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Various speakers

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For those of you who can't figure it out or don't know, I'm not Liz Boyland. It is my great pleasure to welcome you here to this conference in honor of Joan. It's very nice to see how many young people there are here. Her dedication to her students has been a constant during her career here. I think this is a great tribute to her, the number of her old students I see here. I think we owe her a round of applause.

1 Dale Rolfsen: Are 3-manifold groups virtually orderable?

I've known Joan a little less than thirty years. She wrote her wonderful book and I wrote a book at almost the same time. Neither of us knew about each other and it's a shame, because I should have referred to her book in a number of places.

I was working on the problem that Luis Paris will be talking about, a conjecture of Joan about a certain map from the singular braid group into the group ring of the braid group.

We were working on this, and when you're working in a ring and you want to do calculations, if you have $xy = xz$ and you want $y = z$ then you can't have zero divisors or you're in trouble. No one knew if the group ring of the braid group had zero divisors. One good thing about left-orderable groups is that we do know that their group rings have no zero divisors. We never solved this but Luis did. So Joan suggested this, she has a knack for suggesting exactly the right things.

There are connections between the structure of 3-manifolds and properties of their groups. I like to talk about questions because that gets people more excited.

I chew my fingernails so I have trouble opening things.

Suppose G is a group and $<$ is a strict total ordering on the elements of G. Then we say that G is left ordered if $g < h$ implies $fg < fh$. We say that G is classically orderable, and this goes back to the beginning of the century, if also $g < h$ implies $gf < hf$. The prototype is the additive integers, additive reals, but lots of other groups are biorderable. Many interesting nonabelian groups are orderable, and many of them are not, too.

The first observation is that left orderable group implies torsion free. If $g > 1$ then $g^2 > g$ and inductively $g^k > 1$ for all k.

Orderable implies unique roots, which is a stronger condition. This is because if $g < h$ and $g_1 < h_1$, then you can multiply inequalities in an orderable group; I'll leave that as an exercise. Then $gg_1 < hh_1$. Therefore $g < h$ implies $g^n < h^n$. So powers of distinct things will never coincide.

This is definitely not true in general in left ordered groups.

Let's see, what am I going to talk about next?

[Does unique roots and left orderable imply orderable?]

Probably not.

[This depends on $<$, right?]

Well, this is universalized by talking about orderability. G is left orderable if there exists a P such that $PP = P$ and $G\backslash\{1\} = P \amalg P^{-1}$. G is orderable if and only if the above and $gPg^{-1} = P$ for all g.

If G is left orderable you take P to be the elements greater than 1; if you have these semigroups then $f < g$ if $f^{-1}g$ in P.

Think of the Gaussian integers. You can order them lexicographically, or you can choose any cone. Choose a line through the origin with irrational slope and take everything on one side as P. So there are uncountably many of these.

There is a natural topology on the set of orderings, which is in general an uncountable totally disconnected compact set, like a Cantor set.

Theorem 1 *(Farrell)*
If X is a space with a universal cover and $\pi_1(X)$ countable, so not Hawaiian earrings or something like that, then $\pi_1(X)$ is left orderable if and only if you can embed the universal cover $\tilde{X} \hookrightarrow^e X \times \mathbb{R}$, which is respectful of the projections, i.e., such that the diagram commutes:

$$\begin{tikzcd} \tilde{X} \arrow[r, "e"] \arrow[d, "p"'] & X \times \mathbb{R} \arrow[dl, "\pi_1"] \\ X & \end{tikzcd}$$

So take a point in x, lift to the cover, and then push across to $X \times \mathbb{R}$ and order by $\mathbb{R}$. The other direction is more interesting, but we won't go into it. I don't know too much about applications of this.

[Is there a geometric understanding of this?]

One very nice left orderable group is the space of homeomorphisms of the real line $Homeo_+(\mathbb{R})$, so orientation-preserving. This is left-orderable. One way to see it is to take your favorite well-ordering of the rationals. Look at the first rational upon which the values of the two homeomorphisms disagree, and compare their values on that rational. It's sort of lexicographical.

If you have a group which is countable $|G| \leq \aleph_0$, then G is left orderable if and only if $G \hookrightarrow Homeo_+(\mathbb{R})$. It's a very big group; it contains all of the left orderable groups I'm going to talk about.

