CGP INTENSIVE LECTURE SERIES
SI LI: BV QUANTIZATION AND GEOMETRIC APPLICATIONS

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The BV quantization comes from physics, basically, Batalin–Vilkovisky, and this is a general way to quantize a gauge theory. It unifies many different ways to quantize gauge theory in many different contexts. My plan is the following.

(1) I'll introduce BV algebras and toy geometric models, and I mean some finite dimensional models where you can do some, related to some very classical mathematics.
(2) Quantum field theory examples, related to infinite dimensional models, and I'll explain effective BV quantization.
(3) Finally I want to describe some examples and applications, for example
    (a) I'll describe the simplest case, quantum mechanical (one dimensional) models, I'll explain how to use this to prove Atiyah–Singer index type theorem, this is joint with Grady and Li.
    (b) I'll do a two dimensional chiral theory and get integrable systems. This part is recent, this is "joint with Xinyi Li," she is my daughter who is three months old now, she pushed me so hard to work this out during her birth.
    (c) B-twisted topological string field theory, and this part is joint with Kevin Costello.

Examples will be tomorrow.

Let me start with the basic things about BV algebras. Let me start with some definitions, of (differential) BV algebra.

I'll take this opportunity as well to set up notation. We'll be working with graded algebra $\mathcal{A}$, for simplicity $\mathbb{Z}$-graded, supercommutative, and I'll decompose this algebra

$$\mathcal{A} = \bigoplus_i \mathcal{A}_i$$

with $\deg \mathcal{A}_i = i$. For an element a in $\mathcal{A}$, I'll use $|a| = \deg(a) = i$, and supercommutativity means that $a \cdot b = (-1)^{|b||a|} b \cdot a$. This is the basic background.

The second data is the BV operator Δ, this is a degree one "second order" operator, nilpotent, so this is square zero. This means it behaves like a second order differential operator. In particular it's not a derivation, but you can measure the failure of this to be a derivation, and you get the so-called BV bracket.

$$\{a, b\} = \Delta(ab) - (\Delta a)b - (-1)^{|a|} a \Delta b$$

and this is basically the failure of Δ being a derivation.

Sometimes I'll include a differential Q in this story (so $Q^2 = 0$). This is a derivation of the algebra and is compatible with the operator in the sense that they anticommute: $Q\Delta + \Delta Q = 0$. This is degree $+1$.

I'll collect this $(\mathcal{A}, Q, \Delta)$. By the way, there is the generalization of this notion, Jae-Suk has some versions, where you consider more general order operations.

Typical situation. This will be our main situation, suppose I work with a graded vector space $V = \bigoplus_i V_i$, and I want to set up some notation. The notation $V[k]$, this is again a graded vector space, so $(V[k])_m = V_{m+k}$. I'll say $S^k(V)$ to mean $V^{\otimes k}/(a \otimes b - (-1)^{|b||a|} b \otimes a)$ and I'll use $\bigwedge^k(V) = V^{\otimes k}/(a \otimes b + (-1)^{|b||a|} b \otimes a)$, and I'll give a natural identification between these two vector spaces, $S^k(V[1]) = \bigwedge^k(V)[k]$. So even to odd shifts parity and commutation relations. Bosons become Fermions and so on.

A typical situation is the following. My algebra $\mathcal{A}$ is functions on V, I'll say $\mathcal{O}(V)$, and I'll do formal power series $\prod_k S^k(V^*)$. Second, and by the way, the dual is naturally a graded vector space and it inherits a grading as well.

I'll give the operator in terms of a BV kernel K in $S^2(V)$ of order 1. Then this kernel defines a second order differential operator ∂_K from $\mathcal{A}$ to $\mathcal{A}$, for example if you go from $S^m(V^*)$ it goes to $S^{m-2}(V^*)$.

For example, if you apply this to $\alpha_1, \ldots, \alpha_m$, you get

$$\partial_K(\alpha_1, \ldots, \alpha_m) = \sum_{i,j} \pm \langle k, \alpha_i \otimes \alpha_j \rangle \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_m$$

and $\Delta = \partial_K$ and you can check that $\Delta^2 = 0$. So that's a very typical situation.

In many geometric situations, this is a Poisson kernel of a shifted symplectic form. Suppose I have ω a symplectic structure on V of degree -1. I assume this is in $\wedge^2(V^*)$, this is like a 2-form of degree -1. When you play with symplectic geometry, you can get a Poisson kernel. This symplectic structure, for example, gives an identification $V^* \cong V[1]$, and under this natural identification, you see that $wedge^2(V^*)$ can be identified with $\wedge^2(V[1])$, and under our natural identification, this is exactly $S^2(V)[2]$. Under this identification, the symplectic form gives the Poisson kernel K.

