# Birman Conference <br> March 16, 2005 <br> Various speakers 

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## 1 Dan Margalit (Mapping Class Groups)

[Thanks for coming back for the second day of Birmanfest]
We learned about mapping class groups yesterday; today I'll focus on the tools you'll need to understand the talks.

So we talked about the mapping class group yesterday, we didn't talk so much about different kinds of mapping classes, we talked about Dehn twists.

Theorem 1 Nielson-Thurston Classification
Any $f \in \operatorname{Mod}(S)$ is of one the following types:

1. finite order
2. reducible
3. Psuedo-Anosov

This is actually a hard, deep theorem, I don't want to pass it off as something you introduce on the second day. You can't be all three, but you can be, say, one and two. Let me say what these are.

1. Finite order means $f^{n}=1 \in \operatorname{Mod}(S)$. For example, take a three-handled triangle and rotate it one third. That cubed is the identity. Note that this is only isotopic to the identity but there's a theorem that says you can find an isotopic element whose power is actually the identity.
2. Reducible means that $f$ fixes a collection of istopy classes of simple closed curves. One example is a Dehn twist; it fixes itself and anything disjoint to it. The idea is that this is reducible because you can ignore what happens off a particular subsurface.
3. Pseudo-Anosov, I'm not going to give the definition. It's a generalization of Anosov maps on the torus. What's an example? $\left(\begin{array}{ll}2 & 1 \\ 1 & 1\end{array}\right)$.
Let me list some properties:

- length $\left(f^{n}(a)\right) \sim \lambda^{n}$ in any hyperbolic metric, where $\lambda$ is a fixed number called the "stretch factor" and $a$ is any curve, that is, an isotopy class of curves, where length is the length of the geodesic representative.
- $i\left(f^{n}(a), b\right) \sim \lambda^{n}$, where this is the same $\lambda$. If you keep on iterating, it makes everything intersect more and more.
- $Z(f)$ is a finite extension of $\langle f\rangle \cong \mathbb{Z}$.

Let me just say, let me note, that Pseudo-Anosov is mutually exclusive from the first two but the first two aren't If you're stretching curves you can't be finite order; if you're the identity somewhere you aren't stretching curves.

So let's do an example, of $\sigma_{2} \sigma_{1}^{-1} \in B_{3}$. The first question is, braids as mapping classes? Why, yes.


This is like "the" braid. Now $B_{n}=\operatorname{Mod}\left(D_{n}, \partial D_{n}\right)$ where $D_{n}$ is the disk with $n$ punctures. Think of the braid in space as the space-time map of an isotopy of the disk. So this is like the taffee model, the juggling model, the three man weave in basketball.

A claim that I'm not going to prove is that $\sigma_{2} \sigma_{1}^{-1}$ is not finite order or reducible. So how do you get a handle on this? This is the famous example; it was done by Thurston and Sullivan at MSRI, and there's a painting of it at Berkeley. It was on the cover of Notices a while back.
[taffee pictures]
How did I do this without screwing up? There's a picture in my head that comes from Thurston about what this looks like. You start squeezing things together.

You draw a "train track," a 1-complex with well-defined tangent and additive labels. So to recover a curve, you replace a labelled arc with that many parallel arcs.

Okay, these are fun things to play with. Joan and I spent a few days drawing pictures like this last time I came here. We're not done understanding this. So if this is not finite order
and is not reducible then I should be able to produce a stretch factor. If I want to understand the mapping class, it turns out I can just use the train track picture. So we apply $\sigma_{2} \sigma_{1}^{-1}$ to the train track.

If I label my train track from right to left with $x, y, x+y$ and apply this I get $2 x+y, x+$ $y, 3 x+2 y$. So this changes the $x, y$ weights by the matrix above. The stretch factor is $\frac{3+\sqrt{5}}{2}$. That's the famous example.

A very large amount of the work in mapping class groups uses this classification. The real definition of pseudo-Anosov is that there are two transversal foliations, one a stretch, the other a shrink, and their product is one to preserve area.
[Rolfsen, who's talking tomorrow, has a program on his webpage that does this for you for any braid.]

