Dennis Sullivan Course Notes
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Nathaniel asked me a question which made me realize I wasn't discussing obstructions well. Obstruction theory is sort of like a tree. You start at the bottom and try to climb to the top. It could be that the branch you're on stops at some finite level so that you don't get to the top.

At each level there's an obstruction that lets you or prevents you from climbing up one level. I'm describing obstruction theory associated to som inductive process, so the first statement is:

1. There are obstruction cochains θ

If you can't advance, it is because your obstruction cochain is nonzero.

2. Backing up one stage allows us to change θ by a coboundary.

Going back might also give you something nonzero. Real honchos in this bidness can go back two steps, or even more. Most of the real honchos are dead. Some of them are my age, no one can do that any more.

3. θ is always a cocycle.

Therefore, a picture, a terminating tree, corresponds to the homology class of $[\theta] \in H^{[}unintelligible](,\pi_{.}) \neq 0$. Honchos go back two steps but that's too arcane and it's not universally describable. You can do it in concrete problems, but you can't even deal generally with something about Grassmanians.

4. The first "nonzero obstruction" is an invariant of the situation.

I think cohomology theory was given to us by God for this theory, that's what it's there for. Now let's go over again the concrete situation of finding a cross-section. I realized when I talked to Nathaniel that the argument I gave was inadequate.

There are very few obstruction theories where you can actually understand the whole theory all the way through. There's the trivial one and then there's a nontrivial one in K-theory. Other ones, there are higher order obstructions but they're too hard.

Let's go over it again. Suppose you have a base and a bundle over the base. If it's a product bundle, I want a map of the base into the fiber and this is the graph of that map, a section. You can always find a section over the zero skeleton, unless the fiber is empty.

If the fiber were disconnected, I'd get an obstruction when I tried to move up a dimension. I'm going to suppose the fiber is connected. The base will be X, the total space E and the fiber Y.

So okay, if the fiber is connected I can extend it over the 1-skeleton. If $\pi_1(Y) \neq e$ then Y is the circle. I will only consider the cases of the circle or simply connected, $\pi_1(Y)$ is Abelian maybe, or contained in $\mathbb{Z}$.

Now Yasha's remark is very relevant. I could make the cross section $\mathbb{Z}$ ways, it's like a slinky.

Make a choice. Now I want to go around a 2-simplex. Over this I have a product bundle. I have a cross-section on the boundary which has a degree. Any time one of these is nonzero, I'm dead.

Now, that explains, well, let's explain two again. Take any function from the one-skeleton to the integers. If I change by adding something, this changes by the coboundary.

The first obstruction θ is in $H^2(X, \mathbb{Z})$, the integers being $\pi_1(Y)$, the circle.

This is in an oriented circle bundle.

We get these things from complex line bundles. This is your first Chern class, the first obstruction to nonzero cross section.

Suppose C_1 is trivial; then we can extend over the two-skeleton with a cross-section. We get a 2-sphere mapping to the circle. π_2 of a circle is zero. So we go to the three skeleton, and so on. We can continue building a section. We have a cross section of the circle bundle...

This classifies the circle bundle.

That was a fortuitous case, there are no higher homotopy groups. The natural thing to do now is the, well, I still have to prove that it's a cocycle. Let me do another, let me give you a thought exercise.

If you want to write anything about this I'll read it and comment on it.

This is kind of on the edge of research. Now suppose $\pi_1(Y)$ is some nonabelian group. As you move around the group doesn't have to be automorphed.

The obstruction should give you $H^2(X, G)$, where G is nonabelian. What do you get? You get some kind of obstruction. People have gotten around answering this question with some language involving algebra. You need a map from the fundametal group of X the fundametnal group of Y.

When you have a 2-cell, you can sort of say what the coboundary, people talk about $H^1(\ , non-abelian)$.

Only for a 2-cell are cells ordered, the S^1 boundary rather.