In my examples, I should warn you, this Poisson kernel may be degenerate. So it doesn't come from a symplectic structure. The fundamental example comes from string field theory. I'll be more specific later on.

Finally, the Q. The Q usually comes from the following construction. Suppose I have a differential Q acting on my graded vector space V. This one induces a differential on all the associated vector spaces ($V^* \to V^*$, for instance), and so derivations on $S^\cdot(V)$ and $S^\cdot(V^*)$. The structure comes from the following data. If I assume my BV kernel K is Q-closed, for example, this means $(Q \otimes 1 + 1 \otimes Q)K = 0$. Under this condition, the Q will be compatible with the BV operator, and then we get our data $(\mathcal{A}, Q, \Delta)$.

That's our typical situation. In the geometric, symplectic case, the Q-closed condition is the same as saying that Q is compatible with the symplectic form, so $\omega(Qa, b) = -(-1)^{|a|}\omega(a, Qb)$.

Okay, that's our basic data. Suppose I have such data, what sort of games can I play? The first game that is interesting is the so-called *master equation*. This is the most important equation in this story. Suppose I give you this data (A, Q, Δ). Then a degree zero element $I \in \mathcal{A}[[\hbar]]$ (depending on a quantum parameter $\hbar$) is said to satisfy the *quantum master equation* if

$$(Q + \hbar\Delta)e^{I/\hbar} = 0$$

I'll use the shorthand QME for this equation. If you understand that the Δ can be applied either to one or another factor, you see that this is the same as saying

$$QI + \hbar\Delta I + \frac{1}{2}\{I, I\} = 0.$$

You can actually go back to the classical case, when $\hbar \mapsto 0$, and so $I = I_0 + \hbar I_1 + \hbar^2 I_2 + \cdots$, and the leading term satisfyies:

$$QI_0 + \frac{1}{2}\{I_0, I_0\} = 0.$$

In terms of gauge theory this has a very precise meaning as follows. In physics, people want to understand the integral

$$\int e^{I/\hbar}$$

in some sense, and this integration is over some cycle $\mathcal{L}$ in V, a super Lagrangian submanifold. In this super Lagrangian, this plays the role of a closed cycle, and this is related to a so-called gauge-fixing condition. The key point is that if you do this BV geometry, and suppose, let me just say that if I satisfies the QME, this implies that

$$\int_{\mathcal{L}} e^{I/\hbar}$$

is invariant under smooth deformation of $\mathcal{L}$. So this is just like de Rham theory. You want to integrate a form over a cycle. If you choose a closed form and a closed cycle, modifying the closed form the integral is invariant. So this is analagous.

In particular, if you think of $\mathcal{L}$ as gauge-fixing, then this means it's independent of gauge. So the quantum master equation is the quantum consistency for gauge theory.

This arises naturally then if you try to quantize gauge theory. What about the classical case? If you define the classical master equation as the leading term as $\hbar \to 0$, this implies that $Q + \{I_0, \quad\}$ is square zero, this is a flatness condition, this is a vector field, square zero, on my vector space. There are two meanings, this might have, I_0 might have many components, and this gives an L_∞ structure on your graded vector space $V[-1]$, and physically, this means it generates an infinitesimal gauge transformation. In particular, the meaning of this classical master equation is basically the same thing as the classical gauge symmetry, roughly. That's why it naturally arises when you look at a classical gauge theory.

We'll come back to physics. Some examples will be exactly physics models.

Now one thing about the master equation, the second game you play is the so-called observable theory. It arises in the following way. Suppose you want to integrate something $\int ue^{I/\hbar}$. You want some gauge invariance condition, like you wanted something to be closed to be gauge invariant. This is a natural, let me say gauge condition, I want this to be BV closed. So I want $(Q + \hbar\Delta)(ue^{I/\hbar}) = 0$. If you compute this, it's the same thing as saying that

$$Qu + \hbar\Delta u + \{I, u\} = 0$$

and this equation describes a gauge-invariant observable in this theory. This is the motivation for:

Definition 1.1.
$$\mathrm{Obs}^{\mathrm{q}} = H^*(\mathcal{A}[[\hbar]], Q + \hbar\Delta + \{I, \quad\})$$

In particular, if you want to understand $\langle u\rangle$, this is proportional to $\int ue^{I/\hbar}$ and this integral is well-defined on the cohomology.

Again, you can go to classical, $\hbar \to 0$ and define a classical observable as the limit of this guy, $\mathrm{Obs}^{\mathrm{cl}}$ and
$$\mathrm{Obs}^{\mathrm{cl}} = H^*(\mathcal{A}, Q + \{I_0, \quad\})$$
and you try to go back, that's quantization.

The first example is related to singularity theory, it's Landau–Ginzburg. This is a (finite-dimensional) toy model but it illustrates some basic concepts. This is related to Saito theory.

