Birman Conference
March 15, 2005
Various speakers

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March 15, 2005

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## 1 Nate Broaddus (Braids and Links)

I'm glad to be here to honor my teacher Joan. Today we're going to be talking about braids, which are nice because they have a very intuitive picture. Think of them as strings that always move to the right. I can multiply two braids together (if they have the same number of strands).

This was the definition that Artin gave, which dates back to 1925. The group $B_n$ is the group of braids on $n$ strands.

Now, just pictorially, if I reflect a braid through the vertical axis at the end, you can see that each braid has an inverse, because you can flatten out to the identity in $B_n$.

An actual definition is given by looking at configuration space. The configuration space on $n$ points in $\mathbb{C}$ is $\mathscr{C}_{\hat{n}} = \{(z_1, \cdots, z_n) \in \mathbb{C} | z_i \neq z_j \text{ for } i \neq j\}$. So this is $\mathbb{C}^n - \binom{n}{2}$ hyperplanes given by $z_i = z_j$. So now the pure braid group is $\pi_1(\mathscr{C}_{\hat{n}})$.

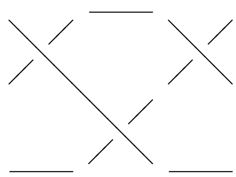
The configuration space of $n$ unordered points is $\mathscr{C}_n = \mathscr{C}_{\hat{n}}/\Sigma_n$, where $\Sigma_n$ is the symmetric group acting on the $\mathbb{C}_n$ by permuting coordinates [?]

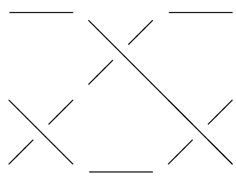
**Definition 1** $B_n = \pi_1(\mathscr{C}_n, *)$, *and often we choose the basepoint to have all the points on a line, but that's arbitrary.*

How do we get from this definition back to the picture we started off with? A loop inside this configuration space allows the points to trace out paths, and in the end we have to return to the picture with our points in a nice straight line, and the points trace out paths which never touch. This gives us a picture of a braid.

Artin gave, again, early on, in 1925, a presentation for the braid group, with generators $\sigma_i$ where $\sigma_i$ is going to be a braid where the $i$ strand crosses over the $i+1$ strand. So the generators will be $\sigma_1$ through $\sigma_{n-1}$.

There are two types of relations. There are commuting relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $i$ and $j$ are not adjacent, $|i-j| \geq 2$. There are also braid relations, $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.





There is a very important representation of the braid group onto the symmetric group $B_n \to \Sigma_n$ by only remembering an ordering on the starting and ending points. Another representation which can be thought of as a deformation of this representation is the Burau representation.

The braid group acts on the $n$-times punctured disk $D_n$ up to isotopy. Now we have the Hurewicz map from $\pi_1(D_n) \to H_1(D_n) \cong \mathbb{Z}_n$, and we can map this to the integers by sending $(a_1, \cdots, a_n) \to \sum a_i$. I'm going to call the map $\pi_1(D_n) \to \mathbb{Z}$ the winding number.

Now the kernel of this map $w$ corresponds to a cover $\tilde{D}_n$. Remember that $B_n$ acts on the disk. If I wrap around a puncture $n$ times, then after doing the isotopy the winding number isn't going to change, so $B_n$ fixes the kernel of $w$. Then $B_n$ acts on $\tilde{D}_n$. That tells us that it acts $H_1(\tilde{D}_n)$. Now, $\tilde{D}_n$ looks like a crazy parking garage. It has $\mathbb{Z}$ levels, and we'll slit it between the points. If I wind around $k$ points and cross a slit, I emerge $k$ levels up. This is a picture suggesting what $\tilde{D}_n$ should be.

The covering transformations are generated by shifting everything upwards once. That makes $H_1(\tilde{D}_n)$ a $\mathbb{Z}[t, t^{-1}]$-module. It turns out to be a free module of rank $n-1$. The $n-1$ comes from the fact that, you get rid of one, if you had one pucture this thing would correspond to a straight line. So if we fix generators for $H_1(\tilde{D}_n)$ as a $\mathbb{Z}[t,t^{-1}]$-module, we can get matrices for $B_n$, $(n-1)\times(n-1)$ matrices. Then those matrices are going to look almost exactly like the matrices for the representation of $B_n$ into $\Sigma_n$, where this is the set of permutation matrices. The only difference is that $\sigma_i$, instead of going to the permutation matrix $\begin{pmatrix} I & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I \end{pmatrix}$ it goes to the matrix $\begin{pmatrix} I & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & I \end{pmatrix}$. Actually, when you reduce the dimension by one, you get slightly different matrices, but I wanted to show that when $t=1$ you get $B_n \to \Sigma_n$, these representations agree.