Let me say what a curve complex is. It was defined by Harvey in the 70s. This is an abstract simplicial complex. The vertices are isotopy classes of simple closed curves in $S$. Edges ( $k$-simplices) are $k+1$ mutually disjoint curves.

First of all, this is an infinite thing. On a genus two surface it looks locally like this:
[picture]
Not only is this infinite, but it's locally infinite. A cool fact is that it's connected. There's a cool Morse theory proof of this. It's infinite diameter (this was shown by Masur-Minsky and Kobayashi). This is a lot of peoples' favorite question these days; take the ball of radius three around a point; is it connected?

There's something else I want to add but I'll go on. This is actually a wedge of spheres. They're all the same dimension.
$\operatorname{Mod}^{ \pm}(S) \rightarrow \operatorname{Aut}(C(S))$ is a natural map since it acts on the vertices and preserves disjointness.

Theorem 2 (Ivanov, Korkmaz, Luo)
The way I've defined things, say $S$ is not the punctured/twice-punctured torus or four times punctured sphere. Then this natural map is surjective.

The idea is to show that intersection number one is preserved. We don't have time to talk about it but discuss it later with your friends. It's not a corollary but a theorem that follows from this:

Theorem 3 The natural map (for the same surfaces) $\operatorname{Mod}^{ \pm}(S) \rightarrow \operatorname{Aut}(\operatorname{Mod}(S))$ is surjective.

This is a cool theorem. You have an automorphism of your group, you want to show it's basically conjugation. Let me say the basic idea of why these things are related. Let $\varphi \in$ Aut $(\operatorname{Mod}(S))$.

Step 1. Prove $\varphi$ takes Dehn twists to Dehn twists, so $\varphi\left(T_{a}\right)=T_{a^{\prime}}$. Then I think that $\varphi$ takes $a$ to $a^{\prime}$.

Step 2. Notice $\varphi$ induces $\varphi^{*} \in \operatorname{Aut}(C(S))$.
Step 3. Now apply the theorem, which tells you this comes from an element $f$ of the mapping class group.

Step 4. Check that $f$ induces $\varphi$.

Step one is where you use Thurston's classification. So you can use the curve complex to tell you about all kinds of neat things.

Thanks.
[Any questions?]
[A comment: some of us sitting here have given many lectures on this and were busy taking notes because he did such a good job.]

## 2 Nate Broaddus (Braids and Artin Groups)

[Welcome back, we're going to hear about Braids and Artin groups.]
Last time I told you everything there is to know about braids, so today we'll talk about a generalization called Artin groups. I can't talk about those without talking about Coxeter groups, which were defined by Tits in 1966 but popularized by Brieskorn and Saito in 1972. This was the first place where it was suggested that Artin groups be studied as a collection in themselves.

So the definition is, let $S$ be a finite set $\left\{a_{1}, \cdots, a_{n}\right\}$, and then a Coxeter matrix is a symmetric $n \times n$ matrix with entries $m_{i j} \in\{2,3, \cdots, \infty\}$ for $i \neq j$ and $m_{i i}=1$.

An Artin system $(A, S)$ is going to be a group $A$ with generating set $S$ and relations $a_{i} a_{j} a_{i} \cdots=a_{j} a_{i} a_{j} \cdots$ where the lengths of these are $m_{i j}$. If $m_{i j}=\infty$ then there is no relation.

A Coxeter system $(W, S)$ with Coxeter matrix $\left(m_{i j}\right)$ is going to be a pair $w$, which is a group generated by $S$ subject to relations $\langle S| a_{i}^{2}=1,\left(a_{i} a_{j}\right)^{m_{i j}}=1\langle$.

If we add the relation $a_{i}^{2}=1$ to $(A, S)$ we get the relations of $(W, S)$ so $A$ always surjects onto $W$.

