Birman Conference
March 19, 2005
Various speakers

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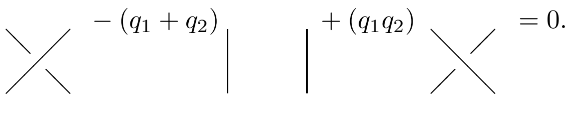
1 Stephen Bigelow (Trace Functions on the Birman-Wenzl-Murakami algebra)

[Our second speaker is from UCSB.]

Thank you very much for the invitation. My first contact with Joan was a flattering thing she said to Rob Kirby that he passed on to me. It was about my first topology paper, she said, it reads like a good novel. That was back when I had time to work on a paper, write it up real nicely. That was on Burau five. We haven't done any collaborative work, unfortunately. I missed that opportunity. One time we were talking about extending my work to mapping class groups, and Joan suggested working together, but I couldn't think of the representation she was suggesting, and nothing happened. Too bad, otherwise the previous speaker might have had to mention the Birman-Bigelow representation.

This is probably well-known, but this talk involves three algebras. There is the Birman-Wenzl-Murakami, the Iwahori-Hecke, and the Temperley-Lieb. It's nice to stick on as many names as you can. I'm not completely up on the history, but I think the first people to define these are Birman-Wenzl, Iwahori, and Jones.

So say K is a field, and we should try to do everything in a ring if we can, but let's just make it a field. Let $q_1, q_2 \in K^*$. The Hecke algebra is $KB_n/(\sigma_i - q_1)(\sigma_i - q_2) = 0$, where KB_n is the group algebra. So this is linear combinations of braid diagrams modulo the Reidemeister moves $RII, RIII$, and

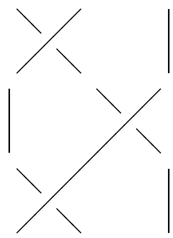


Theorem 1 *This is $n!$-dimensional as a vector field over K.*

Proof. There is an algorithm to "pull strings in front" starting at the bottom left and moving to the right. You use the relation to replace an undercrossing with pair of an overcrossing and a non-crossing. You do this until you get the first string on top in each diagram, and so on. You can see that there are $n!$ of these things; all that matters is where a string starts and ends.

So you get linear combinations of braids T_w for $w \in Sym_n$. I could define this properly as a word, but I'll just leave it as a picture; I hope it's clear. This connects i at the bottom to $w(i)$ at the top such for $i < j$ the string starting from i always crosses over the one starting from j.

For example, T_{13} is



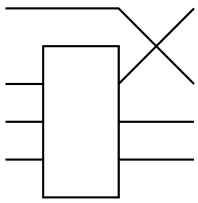
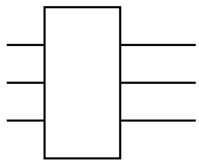
I'd like a nice proof of linear independence.

You have to show that the output of the algorithm is invariant under the Reidemeister moves RII, $RIII$, and the skein relation. It's sort of a pain in the neck. The skein relation is relatively easy.

Now I'm going to talk about trace functions. This came to the attention of knot theorists because of the Jones polynomial. A knot is a braid closure so we want an invariant of braids which depends only on the knot type. Then we want a trace function $tr : Hecke_n \to K$ such that $tr(ab) = tr(ba)$. So take a braid closure in an annulus. Then you get a closed braid diagram. A trace function on $Hecke_n$ is just a function from a vector space I'm about to describe to scalars.

That's V to be linear combinations of closed braid diagrams up to $RII, RIII$, and the skein. You're just modding out by linear combinations of commutators. So I'm going to do the same sort of thing. There are complications on applying my algorithm, because there's no bottom left. So you pick a basepoint on a string. Then for technical reasons I pull that to the outside of the annulus. Later on it comes in handy to allow it to be anywhere on the annulus; since I'm skipping that part I'll just pick it closest to the outside of the annulus. Now I want to start pulling it in front of everything else, progress along until I'm in front of everything else, until I get to a crossing that I've already changed. I don't correct what I've already changed.