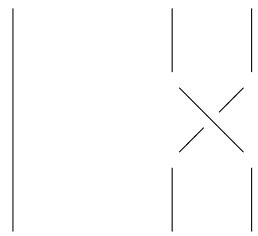
Now let's talk about the Lawrence-Kramer representation. We have $D_n$ again, and now $C = \{\{x,y\} | x \neq y \text{ and both are in } D_n\}$. Now $\Phi : \pi_1(C) \to \langle q \rangle_{\langle} t \rangle \cong \mathbb{Z}^2$. If we ignore punctures of $D_n$, then we get a 2-braid $\alpha = \sigma_i^b$.

If we include the puctures of $D_n$ then $\alpha$ gives a $n+2$-braid. Writing this as a word in $\sigma_1, \cdots, \sigma_{n+1}$ and summing exponents we get $b'$ where $\Phi(\alpha) = q^a t^b$, where $a = \frac{1}{2}(b+b')$. Acting on $D_n$ by the praid group will produce an action on $H_2(C)$, which is a $\mathbb{Z}[q,t,q^{-1},t^{-1}]$-module. This action gives the Lawrence-Kramer representation, and Bigelow showed, and also Kramer, that the representation is faithful, i.e., has trivial kernel. Bigelow also showed that for $n > 5$, it had been shown before him, but the Burau representation isn't faithful.

I want to move on to closed braids. Connecting the two ends of the braid with noncrossing arcs gives a link in $S^3$. We usually choose an $S^1 \subset S^3$ as the braid axis, and we have to continually move, in this case, counterclockwise around the braid axis.

So Markov came up with some nice simple moves, a simple set of moves which take any closed braid representing a link $L$ to any other braid representing $L$.

The moves allowed are braid isotopies where you sent the braid $b$ to $cbc^{-1}$, which doesn't change the knot type, and the other moves are stabilization and destabilization. Stabilization is going to increase the braid index:



Destabilization is the reverse.

For computational algorithms involving braids, increasing the braid index causes slowdown. So Joan Birman and Bill Manescu showed that given any link, there are a different set of moves, a larger set, we can take any link to one with minimal braid index by taking these moves and not increase the braid index along the way.

I'm running out of time. [He is out of time, over by two minutes.] Some very important combinatorial problems are the word problem and conjugacy problem. Artin solved the word problem. Garside solved it in a better way. He found a structure in the braid group that has been generalized quite a bit to a set of axioms.

The word problem is, given a word in the generators, decide if it is the trivial element. I think I'd better stop. The conjugacy problem has been solved but solutions are pretty slow. The word problem, the solution is very fast. The conjugacy problem is to find if two words are conjugate in the group.

[Are there any questions for Nate, who is from Cornell, I forgot to say. The next talk is in twenty minutes. Bathrooms are downstairs, up one flight, and up two flights.]

# 2 Dan Margalit (Mapping Class Groups)

[A few announcements before the next talk. In the back of the book is a list of a few restaurants and some bars. You should probably go to the restaurants first and not the bars. I'd like to introduce Dan Margalit.]

I want to get through a lot of stuff, today I'll introduce things and tomorrow get you ready for the conference. Here are some references. The first one is "Braids, Links, and Mapping Class Groups," it's not just a conference, it's a book that Joan wrote, mid 70s, Princeton Press. It's important. The second one is called "Mapping Class Groups," and is Nikolai Ivanov (at Michigan State). Look at his webpage. It's "A primer on mapping class groups," Farb and Margalit. Email me for a copy in draft form, margalit@math.utah.edu.

I'm going to define the mapping class group in five or six different ways. For today I'll think about oriented surfaces. Let $S$ be an orientable surface.

**Definition 2** $Mod(S) = \pi_0(Homeo^+(S))$, *path connected components of orientation- preserving homeomorphisms in the compact open topology.*
*This is* $Homeo^+(S)/isotopy$. *This is* $Homeo^+(S)/Homeo_0(S)$, *where these are isotopic to the identity.*

You should be able to see that these are all basically the same thing. Now I'm going to write down a theorem. I can write diffeomorphism instead of homeomorphism and homotopy or isotopy to get the same group. This is not true for some low exceptions.

This is a theorem of Baer. Why is this a group? $Homeo^+(S)$ is a topological group, and so $\pi_0$ is a group. Multiplication is composition up to isotopy class. You can multiply and things are nice.

I'm going to say in a second why you would want to think about this group, but first let me tell you about related groups.

$Mod^{\pm S} = \pi_0(Homeo(S))$, so the extended mapping class group includes orientation reversing homeomorphism. Or you can look at $Mod(S, \partial S) = \pi_0(Homeo^+(S), \partial S)$.

Why look at this?

- If you're studying three-manifolds, take a surface, cross it with the interval and glue the ends together; the type of the manifold depends only on the isotopy class of the gluing map.