Sometimes we give a Coxeter group by a graph. Given a finite graph $\Gamma$ with edges labelled in $\{4, \cdots, \infty\}$ we can specify a Coxeter or Artin system as follows: let $S$ be the vertices of $\Gamma$ and let $m_{i j}$ be 3 for an unlabelled edge, 2 if there is no edge between two vertices, or the
label if the edge is labelled. For example,

has $W(\Gamma)=\left\langle a, b, c \mid a^{2}=b^{2}=1,(a b)^{3}=1,(b c)^{4}=1,(a c)^{2}=1\right\rangle$.
If I could write $\Gamma$ as a disjoint union $\Gamma=\Gamma_{1} \amalg \Gamma_{2}$ then both the Artin and Coxeter systems are the products of the groups, that is, $A(\Gamma)=A\left(\Gamma_{1}\right) \times A\left(\Gamma_{2}\right)$ and $W(\Gamma)=W\left(\Gamma_{1}\right) \times W\left(\Gamma_{2}\right)$.

In this case we call $A(\Gamma)$ and $W(\Gamma)$ reducible; otherwise we call them irreducible.
I claim that this is a generalization of braid groups. The way to get braid groups as Artin groups is as follows:

If $\Gamma$ is the graph which is just a straight line with 4 vertices

then $A(\Gamma)=\langle a, b, c, d \mid a b a=b a b, a c=c a, a d=d a, \ldots\rangle \cong B_{n+1}$. Recall that the braid group has one fewer generator than strands.

Note that $W(\Gamma)$ in this case is $\Sigma_{n+1}$.
I'm going to say a little bit about Coxeter groups specifically. Look at the regular pentagon, which looks, up to isotopy, like this picture. Look at the symmetries of this pentagon, which are generated by reflections in two lines $a, b$. So the symmetry group is $\langle a, b| a^{2}=b^{2}=$ $1,(a b)^{5}=1$. This is a presentation for the group, which you can figure out by counting things.

So now let's do a representation of $W(\Gamma)$ for all $\Gamma$.
Let $H$ be codimension one in $V,\langle$,$\rangle , with \vec{n}$ the unit normal. If we project $x$ onto $\vec{n}$ then we get $\langle\vec{x}, \vec{n}\rangle \vec{n}$. So we want to subtract two of those vectors from $x$. So $R(x)=x-2\langle x, n\rangle n$. So this is straight linear algebra, nothing fancy.

We want to design, given $\Gamma$ with $n$ vertices, $\langle,\rangle_{W}$ on $V=\mathbb{R}^{n}$, so that reflection in the planes normal to the standard basis $e_{i}=\left[\begin{array}{c}0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0\end{array}\right]$ generate $W(\Gamma)$.

Say we have two generators $a$ and $b$ with normals $e_{b}, e_{a}$, and we want that reflection in $a$ and then in $b$ to the $m_{i j}$ be the identity. So then we want these to intersect with solid angle to
be whatever is like $2 \pi$ over $m_{i j}$. So $\left\langle e_{b}, e_{a}\right\rangle=-\cos \frac{\pi}{m_{i j}}$. Let $N=-\cos \frac{\pi}{m_{i j}}$ with the obvious extension for $m_{i j}=\infty$.

So we let $\langle x, y\rangle_{W}=x^{T} N y$. Then $R_{i}(x)=x-2\left\langle x, e_{i}\right\rangle_{W} e_{i}$. Then to get a representation of $W(\Gamma) \rightarrow G l_{n}(\mathbb{R})$ we sent $a_{i} \rightarrow R_{i}$.

The punchline is that this is a faithful representation (Tits).
So now let me say quickly what this does for $D_{\infty}=\left\langle a, b \mid a^{2}=b^{2}=1\right\rangle$. If you work through the construction, you get $a$ to be a regular reflection and $b$ is a skewed reflection. To get $W(\Gamma)$ acting on a nice space we look at the orbit of the first quadrant (octant or whatever). So $D_{\infty}$ acts on the upper half-plane (open) with finite point stabilizers.

