Dennis Sullivan Course Notes
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At tea yesterday Travis asked an interesting question. I thought I'd answer it in the lecture, but it turns out it's less trivial than I thought. Let me embed it in a more difficult question. Let M be a closed d-manifold. Well, maybe I should get rid of, what can it be, $H_*(M)$? If M has boundary, then there's really no condition on the additive homology because you can just take any polyhedron, build an arbitrary polyhedron with any homology you want, and then build a complex with whatever you want as homology, embed it in a high dimensional Euclidean space, and take a neighborhood. Then you get all finitely generated Abelian groups.

So let's ask that it be a closed manifold. Or actually we can go back to the first question; we can ask about it being a coalgebra and an algebra, which satisfy the bimodule compatibility I was talking about. So you can ask which of these structures arise, which is a more interesting question. That level of question is pretty hard. So you can ask that over $\mathbb{Z}$ or $\mathbb{Q}$ and look at it over the intersection ring, but let's just keep sticking to the first question, which is the additive structure. Let's take it to be oriented. There is no linear condition; all finitely generated groups are realized. For which dimensions can you realize a set of groups? There's still a little bit left here. The real projective plane cannot be embedded in $\mathbb{R}^3$; this is because a compact embedded submanifold of $\mathbb{R}^3$ is orientable. It's pretty clearly embeddable in $\mathbb{R}^4$.

[Hand motions, pained expression]

It just is, you have it wrapping around a few times, it's just linked. Take a M obius strip, so all of these things embed in $\mathbb{R}^4$, with codimension two. Then you can get codimension two. The dimension is two plus the homological dimension. The homological dimension is the dimension of the top cell. Two more and you can make a manifold. Three more you cross with the interval. If you suspend you move up a dimension in Euclidean space. The little question left is codimension one. Maybe you can do it in higher dimensions than two. What about codimension one thickening? Sometimes you can't but then sometimes you can. Finish analyzing it.

There's a suspension isomorphism. Take two points and take two cones on X and add them together. Any cycle in X suspends to a cycle in ΣX of dimension one higher. Say reduced homology and it's true. You drop a base point. If it's connected it throws away the zero group. Some of the formulas work better with, well, if you take two points you get one $\mathbb{Z}$ in dimension zero, then suspend it and we get the circle. The reduced homology is $\mathbb{Z}$ in dimension one, getting rid of the homology in dimension zero. That's the main property of the homology functor, actually.

That's sort of interesting, can I have a four manifold, like a five manifold with boundary where the homology, $H_3(M^5) = \mathbb{Z}/2$? Can I have that? If it's six I can do it. I take the $\mathbb{RP}^2$ in $\mathbb{R}^4$ and suspend it twice. So what about five?

Oops, I'm being a little too fast. The circle with e^2 has homology $0, mathbbZ/2, 0$ but this is homologically dimension two because it has $mathbbZ/2$ homology. Torsion in integral homology gives you two torsion classes. In $\mathbb{R}^4$ we've got $\mathbb{Z}/2$ in codimension three, but the homological codimension is two.

You can have $H_1(M^3) = \mathbb{Z}/2$. Take $\mathbb{RP}^3$. So take that minus a three ball. That will have $H_1 = \mathbb{Z}/2$. So we can get, let's use these instead. We can also use lens spaces for $\mathbb{Z}/\lambda$. Suspending doesn't preserve the class of manifolds.

Exercise 1 *Construct M^{i+2} with $H_i M^{i+2} = \mathbb{Z}/2$ or show one cannot. This is integral homology.*

Suppose the answer is yes there. What about one? The claim is no. The cases are according to the boundary of M being zero or not. All of these should be orientable manifolds. The reason is that H^1 of any space with integer coefficients is torsion free. I used to use this every day. You have a chain complex $C_3 \to C_2 \to C_1 \to C_0$, and cohomology is you just reverse the arrows $C_3 \leftarrow C_2 \leftarrow C_1 \leftarrow C_0$. You have a cell complex so you can take a maximal tree and crush it to a point so you only have one vertex. Then the boundary map is zero so the kernel of the dual is a subgroup of a free group so is free. Then with integer coefficients and Poincaré duality you get $H_i(M^{i+1}, \mathbb{Z}) \cong H^1(M^{i+1}, \delta M^{i+1}, \mathbb{Z})$. The statement for $\mathbb{Z}/3$ will be different for $\mathbb{Z}/2$.

Now $H_d(M^d, \mathbb{Z}/2) = \mathbb{Z}/2$, while over $\mathbb{Z}$ for M nonorientable is zero. This implies that $H_{d-1}(M^d, \mathbb{Z}) = \mathbb{Z}/2$.