  You can glue together two solid handlebodies along the boundary. This is called a Hegaard splitting.

- There are connections to Braid and Artin groups
- If you're interested in Teichm uller and moduli space. I'm not going to talk about this.
- This is related to arithmetic groups, $Out(F_n)$. This is not an arithmetic group.

At the end of the day you study a group because you think it's fun.

- $Mod(D^2) = Mod(S^2) = 1$. This is a non-example.
- $A$ is the cylinder. Now $Mod(A)$ is $\mathbb{Z}_2$. The relative mapping class group $Mod(A, \partial A)$ is more interesting. I can twist one or the other end through 360° and get something nontrivial.

  To get this back to the identity you'd need to pass through something not the identity on the boundary.

  This is easy to understand, and is in every surface. It's just $\mathbb{Z}$. This is called a Dehn twist. We'll see that the mapping class group is finitely generated by these. It's kind of the goal of the day.

- I'm only talking about orientable, but a cool example is that the twice punctured projective plane has mapping class group the dihedral group with 8 elements. This was proven by Mustafa right here.
- $Mod(T^2) = SL_2\mathbb{Z}$. This gets harder very quickly. If you want, as an exercise work out the pair of pants.

Let's do the torus. Define a map, there is a natural map $Mod(T^2) \to SL_2\mathbb{Z}$ where you think of this as kind of like $Aut(\mathbb{Z}^2)$. We'll call it $Aut^+(\mathbb{Z}^2)$, and secretly know what that means.

In higher genus surfaces, you don't get a nice action on the fundamental group, but here conjugation is trivial, which is why I can put $Aut$ and not $Out$.

So for surjectivity, you get $SL_2\mathbb{Z}$ action on $\mathbb{R}^2$ desceding to $T^2$. I should probably put here that this is a theorem of Alexander. What does injectivity mean? I have an element of the mapping class group of the torus, and this gives me a trivial action of $SL_2\mathbb{Z}$. We pick a representative that fixes a base point. The things are isotopic to where you started. First, you have to show that you have an ambient isotopy to straighten the first guy, and then you have an ambient isotopy that fixes the first guy and straightens the second guy. Then anything you're doing on the complement is just in $Mod(D^2)$, which is trivial. We need to show that if $f$ takes $(1,0)$ and $(0,1)$ to curves isotopic to them, then $f$ is isotopic to the identity.

More generally, the Alexander trick is, if $C$ is a collection of isotopy classes of curves and arcs on $S$ (an arc is an embedding of the interval with endpoints on the boundary) and $S\backslash C$ is a union of disks and once-punctured disks (with no bigons) [!?] and if $f(C) = C$ then $f = id \in Mod(S)$.

What are the generators? We kow the generators of $SL_2\mathbb{Z}$, they're $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. These are Dehn twists, and in general this is generated by Dehn twists. Think in your head about how what I'm defining gives you these matrices.

Let $C$ be an isotopy class of simple closed curves in $S$. Choose a representative of $C$ and take an annular neighborhood $A$. I can act by my generator of $Mod(A, \partial A)$ on that neighborhood and act identically everywhere else. Call this $T_C$. If I have another curve passing through, then I twist that curve around the annulus. We're going to see that these things generate the mapping class group.

Is the definition of a Dehn twist okay?

I have to tell a little story, last night I got to my hotel, which is very nice, but I had a little trouble because it was cold and I wanted to turn the heat up, and to figure out how to turn the heat on, there was a knob on the top. It had CLOSE and OPEN written on it, but both arrows pointed from OPEN. So it was hard to tell which was which, but I started thinking about lefty-loosy, righty-tighty. You might think that the orientation of $C$ would matter for $T_C$, but it depends only on the orientation of $S$. This is an important thing to understand.

By the way, the only thing weirder than that in my hotel room was the combination telephone/toilet paper dispenser. Anyway, this is an important thing to understand.

We'll show, seeing the future, that

**Theorem 1** *(Dehm, 1920s)*
*$Mod(S)$ is (finitely) generated by Dehn twists around nonseparating curves.*

One of the fun things in mapping class groups is looking at the relations between Dehn twists.

Well, I can start with a pretty easy relation, relation zero

0. $T_a = T_b$ is equivalent to $a = b$.

1. $i(a, b) = 0$ implies $T_a T_b = T_b T_a$, where $i$ is the geometric intersection number.
2. $i(a, b) = 1$ implies a neighborhood looks like a punctured torus, so $T_a T_b T_a = T_b T_a T_b$. If anyone here was at Nate's talk, this is the "braid relation." It's not a coincidence that this is showing up. If you put parentheses here and move this over you'll see that $T_a$ and $T_b$ are conjugate to one another by $T_a T_b$ so that $H_1(Mod(S))$ is a quotient of $\mathbb{Z}$.
3. The lantern relation relates three particular curves intersecting one another on a four times punctured sphere, $T_x T_y T_z = \prod T_{d_i}$ where the $d_i$ are the twists around the boundaries, and this implies that $H_1(Mod(S)) = 1$ for $g \geq 3$.