Putting this in words, pull it in front in a positive direction around the annulus so that the first time we meet a crossing it is as an overcrossing. Now it's in front of everything. So you have a closed loop in front of everything else. So we can push it off to the outside of the annulus and so it's disjoint from everything else, and furthermore, it's in a very special form.

Think in three dimensions. You can think of the starting point as the furthest toward you and as it goes around it gets closer to the blackboard. To make it a closed loop at the last minute you bring it back up. So the disjoint loop, the description of what this knot is depends only on the number of times you spiral around. An exponential tree will be generated of different looking things, but they'll all break up into components like this.

So I get a linear combination of these things, and then I push it to the inside or outside and have some other stuff left over, and then I do the same thing to the stuff. And at the end of the algorithm I get a linear combination of diagrams which are disjoint unions of these things. There is one of these for every partition of n.

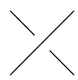
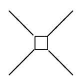
Now again there's a pain in the neck job checking that these are linearly independent. By the way, you can switch the order of these spirals, so all that matters is the partition. How do you know there's no extra relation? You check that it doesn't depend on which basepoint and also is invariant under the relations, the Reidemeister moves and the skein relation. If the output is invariant, then I should get zero if the thing I start with is zero.

So this is not hard, just tedious.

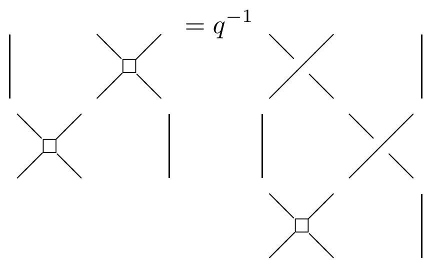
Now to get a knot invariant, we have a vector space worth of trace function, the dimension being the number of partitions of n. I can take the trace of the closure of my skein relation, and then I get $tr((1+q_1q_2)(U) = tr((q_1+q_2)(2U))$ where U is the unknot and $2U$ is two disjoint circles. So if $q_1+q_2 \neq 0$ you need $2U = \frac{1+q_1q_2}{q_1q_2}$. Then you need to unweave your spirals because each of these spirals is isotopic to an unknot, so you get no choice and you have to define that addition of a loop is $\frac{1+q_1q_2}{q_1q_2}$.

So I forget everything except the number of partitions. The usual $Hecke_n$ satisfies $(\sigma_i+1)(\sigma_i-q)=0$. You regain this by sending $B_n \to Hecke_n,\ \sigma_i \mapsto z^{-1}\sigma_i$.

Now let me defin the Birman Wenzl algebra. My skein relation is

q  $+(q-1)$  $-$  $=(1-q)$ 

where the box gets another relation



So for a basis you pull strings in front; every time you get a box, zip it up and pull it in front. You eventually get a trace in the same way with a basis corresponding to the number of partitions of $n - 2k$. I'm sorry I'm going over, but I want to do some history.

First, Jones defined his polynomial as a trace, and then Kauffman defined his polynomial and this algebra arose in reverse as something to take the trace of to get it. I'll stop here.

[Questions?]

[The BMW algebra splits as a direct sum of the Hecke algebra and another piece. How does that split work here?]

...

[It seems to me the number $1 \cdot 3 \cdot 5 \cdots$ comes into play here. Is that easy to see here?]

No.

[Are there other ... for other Artin groups or simple Lie algebras? I mean for BMW?]

[Is algebraic language always covered by the Yang Baxter?]

I don't know enough physics to answer that.

[My impression from book of Kauffman is that this is always a special case of the Yang-Baxter.]

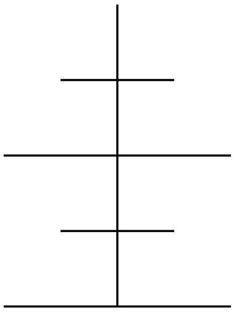
You probably know more than me.