Assume I have a linear space $X = \mathbb{C}^n$, and suppose I have a polynomial $f : \mathbb{C}^n \to \mathbb{C}$, called a superpotential, and assume it has an isolated singularity at 0 (for simplicity), $\mathrm{Crit}(f) = \{0\}$. I'll describe from this what is V, and $\mathcal{A}$ and so on.

So V is basically the shifte cotangent bundle $T^*X[-1]$, and I'll write this $\mathbb{C}^n \oplus \mathbb{C}^n[-1]$, and this is a graded vector space, and $\mathcal{A} = \mathcal{O}(V)$, in terms of coordinates, this can be written as $\mathbb{C}[[x_i, \theta_i]]$, this could be analytic or polynomial. So x_i is a coordinate of $\mathbb{C}^n$ and θ_i is like ∂_i, my polyvector field, and in particular $\deg\theta_i = -1$. This space is basically called polyvectors, if you like. That's my $\mathcal{A}$. I want to tell you the BV operator. This will be a shifted symplectic structure, the symplectic form is the canonical one $\sum_i dx^i \wedge d\theta^i$, this is an odd symplectic structure, which leads to a natural BV kernel and the corresponding BV operator is
$$\Delta = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \theta^i}$$
sometimes people call this the "odd Laplacian" because it looks like a Laplacian. But it's not.

If we use a volume form $\Omega = dx^1\wedge\cdots\wedge dx^n$, we can identify $\mathcal{A}$ with some differential forms on X, $\Omega^{\cdot}(X)$, so $\theta_i \mapsto Idx^1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx^n$ and so on. Then Δ goes to the de Rham differential d.

You may wonder why we use this point of view if we could work with forms, and one answer is that this is different in the infinite dimensional case.

So note that $\Delta(e^{\frac{f}{\hbar}}) = 0$ naturally since f contains no θ, so we have a solution to the quantum master equation in a very stupid sense.

For example, you find the observables $\mathrm{Obs}^{\mathrm{cl}} = H^*(\mathcal{A}, \{f, \quad\})$, and the cohomology is the Jacobian ring of f, $\mathrm{Jac}(f)$ which is
$$\mathbb{C}[[x_i]]/(\partial_i f)$$
What about the quantum observables? By definition, it is
$$\mathrm{Obs}^{\mathrm{q}} = H^*(\mathcal{A}, \hbar\Delta + \{f, \quad\})$$
and you can work this out, this is
$$\mathbb{C}[[x_i]][[\hbar]]/(\forall g, (\partial_i f)g + \hbar\partial_i g \sim 0)$$
This is isomorphic to the (formal) Brieskan lattice.

By contracting with Ω this can be identified with
$$H^*(\Omega^{\cdot}[[\hbar]], \hbar d + df\wedge)$$

and this is

$$H^*(\Omega^\cdot[[\hbar]], e^{-f/\hbar} d e^{f/\hbar})$$

and what's happening, is suppose I have u a quantum observable, and I want to evaluate $\langle u\rangle$, so $d(e^{f/\hbar} u \lrcorner \Omega)$, and

$$\langle u\rangle \sim \int e^{f/\hbar} u\Omega$$

and that's conceptually what's happening, and you can get more about this quantization.

If you go to $\hbar \to 0$, you get the classical observables, which is basically the quotient $\mathrm{Obs}^q/\hbar\,\mathrm{Obs}^q$. You want a section, a quantization. On Obs^q you have a pairing, Saito's higher residue pairing, and you have a natural pairing on Obs^{cl} as well, the residue pairing, and the leading term of the higher residue pairing is the residue. This gives a natural inner product and you want a splitting respecting the higher residue pairing.

So suppose you want to do it this way, then such a quantization map is the same as the notion of a so-called "good section" (this notion due to K. Saito) and once you find a quantization map, this leads to a canonical unfolding of the singularity and somehow this leads to so-called "primitive form."

If you try to understand this higher residue pairing, you're in the following setting. You should integrate over some cycle Γ_i, and then we could say this is $\langle i, u\rangle$. So what's the higher residue pairing? It's like

$$\langle u|v\rangle = \sum_i \langle u|i\rangle \eta^{ij}\langle j, v\rangle$$

and each one of these depends on the choices of cycle but the sum does not depend. Here $\eta_{ij} = \langle i|j\rangle$. This is intrinsic.

Let's take a break.

A quick remark about the splitting, it's related to splitting of the Hodge filtration. In the isolated singularity case, there's a canonical splitting related to the splitting from the mixed Hodge structure. Let me move on to the quantum field theory case.

There's, in the previous discussion it was a toy model. If you move to quantum field theory you have an essential difficulty related to infinite dimensionality. Let me explain what's going on here. You had V a vector space, and it will be replaced by a space $\mathcal{E}$, say smooth sections of a (complex) vector bundle $\Gamma(X, E)$, this is very large, and let's see how much we can extend our discussion from before.