I'm going to leave as an exercise that the matrices represent the twists around the torus.

Okay, so we'll do a proof of finite generation after Birman.

[Oh, I didn't do that.]

Joan doesn't like her name up here. I'm using the Birman exact sequence.

[Dehn did it.]

But I'm using the Birman exact sequence. This is the proof of finite generation after, well, I read it in Joan's survey. I'll say following Lickorish. There are two main ingredients.

Ingredient #1 The Birman exact sequence says that if $S_{g,n}$ is a surface with genus $g$ and $n$ punctures, then there is an exact sequence. There's a map $Mod(S_{g,n+1}) \to Mod(S_{g,n})$ by forgetting a puncture. The sequence is $1 \to \pi_1(S_{g,n}) \to Mod(S_{g,n+1}) \to Mod(S_{g,n}) \to 1$.

Ingredient #2 Let $a$ and $b$ be any two isotopy classes of simple closed curves. Here's an amazing thing. I can find a sequence of nonseperating curves so that every one is disjoint from the next one, and I get from $a$ to $b$. So there exists $a = c_1, \ldots, c_k = b$ where these are isotopy classes of simple closed curves, such that $i(c_j, c_{j+1}) = 0$.

Let me give the sketch of how this all pieces together.

Embedded in here is Elia's questions about punctures, I just don't have time to get to it. Let $f \in Mod(S)$. I want to write it as a finite product of Dehn twists. Let $a$ be an isotopy class of nonseperating simple closed curves on $S$. Consider $f(a)$. Now $f$ takes isotopy classes of curves to isotopy classes of curves. Now construct the sequence from ingredient two. The idea, say, is let's say the sequence is very short, $a$ and $f(a)$ are disjoint. Then using the braid relation, I can connect these two with something intersecting each of them once. So I can take $f$ times a product of Dehn twists that fixes $a$. The idea is to apply induction plus ingredient number one. This isn't really a proof. You can piece it together and sit down for a much longer time and write down a proof.

[Any questions?]

[Relations zero and one are if and only if. What about the others?]

Two is, from $fT_af^{-1} = T_{f(a)}$ and $t(T_ab, b) = u(a, b)^2$. For three, that's something I did for my thesis, thanks for asking, and you can move backward to the picture like this.

[What about intersection number two?]

If two curves intersect more than once they generate a free group, that's an if and only if, that's what Joan is aiming toward.

[Here's a question, how come the kernel comes from Dehn twists?]

That's a great question. If I push a curve around this path, it's the product of these two Dehn twists.

There are many other relations, I left off two relations. There are some other cool relations about squares of Dehn twists.

## 3 Nancy Wrinkle (Contact Structures)

Can you hear me okay? Can you hear me okay?

Introduction to contact structures

Example: $\alpha = dz + xdy,\ \xi = \ker\alpha$

Idea: A plane field $\xi$ that twists somuch that ist is nowhere, not even in a neighborhood of a point, the tangent plane to a surface.

More precisely, a contact structure on a 3-manifold is a nowhere integrable plane field that can be described by $\ker\alpha$, where $\alpha$ is a 1-form and $\alpha \wedge d\alpha \neq 0$.

**Theorem 2** *Martinet, 1971*
*Every 3-manifold supports a contact structure.*

I heard some of you saying that contact structures are kind of like foliations, but this is more of an anti-foliation. With a foliation, I think $\alpha \wedge d\alpha = 0$. This is a nice handy way of visualizing. I have drawn a ray through the origin on the $x$ axis. This is invariant on the $y$ and $z$ direction. The kernel of the form $\alpha$ is $x = -\frac{dz}{dy}$. The coordinates $y$ and $z$ in this description are irrelevant. So the picture in this example totally describes it.

Now I'll come up with one with $r, \theta, z$, in cylindrical coordinates. So take this picture and rotate it, like a propeller. So $\alpha = dz + r^2 d\theta$. You can play the game; locally along a line this will look the same, but along a ray. It's rotationally invariant and is the same if you go up or down. The interesting thing is that these two contact structures are actually the same.

A nice theorem is this theorem of Martinet which says that every (closed oriented) manifold supports a contact structure. This means that you can fill the 3-manifold with such 2-plane fields and it's described by this one-form.

The sphere has a nice contact structure given by $dz + r^2 d\theta$ if you add the point at infinity so that your $z$ axis comes around.

**Theorem 3** *Darboux's theorem: Locally all contact structures look alike.*

The implication is that all the interesting stuff is of a global nature, i.e., is related to the global topology of the manifold supporting the structure. So we think slightly bigger.

