4n$ has a p-adic extension $\hat{Q_p} \to Q_p$ for every p. Why are these things regular, though, that's not understood.

It's not a discrete parameter. Integers are dense in the unit ball. It's part of an analytic function.

This is not what I meant to get into, it's complicated and deep it sounds, but it's not. You use an exact sequence and then they deal with the self-duality in the middle dimension. Then this formula comes in, and they start with an exotic sphere and show that when you embed it Euclidean space its normal bundle is trivial. Then you embed it in a high dimensional sphere, trivialize the normal bundle, and then take a sphere with dimension the normal space and map each of these disks onto that disk, mapping everything outside to one point. That constructs a map from one side to the other. From i to n. Now take two spheres; if they're homotopic, then that homotopy transversal to the point is a cobordism, and you go in there and kill the topology with surgery, cutting them out and gluing in disks, and now it looks like a homology cobordism, when you kill a homology class you also kill its Poincaré dual. You kill both the high and low at the same time. Finally you worry about the middle. If it's even dimensional there's no sweat. When they're odd and these are even, you're left with stuff in the middle. There's a subtle invariant called the Arf invariant, a possible cokernel. In dimesnion $4k$ there's a signature, and that leads to the extra factors. That's kind of a summary, but that exactly uses a little more detailed knowledge of what I've been describing. Take this general cobordism and make it an h-cobordism and that shows that these things are diffeomorphic.

As an exercise show $H_*(SU(3)/SO(3), \mathbb{Z})$ is $\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z}$. π_2 of a Lie group is 0, π_1 of $SO(3)$ is $Z/2$ and the other is simply connected.