Dennis Sullivan Course Notes
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I would like as many people as possible to do the homework. Even if you're not registered I'd like you to participate. It's an excellent way to learn.

The whole thirty two ounces

[Lunch and dinner.]

Anyone seen "Supersize Me?" Amazing.

[Hilarious.]

In some sense, the way death is hilarious.

So, uh, let me just recapitulate what I said last time. If you have any space X then the induced map of the diagonal $H.X \to H.X \times X \cong H.X \otimes H.X$.

Did anyone check this for $\mathbb{Z}_n$ coefficients?

[Scott tries.]

If you take coefficients so the Kunneth formula is true then you have this structure. Then $\Delta(h) = 1 \otimes h + h \otimes 1 + \sum h' \otimes h''$.

This is because you can write $H_n(X \times X) = H_0 X \otimes H_n X + H_1 X \otimes H_{n-1} X + \cdots$

These summands are defined by including the cycles in the horizontal or the vertical into the product. So one can ask what the meaning of the intermediate terms is. An example of when the intermediate terms are zero is if H is in the image of the Hurewicz homomorphism $\pi_n \to H_n$.

This is easy to prove by naturality.

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & S^n \times S^n \arrow[d] \\ X \arrow[r] & X \times X \end{tikzcd}$$

So since this diagram commutes and the sphere doesn't have anything of middle degrees, a spherical homology element can't have anything of middle degrees.

There's a class of spaces, H-spaces, with a converse of that. There's a continuous multiplication $X \times X \to X$, and it follows from a theorem of Milnor in 1965 that with rational coefficients $\pi_n X \otimes \mathbb{Q}$ maps isomorphically to the subset of $H_n X$ with $\Delta h = 1 \otimes h + h \otimes 1$.

So take B_U, which has $\mathbb{Q}$ for even π_n. The homology is huge, dual to a polynomial algebra, but it has a special element at each second level.

Anyway, so these continue the example a little bit. I want to discuss this like a coin, with two sides. You have, in an H-space, this other map, and then you get a ring structure $H.X \otimes H.X \to H.X$, and the relationship between these two structures, the multiplication and the diagonal, is that the diagonal is a map of algebras.

You have $x \to (x, x)$, and $y \to (y, y)$ and then multiply and get (yx, yx) or yx and the diagram commutes

$$\begin{tikzcd} X^2 \times X^2 \arrow[r] & X \times X \\ X^2 \arrow[r] \arrow[u] & X \arrow[u] \end{tikzcd}$$

Under reasonable assumptions (connectivity, associativity, cocommutativity) this algebra is generated by primitive elements, i.e., those without extra terms. When you dualize this for cohomology you get a free polynomial algebra. The breakthrough is to know the Hopf algebra forces a lot of precision on what you get. I mean there are a lot of Hopf algebras but the cocommutative coassociative ones, well.

The quantum group stuff comes when you drop commutativity and associativity and deform things a little. If you ask that the terms vanish you get into the relationship with spheres, and sometimes it's an if and only if.

So $\Delta_i \in H_i \otimes H_{n-i}$, which is it? $A \otimes B$ is the same as an element in $Hom(A^*, B)$. One thing to keep in mind as you go through life, there a lot more true for finite dimensional vector spaces. I'm not even sure of this one. Sometimes you add a functional analytic consideration, i.e., make it a Hilbert space, or grade it and let the pieces be finite dimensional.

So take finite dimensional homology so that this is $Hom(H_i^*, H_{n-i})$, so that this is $Hom(H^i, H_{n-i})$. So you can call this $\cap h$, which takes u to $u \cap h$. In a manifold, if h is the fundamental class of the manifold, then the components of the diagonal, viewed this way, this is the Poincaré duality map, it's an isomorphism, so the components are nontrivial and give you duality. It's probably due to Lefschetz.

Okay, we haven't really gotten to the way that cohomology acts on homology.

Then there's this paradox, which is, if you have a cycle, some geometric thing, and then you want to put it in $X \times X$, then you can project either way and you get two maps of the cycle into X and the paradox is that these two maps determine the cycle but if you just know the homology classes of the two cycles, that doesn't determine the homology class of the product cycle. It's the difference between geometry and algebra. Homologically, it's not determined by the two projections.

I make a lot of mistakes sometimes. I don't understand the geometry so I pass to homology, and then I try to go back to geometry and things go wrong.

The Kunneth theorem is not natural geometrically.

I wanted to go over a little more the Frobenius condition for oriented manifolds. Say X is a manifold; then besides the diagonal structure $H_. X \to H_. X \otimes H_. X$, we also have the intersection algebra $H_. C \otimes H_. X \to H_. X$, which is additive in codimension.

We don't know the essence of being a manifold, but I think it's the niceness of the dual cell decomposition. You take two cycles and move them into opposite decompositions and then intersect them. Then we have this, from this viewpoint it doesn't look very associative or commutative. Up to homotopy it's commutative and associative. You can also define the intersection algebra for–this won't have a unit when the manifold is noncompact, but the whole space is naturally a unit, which leads you to infinite cycles, which look much like cycles locally but go off to infinity.

Later I want to go over this more slowly, but we'll combine these two ways of getting algebras. You have a space over the manifold where the fiber is a continuously varying group, then you can take two cycles, project, intersect, so that $z_1 \times z_2 \to \mathscr{E} \times \mathscr{E}$ withe $z_{12} \to M \to M \times M$, and then pull back to the diagonal.

I've been asking myself for some time, "if I consider $H.\mathscr{E}$, it has the diagonal and a multiplication; what is the relationship?" If the manifold or the group is a point, I need to know simpler cases. If the group is a point we get a Hopf algebra. So what are the possibilities? You can compose these in certain orders.

Most algebraic structures that one sees are expressed by relations among these quadratic compositions. The associative law is a statement about these trees. The Jacobi identity is the sum of three of these. There are a lot of possible structures, you can get twelve things like this.

You can first look for relations among quadratic compositions, and then look at all the operations we can get, and then we should be looking at any number of inputs and any directed trivalent graph. You multiply or comultiply and then ask for a linear combination of the operators to vanish.

It looks sort of difficult; there are a lot of possible answers; what is the natural algebraic structure. Then there's a famous one which is, there are actually two ways, kind of thicken this up to a planar surface. Then erase the original graph. Then you can ask that any time you do make an operation, the two thickenings are homeomorphic means that the operations are equal, that's kind of a natural, but difficult question to ask.

The second possibility would be to take the boundary of a neighborhood in three-space, and then ask that when you take your two structures that the operation depend on the topological type of the surface modulo the boundary.

This is the H_0 level of something. You could imagine that there is a well-defined operations parametrized by all the different Riemann surface structures, or chains on the space of all Riemann surfaces. If you imagine this more general thing and take H_0 you would get this.

So for these, I have for the first one associativity but not commutativity (with orientation of my surfaces) But the second one does have commutativity.

Lastly, there's the Frobenius condition. Multiplying on one side of the diagonal map is the same as multiplying on the same side after applying the diagonal map. So $a\Delta(b) = \Delta(ab) = \Delta(a)b$.

Supposedly, these conditions complete the characterization of the first type; adding commutativity and cocommutativity gives us the second type.