Contact structures and Surfaces.

The characteristicd foliation of a surface $\Sigma \subset (M^3, \xi)$ denoted $\Sigma_\xi$ is, for all $x \in \Sigma$, look at $\xi_x \cap T_x\Sigma \triangleq \ell_x$.

Another thing we'll be talking about later is a smaller set of curves, recently developed, called dividing curves. These are places where the planes of $\xi$ are orthogonal to the surface. More details later. You want the plane field to be oriented so that you have a normal.

So this is where the normal of the plane is tangent to the surface.

He asked how quickly the things rotate. I don't have the words to talk about this yet. You're psychic, the next thing is the fundamental dichotomy of contact structures, overtwisted versus tight. If you have an embedded disk and the contact planes are tangent to its boundary.

This dichotomy is not obviously useful, but Eliashberg showed, I think, in 1989, that:

**Theorem 4** *Eliashberg*
*Isotopy classes of overtwisted contact structures on $M^3$ is in bijection with homotopy classes of oriented plane fields on $M^3$.*

So homotopy classes of plane fields are understood, I think it's a plane field. So we're interested in tight ones. So any time you hear a talk about plane fields it's about classifying tight ones or recognizing tight versus overtwisted.

I wanted to show what is known. What is classified? Which to we know about the contact structures on?

- $B^3, \mathbb{R}^3, S^3, S^2 \times S^1$ (Eliashberg)
- $T^3$ (Kanda)
- solid torus (Maken,Limarov)
- lens spaces (Etnyre, Giroux, Honda)
- torus bundles over $S^1$ (Giroux, Honda)
- circle bundles over $\Sigma$ (Giroux, Honda)

- $\Sigma \times I/\varphi$ [with restrictions on genus, Euler characteristic] (Hond, Hazez, Matic, Cofe)

What is open?

- Handlebodies (higher genus)
- $\Sigma \times I$ (higher genus, and without restrictions.
- Seifert fibered spaces as a category
- hyperbolic manifolds as a category
- all Haken manifolds as a category
- There are some manifolds with no tight contact structures; others have infinitely or finitely many; if the manifold is atoroidal then there are only finitely many tight contact structures.

What methods do we have?

- Symplectic methods
- Haken Decomposition methods
- Open book decompositions and foliations

I get agoraphobic so I stay in three dimensions, but an easy and common way to construct them symplectically is, if $\omega$ is a closed 2-form on $X^4$ with $\omega \wedge \omega \neq 0$.

**Definition 3** *A closed compact symplectic 4-manifold $(X, \omega)$ fills a contact manifold $(M, \xi)$ if*
*$\delta X = M$ as oriented manifolds*
*and $\omega|_M$ is an area form on $M$.*

**Theorem 5** *(Eliashberg-Gromov)*
*If $(M, \xi)$ can be filled by a contact symplectic manifold then $\xi$ is tight.*

Contact Haken Decompositions

Haken decomposition is a decomposition of $M$ into a disjoint union of 3-balls achieved by cutting along incompressible surfaces. The idea is that we understand these on $B^3$. We have som tools for understanding them in a neighborhood of a surface. Glue it back up and we can figure out the contact structure. The problem is what if we lose an overtwisted disk.

The partial solutions are gluing theorems, understanding isotopies of the cutting surfaces. First, how do we cut, how to analyze the cut-up pieces?

A vector field is contact if its flow preserves the contact structure. A surface is convex if there exists a contact vector field transverse to it. Equivalently, if you have a neighborhood of your surface homeomorphic to $\Sigma \times I$ such that $\xi$ on the neighborhood is invariant in the $I$ direction. Any closed surface is $C^\infty$ close to one; this is Giroux's perturbation lemma.

**Definition 4** *Dividing curves* $\Gamma_\Sigma = \{x \in \Sigma : v(x) \in \xi_x\}$. $\Gamma_\Sigma$ *is transverse to the characteristic foliation and they divide the surface into positive and negative regions.*

So the idea is another long slide, convex surfaces and tight contact structures

We have a convex surface and that's what we cut along.

Giroux's flexibility.

Given a convex surface $\Sigma$, it is the dividing curves that carry the essential information about the contact structure in a neighborhood of $\Sigma$, not the specific characteristic foliation.

(The corollary to) Legendrian Realization is:
Suppose a closed curve $C$ on a convex surface $\Sigma$ is transverse to $\Gamma_\Sigma$ and nontrivially intersects $\Gamma_S$. Then $C$ can be realized as a Legendrian curve, i.e., tangent to $\xi$ (Kanda)

The main main things are Eliashberg's uniqueness theorem: If $\xi$ is a contact structure in a neighborhood of $\delta B^3$ that makes $\delta B^3$ convex and the dividing set on $\delta B^3$ is a single closed curve, then there is a unique extension of the contact structure to a tight contact structure on $B^3$.