You need $\mathcal{A}$ which comes from the dual of this algebra. The dual means the dual with some topology. The most natural topology is given by distributions on E, linear functionals on $\mathcal{E}^*$. The third data is that we need a symmetric tensor product $S^m(V^*)$, there's a natural tensor product for these distributions, a completed tensor product with respect to the topology. What this means is that, work with an element, distributions on m copies of X on the bundle $E \boxtimes \cdots \boxtimes E$, okay, and what else do you want? A Poisson tensor or BV kernel from a symplectic form. So Q, our differential, this will be replaced with the differential of a complex, you have E be actually a complex of bundles, and you require that this is an elliptic complex and we require that Q behave like d or $\bar\partial$ or something like that. This gives a differential on smooth sections.

Finally you need a symplectic structure to define a BV operator. The natural thing is to start with a pairing on the bundle and then integrate to get something over the manifold. So start with $\langle , \rangle : E^\cdot \otimes E^\cdot \to \mathrm{Dens}(X)$, of degree -1. That's okay, we know what it means for vector bundles, and we want this such that $\omega(\alpha,\beta) = \int_X \langle\alpha,\beta\rangle$.

Next, the trouble is the following. We want the BV kernel, in the finite dimensional case you define $K = \omega^{-1}$, but in infinite dimensions, the inverse of integration, that's roughly a δ function. The first thing is that this is very singular. This is a distribution supported on the diagonal part of $X \times X$. Then you find trouble. Remember that $\Delta = \partial_K$ in the finite dimensional case, but ∂_K in the infinite dimensional case is ill-defined on $\mathcal{A}$. The reason is simple. If you look at the algebra, this is distributions, and this is given by pairing elements K with my distribution. The problem is that I cannot pair two distributions.

This is a famous difficulty in quantum field theory, called the UV-problem, and this is the problem with infinite dimensionality.

Okay, so the BV operator becomes trouble and we need to resolve this problem. The natural solution is by the so-called method of renormalization, and there are many different ways of doing this renormalization. Today I'll explain a homotopic method by Costello. This is very convenient for some mathematical reasons. I should mention there are many different theories of renormalization.

So what's the basic idea? Let's look at the situation? You cannot pair a distribution with a distribution. You can pair a distribution with smooth data. So homotopically, the basic idea, they are the same. If you look at H^* of the space of distributions with respect to Q, this is basically the same thing as H^* of smooth data with respect to Q, that's regularity of elliptic operators. So what can we do for this? We can start with K, the pairing respects the de Rham, integration by parts, this is compatible with Q. This means I can always replace the distribution with something I call the regularized kernel $K = K_r + QP_r$. Then K_r is smooth and P_r is a distribution. This is allowed. Now you say, it's smooth, we can define our BV operator using the smooth kernel, call it $\partial_{K_r} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$, this is well-defined.

Secondly, we have this operator, and you find that this is also compatible with Q, so $Q\Delta_r + \Delta_r Q = 0$. Now what's happening? Call this the *effective BV complex*, we have $(\mathcal{O}(\mathcal{E}), Q, \Delta_r)$, this is the effective or renormalized BV algebra. This is the first step.

What's happening if you choose different regularizations? Suppose we choose a different regularization, a smoothing $K = K_{\tilde r} + QP_{\tilde r}$, so let's look at the difference of these two kernels

$$K_{\tilde r} - K_r = Q(P_r^{\tilde r})$$

where $P_r^{\tilde r} = P_r - P_{\tilde r}$. So the difference on the left is smooth, so by ellipticity, the difference $P_r^{\tilde r}$ is smooth, so this is an equation in the smooth category, the distributions cancel with each other. The exercise is then that the two BV kernels are related by a homotopy, which can be made explicit,

Lemma 1.1.
$$(Q + \hbar\Delta_{\tilde r}) = e^{-\hbar\partial_{P_r^{\tilde r}}}(Q + \hbar\Delta r)e^{\hbar\partial_{P_r^{\tilde r}}}.$$

If you work with all regularizations, you have many choices, but if you work with a special type of regularization, you can have a unique choice of P_r.

