Dennis Sullivan Course Notes
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Maybe I should start. I'm going to not, I'm going to talk about two folk theorems today, which are surprising in their statements, easy in their proofs, and you won't find them anywhere written down except in my notebook because they're so easy.

Last time I talked about cochains and forms and described how cochains are a good approximation of forms. Today I want to talk about products. You all know there is a wedge product of forms.

Definition 1 *A simplicial cochain product on a positive dimension simplicial complex K is a collection of maps $\cup : C^j(K) \otimes C^\ell(K) \to C^{j+\ell}(K)$. Note that the degree adds like the wedge product. This should satisfy two assumptions:*

1. *$\delta(x \cup y) = \delta x \cup y + (-1)^{|x|} x \cup \delta y$, which means that $\cup$ induces a product on $H^{\cdot}(K)$.*
2. *This is less algebraic and more geometric. I want this to be local. This means that if x and y are j, ℓ simplices, then $x \cup y$ is nonzero only if they are faces of a $j + \ell$ simplex. That's not quite right. It's nonzero only if the two simplices share precisely one vertex. If they do so span then their product has to be a multiple of the unique thing they both span.*

   This is geometrically reasonable if you think about wedge product of forms. This involves multiplying coefficient functions pointwise.

Let me say two theorems.

Theorem 1 *Let K be a simplicial complex with at least one edge.*

1. *There are no "interesting" commmutative associative simplicial cochain products on $C^{\cdot}(K)$ over $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Here commutative means graded commutative.*

2. With $\mathbb{Z}$ coefficients there is no "interesting" commutative cochain product.

The proof is, calculate for the single segment. I'm going to do that for the rest of the class. It has an edge e and vertices a, b.

$$a \overset{e}{\text{———}} b$$

Before I do that, these statements should be surprising, because even though I argued last time that they are a good approximation for forms, they don't have a commutative associative product like forms do.

I'm going to use the same letters for the cochains. Is this board visible? By locality, $e \cup e$ is zero and so are $a \cup b$ and $b \cup a$.

Now $0 = \delta(a \cup b)$. This is $-e \cup b + (-1)^0 a \cup e$. So $a \cup e = e \cup b$.

I don't know what $a \cup a$ or $b \cup b$ are but we know them up to a constant because there is only one zero simplex spanned by a or by b.

Now $-\alpha e = \delta(\alpha a) = \delta(a \cup a) = -e \cup a + a \cup -e$ and $\beta e = \delta(\beta b) = \delta(b \cup b) = e \cup b + b \cup e$

These give $\Box + \triangle = \beta = \alpha$.

	a	b	e
a	αa	0	$\triangle$
b	0	βb	$\Box$
e	$\Box$	$\triangle$	0

So now we have one constant for this whole global thing, for this component, that any vertex times itself gives the same coefficient.

Now let's start proving the theorem. Assume that this is associative. Then let's compute $(ab)e$ and $a(be)$. These are $0e = 0$ and $a\Box e = \Box\triangle e$. So $\Box\triangle = 0$. Then in a domain, $\Box$ or $\triangle$ is zero.

So for associativity we then have

	a	b	e
a	γ	0	0
b	0	γ	γ
e	γ	0	0

If we assume no associativity but commutativity then $\Box = \triangle$ and $\Box = \triangle = \gamma/2$. Oh, so my theorem's wrong. Let me tell you now why γ should be one. Take the differential zero form which is a constant function one, and multiply it by itself. We get the constant function one. Similarly with cochains. You don't want to multiply the constant cochain by itself and get 43. Well, you could. Whitney classified all cohomology products induced by these simplicial cochain products. He showed there's a unique product up to a constant γ, what you get when you multiply 0-simplices by themsleves.

You know the theorem that $H^{\cdot} \cong H^{\cdot}_{DR}$. If $\gamma = 1$ then the cup product is the same as the wedge product. And the product is commutative and associative at the level of homology.

So there's not a lot of leeway. Now, he didn't solve the problem at the cochain level. We just solved the problem of characterizing them for the interval on cochains.

I should have put in my theorem, let $\gamma = 1$. How do you show there is no commutative associative one? If it's associative, then you get 0 everywhere.

The failure is nice because of the homotopies that account for the lack of commutativity or associativity.

Let me give examples of simplicial cochain products. The first one I'll do is in every book on algebraic topology. It's the "standard" associative product. This is what they mean by the cup product 98 percent of the time. Start with an ordering of your vertices. The product will depend on the ordering; the induced product will not. The product is given by concatenation. Then the product $[p_{\alpha_1} \dots p_{\alpha_j}][p_{\beta_0} \dots p_{\beta_k}]$ is zero unless $\alpha_j = \beta_0$; otherwise it is $[p_{\alpha_1} \dots p_{\alpha_j} p_{\beta_1} \dots p_{\beta_k}]$.

This is associative and noncommutative. It's only commutative when the dimension is zero.

Travis' idea is to let $\sigma_k \cup \tau_k$ be 0 if σ_j and τ_k do not span a $j+k$-simplex υ, and otherwise it should be a coefficient with a sign times υ. Again I assume I have an ordering of vertices. I take the sign of the permutations to put the common vertex at the end of the first word and the beginning of the second. Then concatenate. What about the number?

The coboundary is not a derivation of this, but it can be if you multiply by $\frac{j!k!}{(j+k+1)!}$. This is local, satisfies the derivation property, and has $\gamma = 1$. It's nonassociative since $(ab)e = 0$ but $a(be) = -1/4e$.

Where did this product come from? Let $\cup$ be the commutative product. Let $\sigma \cup \tau = (\int_\upsilon W\sigma \bigwedge W\tau)\upsilon$. This integral gives me a number. This could be a lemma or a fact but not an observation. Let me tell you why factorials have to do with this. The volume of an n-simplex is $1/n!$. That's a suggestion of where this might come from. Things are related to volumes of simplices.