Dennis Sullivan Course Notes
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Gabriel C. Drummond-Cole

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I went sledding Friday night and I lost the car keys. Moira bought me the extra set and a flashlight, and I looked and looked but I couldn't find them. So I went and had copies made, they cost a hundred and twenty five dollars each, and then I went to look and I found them, some nice person had hung them on a fence.

[My father carried two keys, then he lost them both in a day, so he went out and bought three.]

I think three is too many, with two you want to find it if you lose one. If you have three you don't mind losing it.

So given, I'm going to try to return to more straightforward considerations, given M^d, an oriented manifold, then for each surface Σ which has in boundary labelled with $h_i \in H_. M^d$, I'm going to say a procedure for getting an output $\sum h'_\alpha \otimes h''_\beta$ as follows.

Choose a pants decomposition of Σ and use $\cap : H_. \otimes H_. \to H_.$ and $\Delta : H_. \to H_. \otimes H_.$ This gives a linear map from the tensor product to the in boundary of homology to the tensor product of the out boundary.

Theorem 1 *The composed transformation only depends on the triple* $(\#in\ boundary, \#out\ boundary, g(\Sigma))$, *and is independent otherwise from the particular decomposition.*

That summarizes what I was saying last time. So let's see a sketch of a proof. In modern language this says that the natural structure on the homology of a manifold only depends on the topology of the surface.

1. If two directed trivalent graphs between n and m points have the same number of loops then they are related by vertex moves. A vertex move slides a vertex past another vertex.
2. vertex moves are generated by the conitions below.

Exercise 1 *prove step one.*

$(\Delta \cap (a_1 \otimes a_2)) = (\Delta a_1) \cap a_2 = a_2 \cap (\Delta a_1)$ (up to sign). This is the Frobenius condition. Also, you need associativity and coassociativity, corresponding to $(1,3)$ or $(3,1)$.

The reason this is true is because Δ is symmetric. No, this is not true because Δ is symmetric. All of the permutation equations are true. $h_1 \times M \cap \Delta h_2 = \Delta h_1 \cap h_2 = \Delta h_1 \cap M \times h_2$.

That's more than the sketch of a proof, I would say. Up to signs, that's basically it.

Further, only surfaces of degree zero and one give nontrivial operations; one is trivial if it is not a closed manifold, since it goes from top homology to zero homology. So the tree is kind of the only interesting one.

You start with n copies of the top cycle, then intersect them to get the top cycle, and then take the diagonal map and join together by multiplication. So this is like the trace.

All right. So if we use homology theory that doesn't vanish above the dimension this might be more interesting.

I have a conjecture that there is a more precise chain level version of the statement which yields the Pontryagan classes of M. I have another conjecture with is more vague. See, these identities are holding at the geometric level, not just at the homology level, but they only hold when the cycles are transversal. Remind me to tell you, Scott, I thought of a way to think about the canonical problem. I think there's a more intrinsic statement. If you just take the diagonal and treat it over the integers, it determines the entire homotopy type if you say it correctly. So this double version has Poincaré duality and should contain the Pontryagan classes. Each manifold has these natural specific classes in codimension multiples of four.

So all right, then I have a problem, when X is not a manifold but a space with multiplication. You have a map $X \times X \to X$ with an identity so that $m(i \times id)$ and $m(id \times i)$ are homotopy equivalent to the identity. Then $H_. X \to H_. X \otimes H_. X$ and $H_. X \otimes H_. X \to H_. X$ are degree zero. We know in this setting Δ is a map of algebras. So the problem is, what is the analog of the Riemann surface statement in this setting? Is there a geometric picture of these identities, for the Hopf algebra identities?