# Dennis Sullivan Course Notes April 8, 2005 

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I have trouble preparing this class because I have this other class, and I'm used to preparing a class just before I teach it. So I prepare both and get emotionally detatched from this one.

No class on Monday, the next class is on Friday. If some of you don't mind, I'd like to continue a little bit beyond the last day. I don't know what my feeling is going to be about the final. Usually these classes end officially on May eight.

Does every totally disconnected space of dimension zero? The proof I thought of really uses local compactness.

You can define dimension inductively, as a space has dimension at most $k$ if it has a neighborhood basis of sets with boundary of dimension $k-1$.

Then there's fractal dimension. In the 1980s everyone was interested in the Mandelbrot set. This is a technically difficult thing because you can change the definition slightly. This is one of the subjects I worked hardest in and got least far in. You can get good so that you can look at a picture and tell its dimension to within one decimal place, 1.3, 1.4, 1.5. Does that seem odd?
[It seems amazing.]
When you look at a person you know how old they are, how pretty they are, how handsome they are. You look at a couple of apartments and the next apartment, you know how much the rent is, $\$ 600, \$ 700$.
[Can you define a continuous family of fractals, so that the dimension goes up with a function of the parameter.]

Hello? Hi, but I'm teaching. Okay, but make it fast. Okay, how much. Okay. I'll call it in. Ex-wives and children.

Okay, if you look at the example $z \rightarrow z^{2}+\epsilon$. Make that $z_{n+1}=z_{n}^{2}+\epsilon$. If $\epsilon$ is zero, then things outside a circle go to infinity and things inside go to zero. The other things move around in a very chaotic fashion. The process of wrapping around twice is very chaotic. You shift
the sequence of a binary expansion. If you shift something that you know only up to some decimal places, if you shift it enough you don't know it at all.

As you vary the parameter, you get the same behaviour, but you get a Jordan curve. If you vary $\epsilon$ from zero then the Hausdorff dimension is a real analytic function of it. These are simply connected regions so there's a Riemann map to bring it back. Something about the function Riemann wrote down for a continuous nowhere differentiable function.

In some sense the integral dimension is like an infimum of the Hausdorff dimensions of the realizations.

There's a talk somehow related to this this weekend. They're analyzing the Perelman proof of geometrization. He's doing it because he wants to show that a negatively curved three manifold admits a metric of constant negative curvature. The universal cover looks like a two sphere but has a fractal dimension; if you can get it to two then you can do what he wants to do. I'm imagining what he's going to talk about.

So I want to, I'll be repeating what I said at CUNY, more or less, with some changes. We've been discussing the realization problem. Now I want to stop working on that and concentrate on the duality structure on a manifold, not just at the homology level but on chains. We'll start with an open manifold, and we'll assume it's divided into cells and if it's an open manifold we have infinitely many cells. It's locally finite, it will be locally finite, the combinatorial structure.

Of course we have $C_{f}$, the finite chains, finite linear combinations of oriented cells. Then we have a boundary operator, the boundary of an oriented cell is a sum of oriented cells.

This is a chain complex, we're working with a chain complex. Now, chain complexes are interesting mathematically. They form a category; there's a notion of chain morphism and also chain homotopy. A homotopy would be a map of degree +1 with $f-g=\delta A+A \delta$. A chain homotopy would be a pair of maps both of whose compositions are homotopic to the identity.

This notion of chain equivalence turns out to be the same, if the complexes are free in a certain sense, to having isomorphic homology.

These are used in all of homological algebra. You find a chain complex of free modules $F_{3} \rightarrow F_{2} \rightarrow F_{1} \rightarrow M$. I never understood this until someone drew it for me like this, where you see the resolution as a chain map:


Anyway, what is, sounds sort of simple and everyone knows it, is the idea of maps between chain complexes becoming isomorphisms of homology. This is quasi-isomorphism. Most of the subtle things I know about geometric invariants, $98 \%$ of what I've heard of, can be embedded in quasi-isomorphism classes of chain complexes.

As an example, if you have two subdivisions of the same chain complex, then they're quasiisomorphic. Sometimes you can get one as a subdivision of the other, in other times you have to use cellular approximation.

I saw some students yesterday, they we're talking about Khovanov homology, and they think that this is a chain complex level of the Vaughan Jones knot invariant. This is somehow related to Floer homology.