All right, anyway, you can have fun playing with nonorientable cases. This also shows that Poincaré duality doesn't work for nonorientable manifolds. Let's summarize. We're starting to get some constraints.

Let's say this works out; then we've analyzed the additive homology. You can build a manifold of any dimension but if there's torsion, you have to go up two more. If the top group has torsion you have to go up two more. That's using the fact that we have boundary. The original question was closed manifolds, right?

[I was thinking of manifolds without boundary but I didn't say so when I asked the question].

I didn't even think about the question before I came in the room, oh yeah, I did, but I didn't think there would be any constraints. The second case is $\delta M = 0$. So now we have a closed manifold. The first thing that's interesting is that a closed manifold can't be contractible, has to have homology. So $H_d(M^d, \mathbb{Z}) = \mathbb{Z}$ and $H_{d-1}(M, \mathbb{Z})$ is torsion free. We have the secret fact inside the manifold that $H_i(M, \mathbb{Z}) = H^{n-i}(M, \mathbb{Z})$. We also know that $H_i(M)$ modulo torsion is isomorphic to $H_{d-i}(M)$ modulo torsion. If you ignore torsion, one is Hom of the other. It's not functorial. One is dual to the other. There's a pairing, given an i-cycle and an $n-i$ cycle, you have an intersection number, so a pairing $H_i \otimes H_{n-i} \to \mathbb{Z}$. This factors through division by torsion. So every homomorphism from one of the two is given by selecting a complementary chain and applying the pairing. A map that is orientation preserving will commute with this.

[Sullivan tries to communicate with Zeng and then answers the phone.]

Hello, hi, I'm teaching a class right now, can I call you back?

Well, actually. If the map has nonzero degree, they're orientable, you can pull back the orientation and it will be orientation preserving. Then what's true is that if you take x and y, two cycles here, and let $\bar{x}, \bar{y}$ be the pullbacks then $\bar{x} = f^{-1}(x)$, then $f^{-1}(x \cap y) = \bar{x} \cap \bar{y}$. So intersection pulls back, just by set theory. On the other hand, if you have λ points, so you get λ times whatever you have here. This homology mod torsion gets embedded in this one. It's not equal, but it's λ times. This is actually an injection of the groups mod torsion.

A lot of these statements are clearer in cohomology. The fundamental class is multiplied by λ, and along with the ring structure this determines homology. You also get that $tor(H_i) \cong tor(H_{n-i-1})$. Actually this is the dual. It's $Hom(\cdot, \mathbb{Q}/\mathbb{Z})$. This can be shown with the dual simplical structure. It's my favorite proof.

We've heard these things, but let's use them. This duality presents some interesting problems.

So I have, I go til forty after the hour, is that when I go? So let;'s discuss some simple examples. Suppose we just have manifolds and we just have in 0 and d we have $\mathbb{Z}$. If I put it down too low I have to put it up high. If $d = 2l$ then I could try putting it in l and the rest zero. There's a possible candidate that satisfies our ground conditions. How about $l = 1$. Is there a two-manifold with $\mathbb{Z}, \mathbb{Z}, \mathbb{Z}$ homology. It's also true that you can't have it with a six, $0, 3, 6$, and you can't have $0, 5, 10$, or any of those. In the middle dimension this duality says that $H_l \cong H_l^*$. But when l is odd it's a skew symmetric isomorphism. You can $*$ it and get a map from H_l to H_l^*. So intersection is skew symmetric and nondegenerate, so even. Then middle odd dimension has to have even rank. So none of those is possible. That argument doesn't work when even. What about $0, 2, 4$ or $0, 4, 8$, or $0, 6, 12$ or $0, 8, 16$, or $0, 10, 20$, or $0, 12, 24$. The first one is recognizably $S^2 \cup e^4$, where the attaching map is the Hopf fibration. The next one exists too, it's the quaternionic plane $Q\mathbb{P}^2$, and for 8 it's the Cayley plane. Serious but not too serious shows that you have to be a power of two; more serious shows that these are the only ones. This is the Hopf invariant one problem, and was solved by Adams. This was one of the famous problems solved in the 60s. So what about for torsion. Ignore self-duality issues. Take any groups you want and any of these occur. I have to respect the first rule about H_{d-1}.

[lost]

So $\tilde{\tilde{H}}\cdot M = Im(\tilde{H}\cdot A) \oplus Im(\tilde{H}\cdot B$. I'll finish this next time. Hello? Hello?