So in particular, another way of saying this is you have a complex, and this is related to a parametrix. So you have a map

$$(\mathcal{O}(\mathcal{E})[[\hbar]], Q + \hbar\Delta_{r_1}) \xrightarrow{\hbar\partial_{P_{r_1}^{r_2}}} (\mathcal{O}(\mathcal{E})[[\hbar]], Q + \hbar\Delta_{r_2})$$

and this diagram commutes up to homotopy.

$$\begin{tikzcd}
(\mathcal{O}(\mathcal{E})[[\hbar]], Q+\hbar\Delta_{r_1}) \arrow[rr] \arrow[dr] & & (\mathcal{O}(\mathcal{E})[[\hbar]], Q+\hbar\Delta_{r_2}) \arrow[dl] \\
& (\mathcal{O}(\mathcal{E})[[\hbar]], Q+\hbar\Delta_{r_3}) &
\end{tikzcd}$$

Definition 1.2. An *effective solution to the quantum master equation* is given by a family $I^r \in \mathcal{O}(\mathcal{E})[[\hbar]]$ for each P_r such that

- (effective quantum master equation)
$$(Q+\hbar\Delta_r)e^{I^r/\hbar} = 0$$
- (homotopic renormalization group flow)
$$e^{I^{r_2}/\hbar} = e^{\hbar\partial_{P_{r_1}^{r_2}}} e^{I^{r_1}/\hbar}$$

The second thing is about homotopic renormalization group flow, conceptually another way to write this is that I^{r_2} is a sum over Γ connected graphs of putting I^{r_1} on each vertex and the propagator $P_{r_1}^{r_2}$ on each edge.

This is the basics, and the important thing here is that, another issue in quantum field theory is locality and quantization.

I've just told you a formalism for doing this, to at least write down a quantum master equation in a reasonable way. But this should come from a local action functional classically, and you want this to be reflected in a subtle way on the quantum level.

In classical physics, here's the problem, it's described by local interactions, let me emphasize this a little bit. Let me describe a subset of $\mathcal{O}(\mathcal{E})[[\hbar]]$, and I'll specialize it to a special description $\mathcal{O}_{\text{loc}}(\mathcal{E})$, local distributions, local functionals. First of all this is a distribution, throw in smooth data and you get a number, but it's given by integrating with a Lagrangian density $\int_X \mathcal{L}$, and $\mathcal{L}$ is a map from the jet bundle to the density bundle $\mathcal{L} : S^{\cdot}(\text{Jet}(E)) \to \text{Dens}(X)$. It's local in the sense that if I have two different inputs whose support is disjoint I should get zero.

So $K_0 = K$, the delta function, and Δ_0 is not well-defined, this is bad on $\mathcal{O}(\mathcal{E})$. The bracket $\{,\}_0$ is also not defined because of the singularity. But $\{,\}_0$ is defined on the smaller space $\mathcal{O}_{\text{loc}}(\mathcal{E})$. The reason is simple. Suppose you want to throw into the δ function some local stuff, then it cancels one of the integrations and makes this still local data.

So in particular, we have a natural BV bracket on local functionals, let's call it the BV bracket
$$\{,\}_0 : \mathcal{O}_{\text{loc}}(\mathcal{E}) \otimes \mathcal{O}_{\text{loc}}(\mathcal{E}) \to \mathcal{O}_{\text{loc}}(\mathcal{E})$$
and if we want we can write
$$\{I_1, I_2\}_0 = \lim_{r\to 0}\{I_1, I_2\}_r$$
if I_i is local.

Definition 1.3. A local functional I_0 is said to satisfy the classical master equation if the following equation is satisfied:
$$QI_0 + \frac{1}{2}\{I_0, I_0\}_0 = 0.$$

Another way of saying this is that $Q + \{I_0, \quad\}$ defines a *local* L_∞ algebra, the higher brackets are local expressions. That's the classical story. This is actually the stuff you always have if you start with a classical action functional with a gauge symmetry and so on.

Now the question is, the local functional is usually our starting point. How can I start from here and get a quantization?

$$I_0 \to \text{QME?}$$

The first condition you want to make is that, I_0 is removing the regularizations,

Definition 1.4. A family $\{I^r\}$ defines a quantization of I_0 if

$$\lim_{r\to 0} I_0^r = I_0$$

Remember that every functional I^r has an $\hbar$ expansion $I^r = I_0^r + \hbar I_1^r + \cdots$

This is somehow related to the fact that I_0^r can be written as the sum over trees Γ, with I_0 at vertices and P^r on the edges. This is well-defined at the tree level. This is the condition, that the leading term is obtained using tree diagrams like this.

How do you obtain a quantization in general? You might expect naively that

$$e^{I^r/\hbar} \overset{?}{=} e^{\hbar\partial_{P^r}} e^{I_0/\hbar}$$

but this has trouble, many problems,

(1) *UV* divergence because it doesn't work already for sums over graphs that aren't trees.
(2) the quantum master equation, it's not clear how this is satisfied.

Anyway the general step is the following, say the steps for quantization.