Okay. So you have the chains $C_{f}$, but you can also have the infinite chains $C_{i n f}$. Since this is locally finite, if we have a coefficient on every edge, you still have a boundary operator. So this forms another complex.

So what do we have? We have a chain map from the finite chains to the infinite chains $C_{f} \hookrightarrow C_{\text {inf }}$ and now we have, partially defined, when you have a manifold you also have this dual decomposition. Those are cell decompositions, that's a definition of what it means to be a manifold, that the boundary of every cell is a sphere. Then there are various chains, like this one, and then chains in the dual subdivision, and the intersection of these two, well, that's another subdivision, and the intersection is transversal, so you have at least a partially defined intersection theory. One thesis of Scott's is that axioms for when you can move a partially defined operation up to quasiisomorphic to be fully defined.

Up to quasiisomorphism I have something defined on the chain level. Last time we were trying to make things globally defined, and we had some issues involving dividing by two. The correct viewpoint is not so much that one, but you should think, what are the problems up to quasiisomorphism. So we have an intersection algebra structure on $C_{f}, C_{i n f}$. So this is a map of algebras, clearly, and $C_{f}$ is an ideal because intersection with a finite chain gives you a finite chain.

Also, the chains on any space, since you have the topological diagonal map, you can make a cellular approximation of the diagonal and then apply the induced transformation on chains, and that will give both a coalgebra structure on the finite and infinite chains. So there is a diagonal coalgebra, which is natural so that the inclusion is a map of coalgebras.

You can use the fact that space is locally contractible to show that there are chain homotopies so everything is a quasiisomorphism invariant.

A point that came up in CUNY is, what is dual to being an ideal for the algebra structure?
Exercise 1 What is the dual notion of an ideal?
So $C \otimes C \rightarrow^{\cap} C$ is intersection and $C \rightarrow^{\Delta} C \otimes C$ is the diagonal, which commutes with left and right multiplication.

So the algebra and coalgebra satisfy Frobenius compatibility. We discussed this some at the beginning of the year, namely that $\Delta$ is a map of bimodules. Then we have the unit, counit discussion. A unit in an algebra is a map $u: k \rightarrow B$ so that $B \cong k \otimes B \rightarrow B \otimes B \rightarrow B$ is the identity. So 1 in the ground ring is taken to some element of $B$ which multiplies everything to itself.

A counit just reverses the arrows. It's a map $\eta: A \rightarrow k$ so that $A \rightarrow A \otimes A \rightarrow A \otimes k \cong A$ is the identity, also to $k \otimes A$.

So $C_{i n f}$ has a unit and $C_{f}$ has a counit. So $C_{i n f}$ does not have a counit and $C_{f}$ has no unit. The counit is to project into degree zero. You have a zero chain which is finite and then you sum the coefficients to get into the ground ring. This would have infinite support over the infinite chains.

Before I discuss why this is true, I want to discuss the other property. Recall that $C_{f} \otimes C_{\text {inf }} \rightarrow$ $C_{f}$, and then we can apply the counit. So this is some kind of pairing between the two. So this gives $C_{f} \rightarrow \operatorname{Hom}\left(C_{i n f}, k\right)$.

So from a chain complex you can get a dual chain complex. You can get a map in the same way $C_{i n f} \rightarrow \operatorname{Hom}\left(C_{f}, k\right)$. The proper statement of duality is that these mappings are quasiisomorphisms of complexes. That's the sense in which this is a dual pairing, that each of these maps is a quasiisomorphism.

Historically, this may be how it happened, this construction, you have a chain complex $C ., \delta$, which has a dual $\left(\operatorname{Hom}(C ., k), \delta^{*}\right)$. This is called the cohomology.

This isn't very mysterious. Every chain complex can be built out of these simple models $0 \rightarrow \mathbb{Z} \rightarrow 0$;
$0 \rightarrow \mathbb{Z} \rightarrow{ }^{n} \rightarrow \mathbb{Z} \rightarrow 0$
(over $\mathbb{Z}$. These give homology $\mathbb{Z}$ or $\mathbb{Z} / n$. ( $\mathbb{Z} / 1$ is trivial). Dualizing, you get the same thing for the first one, but shift over for the nonzero maps.