Giroux's criterion is $\Sigma \neq S^2$ is convex; then there exists a tight contact structure on a neighborhood of $\Sigma$ if and only if $\Gamma_\Sigma$ contains no homotopically trivial closed curves. ($\Sigma = S^2$ then there exists a unique contact structure if and only if your dividing number is one.

That's an idea of the tools people use. Now, gluing it all back together

**Theorem 6** *Gluing Theorem (Colim)*
*If:*
$(M, \xi)$ *is irreducible,* $\delta M$ *is nonempty and convex.*
$\Sigma \subset M$ *is a surface which is properly embedded, compact, convex, incompressible.*
$\delta\Sigma$ *is Legendrian, nonempty, and each component intersects* $\Gamma_{\delta M}$ *nontrivially.*
$\Gamma_\Sigma$ *is boundary parallel, so they cut off a half disk.*
$\xi$ *is universally tight on* $M\Sigma$, *meaning it persists along covers.*

*If all of these are satisfied, then the contact structure is universally tight on* $M$.

This is at the end of its usefulness. Well, I shouldn't say that, it needs to be extended. Tanya Kofer proved a generalization of this in her thesis.

Another gluing theorem:

**Theorem 7** *(Honda)*
*Suppose $(M, \xi)$ is overtwisted; take a convex decomposition of $M$. Look at all the nontrivial isotopies of the cutting surfaces. Eventually we can find a decomposition that does not cut through the overtwisted disk.*

This would be a very long check. Honda describes, in this theorem, a "minimal" isotopy, a bypass disk. You have a surface with a disk like a flap in a neighborhood of the surface or in the whole 3-manifold. This is a little like stabilization in terms of braids. The idea is that we can push the surface out past this bypass disk. Then what happens is that you get rid of some singularities in the foliation. Imagine I'm looking down at the bypass disk. Then you lift the surface over the bypass disk and this changes the structure of the dividing curves.

That's the idea of a Haken decomposition. Another 3-manifold idea about contact structures is open book decompositions.

An open book decomposition of $M^3$ is a pair $B, \pi$ where

- $B$ is an oriented link, called the binding, and
- $\pi : M \backslash B \to S^1$ is a fibration, and $\pi^{-1}(\theta)$ is the interior of $\Sigma_\theta$, a compact surface, and $\delta\Sigma_\theta = B$ for all $\theta \in S^1$. Then $\Sigma = \Sigma_\theta$ are pages.

An abstact open book is a pair $(\Sigma, \varphi)$ where $\Sigma$ is a (compact) surface with nonempty boundary and $\varphi : \Sigma \to \Sigma$ is a diffeomorphism such that $\varphi|_{(N(\delta\Sigma)} = id$ (monodromy)

Then Birman-Menasco did a huge work on braids in the context of the open book decomposition on $\mathbb{R}^3$.

So, for $S^3$, two solid tori, You can take the union of the disks and the annuli, and get two pages of the open book decomposition for $S^3$.

A theorem of Alexander, 1920, is that every closed oriented $M^3$ has an open book decomposition.

Here's a theorem of Giroux, 2000. Isotopy classes of contact structures on $M^3$ are in bijection with open book decompositions of $M^3$ up to stabilization.

What are the connections to contact structures? A contact structure $\xi$ is supported by an open book decomposition $(B, \pi)$ if there exists a contact 1-form $\alpha$ such that $d\alpha$ is an area form on the pages of the open book decomposition and $\alpha > 0$ on $B$. Equivalently and more visually, $\xi$ can be isotoped to be close on comact subsets of pages to tangent planes of the open book decomposition in such a way that the planes end up transverse to $B$ and the pages of an open book in a fixed neighborhood of $B$.

You push all of your twisting inside a given radius, and then outside it's close enough to the open book decomposition.

I'll start up tomorrow, talk more about this tomorrow.

Noah Goodman, last year, came up with a necessary and sufficient condition on an open book decomposition to support an overtwisted contact structure. That's the only result I know of so far that uses the classification. The relation between them have been really well-explored.

I'd like to give you a reference. All of this stuff is based on a survey article. There is no source, but if you go to John Etnyre's homepage at *UPenn* (for now), he's written a whole series of articles on contact structures. He's a geometer but he writes so that a topologist can understand.

[The next talk is at 3:30.]

# 4 Abhijit Champanerkar (Invariants of links)

Thank you for inviting me. My time at Columbia was very nice. Joan was always very inspiring and helpful.

Today I will talk about polynomial invariants, tomorrow I will talk about hyperbolic invariants.

If you have a knot or a link, which we will denote by $K$, you can always represent this as a closed braid. This is Alexander's theorem, that any knot or link in $S^3$ can be represented as a closed braid.