(1) Solve the homotopic renormalization group flow, using the counterterm method. The trick is that you can find ϵ-dependent counterterms I^ϵ in $\hbar\mathcal{O}_{\mathrm{loc}}(\mathcal{E})[[\hbar]]$, called *counter term*, such that $\lim_{\epsilon\to 0} e^{\hbar P_\epsilon^r} e^{(I_0+I^\epsilon)/\hbar}$ exists. You do a cutoff and extract the singularities, and add back singular terms to get something convergent. This is the naive quantization, the first step, $e^{I^r_{\text{Naive}}/\hbar}$, and this already satisfies homotopic renormalization group flow. This is always possible.
(2) This is the hard part, find further quantum corrections to I^ϵ, the choice of I^ϵ is not unique, and you want to find a good choice such that $e^{I^r/\hbar}$ satisfies the quantum master equation. This step is *not* always doable. In this step you find so-called anomalies, in math sometimes called obstructions. Suppose you're working with an anomaly-free theory. Then you get a solution.
(3) Study the quantum observables, again defined in an effective way. Remember that I define this homology in a compatible way. If I think of the observables as a deformation of the quantum master equation, then this is natural.

Next time I'll talk about some real examples.

2. April 28

Last time we discussed effective BV quantization. Let me briefly remind you what's happening. Basically we're discussing some data usually given by some section on a vector bundle $\mathcal{E} = \Gamma(X, E^\cdot)$ along with Q and a symplectic structure

ω, and we try to write down a BV algebra in analogy to the finite dimensional case and we run into some trouble, and what happens, we have a singular BV kernel $K = \omega^{-1}$. But you can find one which is homotopic to this one and regular, $K_r + QP_r$, and for each such choice of P_r we define a regularized BV kernel and get Δ_{P_r}, ∂_{K_r} acting on $\mathcal{O}(\mathcal{E})$.

Then we consider the BV quantum master equation, a solution $(Q+\hbar\Delta_{P_r})e^{I^r/h} = 0$ compatible with homotopical renormalization group flow.

So that's a summary.

Today we're going to do examples and examples. The first one is the simplest case, suppose X is a point. Then what you find is you have a vector space, you get back to the space we discussed last time, the toy model. That's not the one we want to discuss. We really want an infinite dimensional model. This is the simplest non-trivial example, this is $X = S^1$, related to quantum mechanical models. I'll call this topological quantum mechanics, which come from a sigma model. We're describing this in some kind of low energy locus, this is a low energy effective theory. This comes from the way we describe the model. We try to understand what's happening for the sigma model, so we have $S^1 \to (M, \omega)$, a symplectic manifold. Using this data, I'll write down $\mathcal{E}$ and K and this stuff in a consistent way.

First of all I need some notation. Let me denote by V_M a vector space, given by all differential forms on M valued in the tangent bundle, $\mathcal{A}^*(M, TM)$. Our space of fields $\mathcal{E}$ is all differential forms on the circle, valued in this vector space

$$\mathcal{E} = \mathcal{A}^*(S^1, V_m) = \bigwedge^{*}(S^1) \otimes_R V_M.$$

Here $Q = d_{S^1}$

Now actually we do something special, our base ring is, we assumed last time that the base ring was $\mathbb{R}$ or $\mathbb{Z}$ but now it's over $\mathcal{A}^*(M)$. My dual $\mathcal{E}^*$ is $\mathrm{Hom}_{\mathcal{A}^*M}(\mathcal{E}, \mathcal{A}^*(M))$. So this is a little different.

Now what's my local pairing? My local pairing $(\quad,\quad)$ is a pairing $\mathcal{E} \otimes_{\mathcal{A}^*(M)} \mathcal{E} \to \mathcal{A}^*(M)$, given as follows, say I have $\alpha \otimes \beta \mapsto \int_{S^1} \omega(\alpha, \beta)$, this is local in S^1.

Now there's a little bit more, we want to describe an action functional for this one solving the master equation. How to find a solution to the quantum master equation? It turns out there is a geometrical way to describe this. Before moving to this, I want to describe this model a little bit.

Remark 2.1. If you look at the "Bosonic" part of the data, the "Bosonic" fields, the leading term is given by smooth functions on the circle tensored with something like the smooth vector fields $C^\infty(S^1) \otimes T_M$. So this has the following interpretation. Look at maps from the circle to the target. Usually in physics we consider a sigma model, that is, all the maps. But look at a locus where the circle is taken to a point, and look at a local deformation of this consntant map. Every point on S^1 gets a tangent vector in M. So $\mathcal{E}$ describes a formal neighborhood of constan tmaps in $\mathrm{Maps}(S^1, M)$.

What is the quantum master equation and the geometry? This has a complete solution in the following sense. First let me introduce a Weyl bundle on my symplectic manifold. This bundle is given by formal power series of T^*M, so $\widehat{S^{\cdot}T^*M}[[\hbar]]$, and $\omega|_{T_pM}$ is linear, which we can use to define a fiberwise Moyal star product $*$. If you haven't seen this before, this is a natural associative algebra which gives a natural deformation quantization of the natural bracket along each fiber. This gives a bundle of algebras $W \otimes W \to W$.