This chain complex has homology and cohomology as shown:


This is more subtle for open manifolds. Let me do this one first.
We can say $H_{i}^{i n f} \cong H^{n-i}$.
We have an inverse limit here, which induces a natural topology, but you need to focus only on continous maps so that Hom of Hom will come back. The finite ones are already an inverse limit; the topology is that two things are close if they agree on compact subsets. So then we get a direct limit $\operatorname{Hom}\left(C_{i n f, k}\right)$ along with the direct limit $C_{f}$. The other ones are inverse limit, so uncountable where the direct ones are countable. I'm sorry, I didn't even think of this before. They're actually the same kind of animals.

So this isomorphism preserves [some] structure.
So the two statements are in the direct and inverse limits of finite dimensional vector spaces. This kind of goes away with finit dimensional homology.

So we have $H_{i}^{\text {ord }} \cong H_{\text {compact support }}^{i}$.
So what is the proof of these things? The theorems took pages to state but the proof takes one line. It's this picture. Going across duality shifts whether you're in Hom or not. If you have an original cell, take its boundary, then dualize, that's the coboundary of the dual to the original cell. This is the proof of whatever you can state.

So chains or cochains, not on a manifold-
Hello, hello? Yes, hi. Okay, good. Okay, perfect. I'll pick you up at the ferry. The first call is from the 16 year old son, the second is from the 32 year old, and I'm 64, that's it.

Think of the adjoint of the boundary. The boundary goes $C_{2} \rightarrow C_{1}$; the coboundary goes $C^{1} \rightarrow C^{2}$.
[...]
A continuous homomorphism on an inverse limit factors through something finite. Otherwise you'd just get bigger and bigger, $\aleph_{0} \rightarrow \aleph_{1} \rightarrow \aleph_{2}$.

This may be a little mysterious, but let me say, first of all, there's nothing mysterious about it. In the solid torus, this finite cycle is dual to this infinite one, so you won't find a dual unless you go to infinite ones. To make sense of $\int_{\text {chain }} \omega$, this could be a bounded $\omega$ over a finite chain or a compact support $\omega$ over an infinite chain.

That's like the situation, so you have a whole lot of structure here, frobenius algebras with unit/counit and two dualities. So what's dual to the ideal?

We were studying the question of which groups appear as the homology of a closed manifold, and there were subtleties and so on, I say we did about eighty percent. We can also ask about realizing this entire structure. The chain complexes have all this interesting structure. With the closed manifold, then $C_{f}=C_{i n f}$ and you just have one statement here, and you get the same statement, and so on. There's a chain complex and you have this natural equivalence $C \cong \operatorname{Hom}(C, k)$. I think that if you put that into the question, then you can finish the last twenty percent. But I want to put it in with the additional structure, and so the next thging I wanted to say is that, given one of these algebraic, let's call this whole structure here an open Frobenius algebra. You have the two things, the arrow, the ideal, and so on. Everything that I've said here, as an algebraic model of an open manifold.

Then you could also have a closed Frobenius algebra. By the way, the dimension is in here because it uses the intersection product, there is a dimension shift. So this should be an open/closed Frobenius algebra (of dimension n). The unit is in that dimension.

Given the open Frobenius algebra $(A \rightarrow B)$ of dimension $n+1$. we will try (motivated by

Renichki) (assume homology is finite dimensional) to construct an associated closed Frobenius algebra of dimension $n$.

He wrote a thousand pages of algebraic surgery theory, and it's many versions of the following construction. Let me give it a name. $C$ for closed. $C_{k-1}=\oplus\left(B_{k} \oplus A_{k-1}\right)$. This is a finite cycle and a homology of it to zero at infinity, a cycle in this. A chain will be a finite cycle and something chosen going off to infinity in one dimension higher. The boundary map is $\delta b-i a \oplus \delta a$, where $i$ is the inclusion.

Time is up, let me say one more sentence. If you have a manifold with boundary, there is a natural exact sequence, suppose $\delta W=M$. The sequence goes $\cdots \rightarrow H_{k+1}(W, M) \rightarrow$ $H_{k}(M) \rightarrow H_{k}(W) \rightarrow \cdots$ Let me write the chain complex... The relative homology is like the homology of an infinite chain. So $B_{k} \oplus 0$, this is a subcomplex, so $0 \rightarrow B_{k} \rightarrow B_{k} \oplus A_{k-1} \rightarrow$ $A_{k-1} \rightarrow 0$. SO you have this for every $k$. This is a short exact sequence of chain complexes, and so this is a good candidate for the boundary. You can't do this if you just know homology. Out of the chain representations of a manifold, you can perform a chain resolution of the boundary.