[Afternoon session starts at 2:00]

2 Jozef Przytycki (Skein Modules to ...)

Our first speaker is from George Washington University.

I will mention Joan's work and that of her grandstudents.



My work is motivated by skein modules. My motivation is the Jones polynomial, Homflypt, Kauffman polynomial, and the three algebras mentioned in the last talk. This was the motivation to make a general definition.

I will work with a three dimensional manifold M and $\mathscr{L}$, links in M. Choose a ring R of coefficients. We take formal linear combinations of links dividing by some relation, as $R\mathscr{L}/(skein)$. These skein relations may be related to a polynomial. The simplest example $\mathscr{S}_2(M) = \mathbb{Z}\mathscr{L}^{or}$, where you set an oriented link equal to its smoothing. This relation does not change homology. When you have two things which are homologous, there is a surface between them. So we can go from one end to another. One can argue from this that you see nothing more than homology. So you get $\mathbb{Z}H_1(M, \mathbb{Z})$.

So the idea is to take such simple things and then deform them. I like to start from a simple deformation where you add in a q coefficient to the smoothing. So then $\mathscr{S}_2(M, q) = \mathbb{Z}[q^{\pm}]\mathscr{L}^{frame,\ or}/(D - qD_0, RI - q)$, where $R1$ denotes a right-handed kink. This is $\mathbb{Z}[q^{\pm}] \oplus Torsion$, where the torsion detects nonseparating surfaces.

This is just motivation.

I am starting from Skein modules, and I will move to Kei, which is a Chinese character that we use to define an algebraic object: So now $\mathscr{S}_{2\infty}(M) = \mathbb{Z}[A^{\pm}]\mathscr{L}^{unor,\ frame}/D = AD_\infty + \bar{A}D_0, D \sqcup U = (-A^{-2} - A^2)D$, and $RID = -A^3D$.

So for S^3 it is the Kauffman bracket. This works for $F \times I$, $F \hat{\times} I$. This is understood for $L(p, q)$ as well, and for those things which they branch cover, $S^3 - L(p, q)$.

My goal is not to describe what we know about skein modules. Here we've deformed smoothings; now we will deform crossing change. If you have the oriented version you get Hecke-style algebras and the Homflypt polynomial; in the unoriented version you get the BMW algebra and Kauffman polynomial.

So $\mathscr{S}_{3,\infty}(M) = \mathbb{Z}[a^{\pm}, x^{\pm}]\mathscr{L}^{unor\ frame}/D_+ - D_- = x(D_0 + D_\infty)$. Then $\mathscr{S}_{3,\infty}(S^3) = \mathbb{Z}[a^{\pm}, x^{\pm}]$ and $L = F_L(x, y)U$. Now, so, allow relative links, arcs properly embedded in $M, \delta M$.

The interpretation is that that we can create a tangle for every connection and you can make a basis. We would say that $\mathscr{S}_{3,\infty}(D^3, 2n$ points on $\partial D^3)$ is a free module of $(2n-1)(2n-3)\cdots 5 \cdot 3$ elements.

The proof I like is to take another copy, close it, and you have a link in S^3. Assume you know the existence of the Kauffman polynomial, ...

You can do this for a surface cross an interval by Liberum, that this is a free module, with a particular basis. Let's see what we're deforming.

The smoothing, or one move, led to the Kauffman bracket. The crossing change, or two move, led to the Kauffman polynomial. Next, we want to look at the three move. You can find some invariants to satisfy this, but address the simplest thing, what you get for S^3. You guess the generating set and guess that it is a basis. So now for the three move, what is the generating set? Before the deformation you get something easier; hopefully a generating set for one will be a generating set for the other. There was a conjecture that this would get from any link to the trivial link.