Now it turns out that if our Coxeter group is finite, the orbit of the first quadrant is our entire vector space; if it's infinite it's a cone in the space called the Tits cone. In this case the Tits cone is the upper half-plane. Now I want to move on to Artin groups. Remember that we had the braid group as $B_{n}=\pi_{1}\left(\mathscr{C}_{n}, *\right)$ where $\mathscr{C}_{n}=\mathscr{C}_{\hat{n}} / \Sigma_{n}$. We can do the same type of thing here. For Artin groups we can do the same type of thing.
$\mathscr{C}_{\hat{n}}=\mathbb{C}^{n}$ - hyperplanes, is a sort of a complexification of $\mathbb{R}^{n}-$ planes of reflection for the symmetric group.

So in general, $A(\Gamma)=\pi_{1}$ of a complexification of the Tits cone modulo the corresponding Coxeter group. A number of questions come up from this description. A big one, I guess, is $\mathscr{Y}$ a $K(A(\Gamma), 1)$ for $A(\Gamma)$ ? This is known to be true in a few cases and is conjectured to be true in all cases.

Sometimes the Tits cone of a Coxeter group $W(\Gamma)$ has a natural "nice" geometry such as spherical, Euclidean or hyperbolic spaces. Some examples are, well, a Euclidean example is plane reflections in the sides of an right triangle with angles $\pi / 4$. If you give me a hyperbolic triangle, the reflections will act on the hyperbolic plane. So this question has been answered affirmatively for Euclidean and spherical cases, well, one Euclidean space (Charney, Peifer 2003).

Oh, okay, I just want to say one quick thing. There are a few results. There is the Tits conjecture, which has been shown to be true by Paris and Crisp. This says the subgroup of the Artin group generated by squares of the generators has a very simple presentation. It basically says that if $m_{i j}$ is anything but two, there is no relation between the squares of the generators. Paris and Crisp showed a bit more, and the proof is nice because it relates Artin groups to mapping class groups.
[Any questions?]
[Can you say a word about the relation to mapping class groups?]
They define a type of Artin group called small type, where $m_{i j}$ is two or three. All your generators are commuting or braid relations. Those type of relations hold in the mapping class groups. Given curves $a, b$, then $a b=b a$ if their intersection is empty and $a b a=b a b$ if
$|a \cap b|=1$. So we send $A_{s}$ into the mapping class group and show that it acts on, we have our original Artin group, and we have a subgroup $B \subset A$ sent into a subgroup of the small Artin group, we're only interested in the group generated by the squares of the generators.

We'll take another two hours for lunch.

## 3 Abhijit Champanerkar (Invariants of links)

Today I'm talking about hyperbolic invariants of links. Let me start by giving you some survey articles.

1. Hyperbolic Geometry: The first 150 years (Milnor)
2. Geometric Invariants for 3-manifolds (Meyerhoff)
3. Hyperbolic Invariants of Knots and Links (Adams, Hildebrand, Weeks)
4. Hyperbolic Structures on Knot Complements (Callahan and Reid)

Before I tell you what hyperbolic means, let's talk about geometry and geometric structures on a manifold.

If you have $X$, a Riemannian manifold such that it is

1. complete
2. homogeneous
3. simply connected

Then $X$ is called a geometry.
Then $M^{n}$ ( $n$ is the dimension of $X$ ) has a geometric structure based on $X$ if you can put a metric on $M$ such that $M$ is complete and such that every point of $M$ has a neighborhood isometric to a neighborhood in $X$.

This condition gives you equivalently that $X$ can be seen as a covering space of $M$ with covering translations isometries of $X$.

You can think of $\Gamma=\pi_{1}(M) \hookrightarrow \operatorname{Isom}(X)$ and $\Gamma$ acts freely and discretely on $X$.
I will assume that $M$ has finite volume geometric structure based on $X$.
In dimension two you have three geometries, spherical, Euclidean, and hyperbolic, denoted respectively $\mathbb{S}^{2}, \mathbb{E}^{2}, \mathbb{H}^{2}$. The closed orientable surfaces are as follows: the sphere is spherical, the torus is Euclidean, and all the others are hyperbolic.

What about dimension three? Thurston came up with eight geometries and made the famous conjecture that all 3-manifolds are based on one of these geometries.