Okay, secondly, let me choose a connection ∇, a sypmlectic connection on T_M. Symplectic means that $\nabla(\omega) = 0$ and let me say also torsion free. This connection makes a natural connection on associated bundles, such as the Weyl bundle, and I'll use the same notation. Since the manifold is highly nonlinear, this is almost never flat. But the remarkable thing is that if you work with the Weyl bundle you can make it flat.

Theorem 2.1. *(Fedosov) There exists a one-form $\Gamma \in \mathcal{A}^1(M, W)$, valued in the Weyl algebra, such that $\nabla + \frac{1}{\hbar}[\Gamma, \quad]_*$ is flat.*

This is one remarkable thing.

It turns out that this model, what a solution looks like is the following. What's the relation between the Weyl bundle and this data? Say I have a map Φ? I get an element, starting from any differential form in $\mathcal{A}^{\cdot}(M, W)$, I can get a local functional on my fields $\mathcal{O}_{\text{loc}}(\mathcal{E})$.

So say you have $\gamma \in \mathcal{A}^{\cdot}(M, S^*(T_M))$. Then what is $\Phi_\gamma(\alpha)$? It's

$$\frac{1}{k!}\int_{S^1}\gamma(\underbrace{\alpha, \dots, \alpha}_{k})$$

here $\alpha \in \mathcal{E}$.

That defines a local functional on the circle, and you start to see that this simplest example is already nontrivial. For example, well, first let me describe the action functional I^Γ, where Γ is a solution to the Fedosov flatness thing. Then

$$I^\Gamma(\alpha) = \frac{1}{[unintelligible]}\int_{S^1}\omega(\alpha, \nabla\alpha) + \Phi_\Gamma(\alpha)$$

Theorem 2.2. *(Grady–Li–L.)*

1. $E^{I^\Gamma[r]/\hbar} = \lim_{\epsilon\to 0} e^{\hbar\partial_{P_r}} e^{I^r/\hbar}$, *you usually need to take into account the counterterms, but here you don't need to, this limit exists (UV-finite)*
2. *There are a lot of $\hbar$ from Γ, but $\{I^\Gamma[r]\}$ is a solution to the effective quantum master equation. This gives a complete solution in this example. The quantum corrections can be described explicitly, recursively, there's an algorithm for doing this.*

Secondly, as we know from last time, once we have this kind of data, what is the situation for the observables?

This is given by distributions. I'll describe a special kind of observable, local on the circle near a point. I choose a small open interval U on S^1, and let's denote $\text{Obs}[U]$ for observables with support lying inside U.

You can compute the corresponding cohomology, and if you compute it you find the following data

Theorem 2.3. *(Grady, Li, L.)*

1. $\text{Obs}^{\text{cl}}[U] = C^\infty(M)$
2. $\text{Obs}^q[U] = \{\nabla + \frac{1}{\hbar} -$ *flat sections of* $W\}$ *which can be identified naturally with* $C^\infty(M)[[\hbar]]$.

There's a fiberwise product preserved because these are flat sections, and so the fiberwise Moyal product on the quantum observables, this makes a factorization algebra. Let me say in pictures. [pictures].

Another way to say it is that the quantum observables give a deformation quantization for the classical ones.

(3) *There's a natural notion of BV integration, and that gives a trace map on the quantum observables, and you can compute it,* $\mathrm{Obs}^q \to \mathbb{R}((\hbar))$*. So* $Tr(\alpha\star\beta) = Tr(\beta\star\alpha)$ *and if you compute this,* $Tr(f) = \frac{[unintelligible]}{\hbar^n}\int \frac{\omega^n}{n!}(f + O(\hbar))$ *so the leading term is the volume. If you compute, you find*

$$Tr(1) = \int_M e^{-\omega_\hbar/n}\hat{A}(M)$$

and this is, $\omega_\hbar = \omega + \hbar\omega_1 + \hbar^2\omega_2 + \cdots$ *and this is an algebraic index theorem.*

This comes naturally from BV integration.

That's the first example I want to describe. You see that something nontrivial happens. Now the second example I want to describe is related to string field theory. It's my favorite example. This is B-twisted topological string field theory. So there's a very general notion developed, for example, by Zwiebach, saying that a closed string field theory is basically described by this BV structure. This is basically his theory. We'll make a concrete description of this theory in the topological sector.

This example was developed with Kevin Costello.

It's something like this, it's related to the [unintelligible]model I described yesterday. The starting point is a Calabi–Yau manifold X. I'll fix a choice of holomorphic volume form Ω_X. I need to describe the space of fields, the Q, the BV kernel, and so on. I'll work with polyvector fields of X, that's $\mathcal{A}^{0,*}(X, \wedge * T_X^{1,0})$, and this can be identified with $A^{*,*}$ by contracting with the holomorphic volume form. You have two operators $\bar{\partial}$ and ∂ on $\mathcal{A}^{*,*}$. We have the two operators, then, $\bar{\partial}$ and ∂ on the polyvector fields.