The next question is, how many ways are there to do this? The answer, again stated by Nate, is that $b_1$ and $b_2$, which represent the same knot, by which I mean isotopic, knot or link, are related by Markov moves.

What are Markov moves? The first one is conjugation, and the second is stabilization and destabilization.

So what we can do, if we have a representation of the braid group in $GL_n(\mathbb{C})$. Since trace is conjugation invariant, if a trace is invariant under Markov moves, then we can get knot and link invariants by using Markov traces, that is, those invariant under the second Markov move.

There were very few representations known before Jones. One was the Burau representation that Nate talked about. Jones generalized this to classify all representations of $B_n$. This is very historic with respect to this conference; the work was done at Columbia and Joan contributed. With Markov traces he defined the Jones polynomial and also explained the Alexander polynomial in terms of the new theory.

[He didn't classify representations, he just found a family of them.]

A huge family. I'm not going to talk about representations, I'm just going to talk about the two more famous polynomial invariants, the Alexander and Jones polynomials.

The Alexander polynomial $\Delta_k(t)$ was introduced by Alexander in 1928. It's kind of ideal because it is very well understood. Let me give you the topological definition.

1. Topological definition: Let $X = S^3 - K$. Then $\pi_1(X) \to H_1(X) \to \mathbb{Z} = \langle t | \rangle$ (last step needed for a link). Then $\tilde{X}$ is an infinite cyclic cover corresponding to the kernel of the above map. Then $\mathbb{Z}$ acts on $\tilde{X}$, and in particular on $H_1(\tilde{X})$.

   Take a Seifert surface for the knot complement and cut it along the surface. Then take a loop in the cut version, and $\mathbb{Z}$ just acts by translation. So $H_1(\tilde{X})$ is a $\mathbb{Z}[t^{\pm}]$-module, so is $\mathbb{Z}\Gamma$, the group ring of $\mathbb{Z} = \Gamma$. This is called the Alexander module of the knot.

   When you have a polynomial, you have a presentation matrix of $H_1(\tilde{X})$. Then the Alexander polynomial $\Delta_K(t)$ is the *gcd* of $m \times m$ minors of $M$, that is, the generator of the first elementary ideal.

   There's a number of definitions, like Seifert's with Seifert matrices. Seifert and Alexander finished most things in this in a few years.

2. Let me give you a definition using Fox differential calculus.

   If $\pi_1(X) = \{x_1, \cdots, x_n | y_1, \cdots, y_m\}$, then $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$ and $\frac{\partial uv}{\partial x_j} = \frac{\partial u}{\partial x_j} + u \frac{\partial v}{\partial x_j}$, which tells us $\frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij} x_j^{-1}$

   The Jacobian $A$ is $(\Phi(\frac{\partial \gamma_i}{\partial x_j}))$. This is a funny product rule, that's why it's called calculus. Here $\Phi : \pi_1(X) \to \mathbb{Z}$.

   **Theorem 8** *(Fox) $A$ is a presentation matrix for $H_1(\tilde{X})$ as a $\mathbb{Z}[t^{\pm 1}]$-module.*

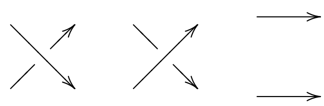
   For the trefoil, since knot theory is the study of one knot, the trefoil, well, until Thurston, then it was the study of two knots, the trefoil and the figure eight.
   So $\pi_1(X) = \langle x, y, z | yxy^{-1}z, xzx^{-1}y^{-1} \rangle$ and

   $$\frac{\partial \gamma_1}{\partial x}$$

   $$= \frac{\partial y}{\partial x} + y \frac{\partial}{\partial x}(xy^{-1}z^{-1}) = 0 + y(\frac{\partial x}{\partial x} + x \frac{\partial y^{-1}z^{-1}}{\partial x}) = y.$$

   Now $A = \Phi \begin{pmatrix} y & 1 - yxy^{-1} & -yxy^{-1}z^{-1} \\ 1 - xzx^{-1} & -xzx^{-1}y^{-1} & x \end{pmatrix}$, which, under the map taking $x, y, z$, to $t$ is $\begin{pmatrix} t & 1-t & -1 \\ 1-t & -1 & t \end{pmatrix}$ so that $\Delta_K(t) = -t - (1-t)^2 = t^2 - t - 1$.

3. **Theorem 9** *(Conway). If $L_+, L_-, L_0$ are locally*

   *then the conditions*

*(a)* $\Delta_0(t) = 1$

*(b)* $\Delta_{L_+} - \Delta_{L_-} = (t^{-1/2} - t^{1/2})\Delta_{L_0}$

*characterize the Conway normalized* $\Delta_L(t) \in \mathbb{Z}[t^{\pm 1/2}]$.