Before I get to 3-manifolds, let me talk about surface groups, $G = \pi_1(\Sigma^2)$. By a surface, I want to assume it's at least a metric space, I don't want the product of the long line with itself. I want everything locally homeomorphic to the plane or half-plane, and let's say connected.

If the surface is noncompact or the boundary is nonempty, then $\pi_1(\Sigma)$ is free, and it turns out that a free group is not only left orderable but orderable. This is not obvious but I don't have time to give a proof.

If it's compact with no boundary then it is a closed surface and those are classified. For Σ the sphere you have a trivial group, which is orderable. $\mathbb{RP}^2$ is finite so not orderable. The next nonorientable surface, the Klein bottle $2\mathbb{P}^2$ has $K = \pi_1(2\mathbb{P}^2) = \langle x, y | xyx^{-1} = y^{-1} \rangle = \langle a, b | a^2 = b^2 \rangle$. If you set $y = 1$ then this maps $K \to \mathbb{Z}$. Then the kernel is also freely generated by y, so you get the exact sequence $1 \to \mathbb{Z} \to K \to \mathbb{Z} \to 1$. This property is closed under extensions, we now see, which is a fancy way of saying that if the kernel and cokernel are left orderable then so is the thign in the middle.

You use the ordering on x first and if two things agree you use the ordering on y.

So then you can see immediately that K is not orderable, since a^2 has a nonunique root.

All other surface groups are O-groups. A paper came out in 1942, the year I was born, in an Indian journal, that said you could not order nonorientable manifold groups. It was thought that such groups have this type of relation. Only two years ago, a partner and I came up with an argument that all such are ordered.

[How to do it for a surface of genus two?]

This problem reduces to ordering $\pi_1(3\mathbb{P}^2)$, which we all know is $T^2 \# \mathbb{P}^2$. So say you remove a disk from T^2 and sew in a Mobius band. Clearly anything nonorientabl with higher genus covers this. So $g\mathbb{P}^2$ covers $3\mathbb{P}^2$. So $\pi_1(g\mathbb{P}^2) \hookrightarrow \pi_1(3\mathbb{P}^2)$, so if this last is orderable so are all the others.

You can also cover with the oriented group, via a double cover, and so this gives all surfaces of higher genus.

Take the picture of the universal cover of the torus, and put a crosscap at every half-integral point. So this gives a covering $\tilde{\Sigma}$ of Σ. This is an infinite surface, so has $\pi_1(\tilde{\Sigma})$ free and is generated by x_{ij}, where this comes from the ij crosscap.

Then you have a short exact sequence $1 \to \pi_1(\tilde{\Sigma}) \to \pi_1(\Sigma) \to \mathbb{Z}^2 \to 1$. You need to know that this order is invariant under conjugation.

That was kind of sketchy, I didn't want to take the time to do it, but since Joan asked I did.

[Are there conditions on a covering space to say that the group of a space is orderable if the group of the cover is orderable?]

The question I want to ask is whether a 3-manifold group has a finite index subgroup which is orderable?

So what about 3-manifold groups? Well, it turns out that with two exceptions all surface groups are totally orderable. But there is a theorem, I'll call it the fundamental theorem of orderability for 3-manifold groups.

Theorem 2 *Assume M^3 is a compact connected 3-manifold, possibly with boundary or nonorientable. Without loss of generosity, as one of my students once said, assume it's irreducible (every tame 2-sphere bounds a ball) or P^2-irreducible if not compact. This is without loss of generosity because orderability behaves well with respect to free products.*

With M as above, $\pi_1(M)$ is left orderable if and only if there exists a nontrivial homomorphism $\pi_1(M) \to$ some left orderable group.

One direction is obvious; the other is astonishing. This isn't true for a group in general. The proof actually uses some real 3-dimensional topology, you need Peter Scott's compact core theorem, which says a noncompact manifold with finitely generated π_1 has a compact core carrying the topology.