Thurston (1980) gave the eight geometries $\mathbb{S}^{3}, \mathbb{E}^{3}, \mathbb{H}^{3}, \mathbb{S}^{2} \times \mathbb{R}, \mathbb{H}^{2} \times \mathbb{R}, N i l$, Sol, $P S L$. There is a good article by Peter Scott.

So imagine you have a 3-manifold; cut along $S^{2}$ (The Kneser-Milnor decomposition) and then on tori (JSJ decomposition). The conjecture is that having done these cuts, so that there are no incompressible tori and every sphere bounds a ball, each piece has a unique geometric structure based on one of the eight geometries. This is the geometrization conjecture. Thurston proved this for Haken manifolds, and recently Perelman has proven it in general.

Most of these can be described in terms of surfaces but the hyperbolic geometry, $\mathbb{H}^{3}$, is not as well understood and is ubiquitous, meaning most manifolds are hyperbolic. What is $\mathbb{H}^{3}$ and what are manifolds based on it?

I'm going to describe to you a model of $\mathbb{H}^{3}$ and give you the metric. There are various models; the one I'll use is $\{(x, y, t) \mid t>0\}$. The metric is $d s_{\mathbb{H}^{3}}^{2}=\frac{d s_{\mathrm{E}^{3}}^{2}}{t^{2}}$. The geodesics are vertical rays and semiscircles perpendicular to the $x y$ plane. The isometry group is $\operatorname{Isom}^{+}\left(\mathbb{H}^{3}\right) \cong$ $\operatorname{PSL}(2, \mathbb{C})$. The sphere at $\infty$ is $\mathbb{R}^{2} \times\{0\} \cup\{\infty\}=\mathbb{C} \cup\{\infty\}$.

If you do a JSJ decomposition you get a manifold with torus boundary. How do you explain a manifold with torus boundary using a hyperbolic structure? The way to do it is by using cusps.

These manifolds with torus boundary, well, a cusp is $T^{2} \otimes[0, \infty)$. The metric is $d s_{\mathbb{E}^{2}} \times \frac{d s_{\mathbb{E}}}{t}$. So you think of the torus as getting smaller and smaller. Such a thing has finite volume. If you look at hyperbolic space, and a torus is something like a square. A pillar in the hyperbolic space starting at height one gives you this torus cross a ray; then the volume is $\int_{1}^{\infty} \frac{1}{t^{2}} d t=1$.

In Euclidean space this is impossible but in hyperbolic space you can have an infinite piece with finite volume.

So a noncompact hyperbolic manifold of finite volume will have cusps. A knot or a link $K$ in $\mathbb{S}^{3}$ if the complement of the link $S^{3}-K$ is hyperbolic, i.e., has a geometry based on $\mathbb{H}^{3}$.

Now you can see that the complement of a knot or a link will have a boundary of tori. If you remove a knot you will see something like $T^{2} \times[0, \infty)$ around it.

So the complement really has such cusps. Why is hyperbolic geometry important for knots and links?

Theorem 4 (Thurston)
Let $K \subset S^{3}$ be a prime nontrivial knot. Then $K$ is hyperbolic if and only if $K$ is not a torus or satellite knot.

A torus knot can be drawn on the surface of a torus; a satellite knot is a nontrivial knot in a solid torus mapped to a nontrivially knotted solid torus.

There are always very nice theorems for alternating knots.

Theorem 5 (Menasco)
Let $K$ be a prime alternating knot which is not a torus knot; then $K$ is hyperbolic. The alternating torus knots are the $(2, k)$ torus knots.

The nice statement is that any sort of geometric invariant is a topological invariant.

Theorem 6 (Mostov-Prusad Rigidity)
If $M^{3}$ has a finite volume hyperbolic structure then that structure is unique.

The way to think about this is that, take the torus, you can rescale it and area, a geometric invariant, is not a topological invariant of the torus. The statement is that any hyperbolic geometric structure is unique, so that geometric invariants like volume, cusp shape, shortest geodesic, etc., are all topological invariants. That gives you a lot of topological invariants for knots and links.