Let me formulate this. The Montesinos-Nakanishi "conjecture" in 1981 said that every link can be reduced to a trivial link for 3-moves. This was shown in 2002 not to hold by Dabkowski. To show this we had to introduce tools which will lead to the kei. I thought it would hold so I will start for braids of small braid index.

There is a beautiful theorem of Coxeter that $B_n/(\sigma_i)^3$ is finite if $n \leq 5$. So I told my student to prove this up to $n = 5$. For three braids it was fairly easy, for four it was hard. For five this is a very big group. My student used computers and found conjugacy classes of this group. There are more than 150,000 elements, but only about 100 conjugacy classes. Some could be done independently, and after some fight he couldn't deal with $(\sigma_1\sigma_2\sigma_3\sigma_4)^10$ In his thesis, he said that every five-braid can be reduced to either the trivial link or this one.

So now what about the 4-move? A small advantage is that it does not change the number of components, so you can work with knots independently. Nakanishi thought it should be true. It is still an open problem. How about two-component links? This preserves linking number modulo two, but can anything be reduced to the Hopf link or the unlink?

The Borromean rings were not reducible, and neither a doubly ringed Whitehead link, even though it is homotopically trivial.

The next picture is the five-move. For the Jones polynomial, evaluating at a tenth root of unity this is preserved under a five-move. So we cannot reduce everything by a five move, so instead of a five move, you take a two and a half-move. So Nakanishi and Hanikae and Uchida formulated this by taking a clasp to a rotation of it.

So now we have a more delicate move. We showed that we cannot reach everything with the five halves move, which is so called because you replace a trivial smoothing with a 5/2 rational tangle.

Anyway, so, this gives the idea that it would be nice to analyze rational moves in general. That's a good idea but hard. I'm at the level now of moves instead of skein modules. Say you have $T_1 \iff T_2$ for two tangles. We try to develop techniques to work in this general setting. The main example is to take the p/q rational tangle to the trivial smoothing.

The first tool is Fox n-coloring. Take a diagram of the link, and then color it with $\mathbb{Z}_n$.

So you get $Col_n(L)$, an invariant of links under n-moves. After going through an n move, you take a, b labelling to $a + n(b-a), b + n(b-a)$. But we know that $Col_n(\sqcup^m U) = \mathbb{Z}_n^m$.

The interpretation of the coloring space is the homology of the double branched cover with $\mathbb{Z}_n$ coefficients with an extra $\mathbb{Z}_n$. So we want to replace H_1 with π_1. This works; you decorate with formal variables a, b. At a crossing you get $a \to ba^{-1}b$. Then $\Pi^{(2)} = \pi_1(\mathbb{M}^{(2)}) * \mathbb{Z}$.

So now for the n moves you get $(ba^{-1})^n a$ and $(ba^{-1})^n b$. Then the Burnside group of a link $B_L(n)$ is $\pi_1(M_L^2)/(w^n)$ universally over π_1. So the closed braid $(\sigma_1\sigma_2\sigma_3\sigma_4)^1 0$ is not trivial, and you can show this by this argument.

Now let me go to kei. In 1942, M. Takasaki introduced an algebraic structure $X, * : X \times X \to X$. This was not motivated by topology; his motivation was to take i, j on the line. This is $i * j = 2j - i$, reflection around j. He was generalizing this kind of symmetry. He axiomatized to

1. $a * a = a$
2. $(a * b) * b = a$
3. $(a * b) * c = (a * c) * (b * c)$.

This paper will be translated soon; for the moment it is only Japanese. Forty years later this was rediscovered by Joyce, Conway, and Wraith. This is called an involutive quandle. From a group you make this action $a * b = ba^{-1}b$, this makes such a quandle.

What made me so interested, and is very interesting, I will leave you with this. After three moves, you get $a = b*(a*b)$. Multiply both sides by b. You get $a = b*(a*b)$ so $a*b = b*a$. This is a commutative quandle. Is $Q(m, 3)$ finite? It's a very interesting question. $|Q(4, 3)| = 81$. I should finish, thank you.