Corollary 1 *If $\beta_1 M > 0$ then $\pi_1(M)$ is left orderable, because you just take the Hurewicz homomorphism to $H_1(M)$. So all irreducible 3-manifolds with positive first Betti number are left orderable, e.g., all knot groups.*

Corollary 2 *Suppose M^3 is a homology sphere, so that the first Betti number is zero and there is no torsion, and is Seifert fibred, then either M is the Poincaré manifold $\Sigma(2,3,5)$ or else $\pi_1(M)$ is left orderable.*

The reason is that you take, Seifert fibred means you're filled with circles. There's a map from $\pi_1(M)$ to $\pi_1^{ORB}(B)$ where B is the base orbifold. This kills the class of a generic fiber. This latter embeds in $PSL_2(\mathbb{R})$. With the exception of $\Sigma(2,3,5)$, the others have hyperbolic structures so map accordingly like this:

$\pi_1(M) \to PSL_2(\mathbb{R})$ nontrivially, and since there is $\widetilde{SL_2(\mathbb{R})}$, the solid torus $SL_2(\mathbb{R})$ has a universal cover and double covers $PSL_2(\mathbb{R})$. Because there is some trivial homology the map lifts

$$\begin{tikzcd} & \widetilde{SL_2(\mathbb{R})} \arrow[d] \\ \pi_1(M) \arrow[r] \arrow[ur, dashed] & PSL_2(\mathbb{R}) \end{tikzcd}$$

where this universal cover acts on $\mathbb{R}$ so that it injects into $Homeo_+(\mathbb{R})$. So we have lots of left orderable 3-manifolds. What isn't? We're going to have torsion free irreducible Haken manifolds.

Let Q be the oriented I-bundle over the Klein bottle. This is also the mapping cylinder of the double cover $cyl(T^2 \to K^2)$.

Now the boundary of Q is the boundary torus T^2, and $\pi_1(Q)$ is π_1 of the Klein bottle, so is $\langle a, b | a^2 = b^2 \rangle$, and $\pi_1(\partial Q) = \langle a^2, ab \rangle$, and let's call these the meridian and longitude. Then $M^3 = Q \cup_h Q_1$, where $h : \partial Q \to \partial Q_1$ has matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

All of these has torsion free fundamental group; it is $K *_{\mathbb{Z}^2} K$, and an amalgamated product of torsion free groups is torsion free. You can use the normal form of words or something like that.

Suppose p and q are positive but r and s are negative. Then I claim that $\pi_1(M)$ is not left orderable. Let me write a presentation for $\pi_1(M)$. This is $\langle a, b, a_1, b_1 | a^2 = b^2, a_1^2 = b_1^2, a_1^2 = a^{2p}(ab)^q, a_1 b_1 = a^{2r}(ab)^s \rangle$.

Suppose this is left ordered. I claim a and b have the same sign. a is bigger than the identity if and only if a^2 is. Suppose both are positive; then if p and q are positive, so is a_1^2, and thence a_1. But r and s negative implies that they have different sign, a big contradiction.

So we can't order any of these groups.

Let me just point out that $H_1(M)$ is finite, $|H_1(M)| = 16|p + q - r - s|$. These are rational homology spheres but not homology spheres, there are an infinite number of these guys and they all have torsion free non-left-orderable fundamental groups. It's Haken because the torus is incompressible; this is the JSJ decomposition. It's nonhyperbolic because of the torus.

Let me just leave you now with this question:

Is $\pi_1(M^3)$, say with the same hypotheses as above, virtually orderable. The answer is yes for Seifert fibred manifolds and others, for seven of the eight geometries. We don't know for hyperbolic.

Is $\pi_1(M^3)$ virtually left-orderable? Virtually means there is a finite index subgroup with that property.

Why are these questions interesting? If you have an orderable group which is finitely generated then its first homology is nontrivial, in fact infinite. Yes to question one implies the virtual first Betti number conjecture.

One way of saying that is that an irreducible 3-manifold has a finite cover which is orderable. And this other conjecture implies yes to question two.

[Can you talk a bit about foliations?]

Suppose you have a taut foliation; there is a closed curve in the manifold intersecting each leaf once transversally. Then $\pi_1(M)$ is left orderable.

The reason is, look at the universal cover; this inherits a foliation $\tilde{M}, \tilde{\mathscr{F}}$, which has leaf space $\mathbb{R}$. Then $\pi_1(M)$ acts on $\tilde{M}, \tilde{\mathscr{F}}$, and therefore $\mathbb{R}$. So $\pi_1(M) \to Homeo_+(\mathbb{R})$, with kernel a surface group because if it acts trivially on $\mathbb{R}$ it acts on the leaf. The kernel and cokernel are left-orderable.

Some people produced a family of hyperbolic manifolds whose groups are not left-orderable, so they don't support taut foliations. The famous manifold of Jeff Weekes, its group is not left orderable so it does not support a taut foliation.

[Let's thank the speaker again and meet down here in fifteen minutes.]