### 3.1 Ideal triangulations and hyperbolic structures

You can see a geometry as the covering space of a manifold based on it; you could look at the fundamental group, and whether it injects into the isometry group. Instead take pieces of $\mathbb{H}^{3}$ and glue them together to get manifolds. The pieces we'll take are ideal tetrahedra, i.e., geodesic tetrahedra in $\mathbb{H}^{3}$ whose vertices are on the sphere at $\infty$.
[picture]
And, and, uh, the volume of a, so, the isometry classes of ideal tetrahedra are parameterized by the upper half plane. You have an ideal tetrahedra, and you can associate a complex number $z$ to an edge and get a tetrahedra. You'll get the same tetrahedron if you use the numbers $1-1 / z$ or $\frac{1}{1-z}$. Here $\operatorname{Im}(z)>0$. So this parametrizes by the upper half plane. Then $\operatorname{vol}(\Delta z)$ is a real valued function of $z$. It's $\operatorname{Im}\left(l i_{2}(z)\right)+\log |z| \arg (1-z)$ where $l i_{2}(z)=$ $\sum_{n=1}^{\infty} \frac{z^{n}}{n^{2}}$ is the dilogarithm.

Now start with $M=S^{3}-K$ written as a union of ideal tetrahedra $\Delta_{1} \cup \cdots \cup \Delta_{n}$. Here $\Delta_{i}=\Delta\left(z_{i}\right)$. Tetrahedra come together along an edge. Now the parameters around an edge have to multiply to one, that is, it has to "come back."

That gives you an equation in $z_{1}, \cdots, z_{n}$, which gives you an equation in every edge. These are called gluing equations. These are not enough to give you the hyperbolic structure, because of completeness. There are also two more equations for every cusp called the completeness equations. The number of edges is the same as the number of tetrahedra by an Euler characteristic argument. You solve these equations, and a solution with $z_{1}^{0}, \cdots, z_{n}^{0}$ gives you a hyperbolic structure. Then the volume of the manifold is the volume of the tetrahedra, $\operatorname{vol}(M)=\sum_{1}^{n} \operatorname{vol}\left(\Delta\left(z_{i}^{0}\right)\right)$.

For an example we will do the figure eight not. That is why I said, knot theory is the theory of the trefoil because it's the example you do. You can't do hyperbolic volume because the trefoil is a torus knot.

Think of two $S^{2}$ sitting on each side of this knot diagram. So we add in edges near each crossing, replacing the undercrossings, and then push the knot off to infinity. Then you identify the bigons and the corresponding edges and get a graph. So then you can see here a tetrahedron, and the other one is on the other side. When you collapse it you get the other tetrahedra. I convinced you that $S^{3}$ minus the figure eight is the union of two tetrahedra. You can solve the gluing and completeness equations with $z=w=\frac{1+i \sqrt{3}}{2}$.

Now you can compute hyperbolic structures for knots called SnapPea, which computes hyperbolic structures and hyperbolic invariants. It's written for Mac, so you can find it right now. You can triangulate it and find a hyperbolic structure. There is another program called Snap which computes arithmetic invariants which we will talk about on another day at another conference.

Let me talk about the volume conjecture, which I promised yesterday.

### 3.2 Volume Conjecture

You can get a family of polynomials called the colored Jones polynomials in terms of cablings of a knot, so let $K^{n}$ be the $n$ cabling of a knot which is zero framed. Then the colored Jones polynomial is the sum of the Jones polynomials of these cablings, $J_{N+1}(k, t)=(\sqrt{t}+$ $\left.\left.\frac{1}{\sqrt{t}}\right) \sum_{j=0}^{[ } N / 2\right] a_{j} V_{K^{N-2 j}}(t)$. This computationally difficult.

The volume conjecture of Kashaev and two Murakamis says

$$
\lim _{N \rightarrow \infty} \frac{\log \left|J_{N}\left(K, e^{\frac{2 \pi i}{N}}\right)\right|}{N}=\operatorname{Vol}(K) / 2 \pi
$$

[I think Joan wants to say some words.]
[I'd like to thank the organizers, who were excellent, and the speakers for wonderful lectures. Thank you all very much.]