**Theorem 10** $\Delta_K(t) = \Delta_K(1/t)$;
$\Delta_K(t) = 1$

**Theorem 11** *Any polynomial which satisfies these two conditions is the Alexander polynomial of $K$ for some $K$.*

**Theorem 12** *There exist nontrivial knots with* $\Delta_K(t) = 1$, *for example the Kinoshita-Tenaka knot or the Whitehead double of any knot.*

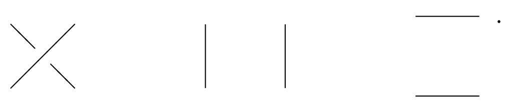
I think the Whitehead construction was probably known to Seifert. The main theorem is that the span of the Alexander polynomial gives a lower bound on the genus of the knot, that is, $span(\Delta_K(t)) \leq 2g$ where $g$, the genus of $K$, is the minimal genus of an orientable surface bounded by $K$. If $K$ is alternating then this is an equality.

Now let me go to the Jones polynomial $V_K(t)$, introduced in 1984.

I'm going to give you a very combinatorial description of this by the Kauffman bracket. It was in representations of the braid group that this originally arose.

If you have an unoriented link diagram $D$ of $K$ then $\langle D \rangle \in \mathbb{Z}[A^{\pm 1}]$ is defined as satisfying

1. $\langle O \rangle = 1$
2. $\langle D \sqcup O \rangle = (-A^2 - A^{-2})\langle D \rangle$
3. $\langle D_K \rangle = A\langle D_0 \rangle + A^{-1}\langle D_\infty \rangle$, where these differ locally as



This is not invariant under the first Reidemeister move, so you multiply by a factor depending on the writhe $w(D) = \sum_c w_c$, where $w_c$ is one or negative one according as the crossing $c$ looks like the respective of the following:

**Theorem 13** *(Kauffman)*
$V_L(A^{-4}) = (-A)^{-3w(D)}\langle D\rangle$.

It's not obvious if you define a polynomial thusly you don't know whether it exists. You can get around this by writing this polynomial in what is called the state sum, which we will use later. A state $s$ of a diagram is a smoothing (0 or $\infty$) of all the crossings. Let $\sigma(s) = $ #0-smoothings - #$\infty$-smoothings and $|s|$ be the number of loops in $s$. Then $\langle D\rangle = \sum_s A^{\sigma(s)}(-A^2 - A^{-2})^{|s|-1}$.

What are the main theorems about the Jones polynomial? Most of the questions answered about the Alexander polynomial are open for the Jones polynomial.

One of the main theorems is

**Theorem 14** *(Kauffman, Menasco,Thistlewaite) If $D$ is an alternating reduced diagram then span$(V_D(+))$ is equal to the crossing number of the diagram.*

The crossing number of a knot is defined as the minimal number of crossings over all diagrams.

Also, you can prove Tait's conjecture, that such a diagram, reduced and alternating, has minimal crossing number.

What about other questions, like those about the Alexander polynomial? How is $V_K(t)$ related to the topology of $S^3 - K$? No one has any clue. Evidence suggests that it is related to the geometry of the knot complement. What does it mean by geometry? I'll talk about it in my next talk, especially hyperbolic structures on knot complements.

The second question is, does there exist a nontrivial knot with trivial Jones polynomial, $V_K(t) = 1$? It's not even clear whether it exists or not. There's not even a conjecture one way or the other. There do exist nontrivial links with trivial Jones polynomial. This was a very recent theorem by Kauffman and someone, in 2000 or 2001.

So how do you compute these polynomials? Not by hand. Some of the programs are

- Knotscape (Hoste,Thistlewaite, Linux).
- Table of Knot Invariants (Livingston). This is a very nice webpage, where you tell it what you want and it prints it out.
- Knot Theory (Bar-Natan). This has an associated webpage.

In the last two minutes I'll tell you about categorification. There are bigraded homology groups which are invariants of a knot whose bigraded Euler characteristic are the Jones or Alexander [or HOMFLY or colored Jones] polynomial.

We have $\chi(M) = \sum_i (-1)^i rank\ H_i(M)$, where $H_i(M)$ is a stronger invariant than the ordinary Euler characteristic $m$.

Now $\sum a_i t^i = \sum(\sum_j (-1)^j rank\ H_{i,j}(D))t^i$.

So there exist such homologies with the right hand side equal to the Alexander and Jones polynomials; these are the Heegaard Floer Homology (Oszvath, Szabo, around 2000), and the Khovanov homology, around 1998.

I was going to give an idea of these but I'm out of time.

[We'll have breakfast again tomorrow at 9:00. You're free for the afternoon. Again, there are restaurants in your pamphlet. This was left out, but you may notice there's a nice sculpture just down the street at 117th. There's a sculpture down Broadway every block or two so that's a nice way to spend the afternoon.]