The space of fields is $PV(X)[[t]][2]$, this variable is related to [unintelligible]and so we shift by 2. We won't discuss the BV operator in the t direction today. So $Q = \bar{\partial} + t\partial$. In this example, the BV kernel is degenerate. You can describe the Poisson kernel. The trace map $\mathrm{Tr} : PV(X) \to \mathbb{C}$ takes μ to $\int_X (\mu \vdash \Omega_X)\Omega_X$. This map is compatible with ∂ and $\bar{\partial}$. Let me describe a pairing $\eta : PV(X) \otimes PV(X) \to \mathbb{C}$, which takes $\eta(\alpha, \beta) = \mathrm{Tr}(\alpha \wedge \beta)$. I'll call the inverse η^{-1}, this looks like a δ-function, it's actually a distribution section, $S^{\cdot}(\mathrm{Dist}(PV(X)))$.

I will describe to you next what the BV kernel is in this example. The singular BV kernel, $K = K_0$ is $(\partial \otimes 1)\eta^{-1}$. This is an element of $\mathrm{Dist}(PV(X) \otimes PV(X))$; in particular there is no t here. This only uses the leading term. All the other stuff, there's no pairing for it. In particular, the Poisson or BV kernel is degenerate.

It doesn't come from a natural shifted symplectic form. But the kernel is well-defined.

Again, you can do regularization, one way to regularize is to use the heat kernel, let's denote this, I want the Laplacian, I'll call it $\square$ because I want to reserve Δ, this is $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, and then $\delta = \eta^{-1} = \lim_{L\to 0} e^{-L\square}$.

For example, what you find is that K_0, the singular kernel, can be regularized by $K_L + QP^L$, with $K_L = (\partial \otimes 1)e^{-L\square}$ and $P^L = \int_0^L (\bar{\partial}^* \otimes 1)e^{-L\square} dL$, so that's the renormalization. We then have the BV algebra

$$\mathcal{O}(\mathcal{E})[[\hbar]], Q + \hbar\Delta_L$$

with $\Delta_L = \partial_{K_L}$. You see that this is a different $\hbar$ than the previous one, there are two directions to quantize.

Now I need to describe to you the solution to the quantum master equation, local interaction and the limits and so on. Now, classical theory. This involves local interaction given by the following. Let I_0 be a local functional on $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ as follows

$$I_0(\mu) = \mathrm{Tr}(\langle e^{\mu}\rangle)$$

where the notation

$$\langle t^{k_1\mu_1,\ldots,t^{k_m,\mu_m}} = \binom{m-3}{k_1,\ldots k_m}\mu_1 \wedge \cdots \mu_m$$

The number $\binom{m-3}{k_1,\ldots,k_m}$ is basically the ψ class

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi^{k_1}\cdots\psi^{k_m}$$

So you can understand this, look at the B model, how it localizes to constant maps.

Remark 2.2. This formula was known to Losev, something about Hodge field theory in some toy model. He figured out the meaning of this in a very general algebraic sense.

So

Theorem 2.4. *I_0 satisfies the classical master equation, in this case*

$$QI_0 + \frac{1}{\hbar}\{I_0, I_0\}_0 = 0$$

where the bracket uses K_0 and is well-defined without regularization.

Then you try to quantize using the effective renormalization method. You get higher genus B-model. You can do this as we described yesterday. I want to describe this to you in a purely geometric way.

I have the viewpoint from the classical method of canonical quantization. What's happening here, let's call $S(X)$ the Laurent series $PV(X)((t))$ with differential $Q = \bar{\partial} + t\partial$, and secondly, let me call $S_+(X)$ the space of fields $\mathcal{E}$, which is $PW(X)[[t]]$ with the same differential Q. Finally, I'll denote $S_- = t^{-1}PV[[t^{-1}]]$, and Q does not act on this, and the failure of this one will give us [unintelligible].

I also denote the corresponding cohomology, I'll call H the cohomology of S and H_+ the cohomology of S_+. All of these are vector spaces, I'll denote the symplectic structure ω as follows. So $\omega(f(t))\alpha, g(t)\beta)$ is

$$\mathrm{Res}_{[unintelligible]}\, dt(f(t)g(-t)\,\mathrm{Tr}(\alpha \wedge \beta))$$

where the internal part is basically Saito's higher residue pairing in the Calabi–Yau case.

You can check, this is a symplectic pairing, also compatible with Q. Now, whenever you see a vector space and a sypmlectic structure, you can study the canonical quantization. Let me give you the following picture, I have this diagram, [picture]

You can see that S_+ and S_- are somehow dual under this pairing, so I can somehow identify S as T^*S_+, and then identify two interesting Lagrangians. $\mathcal{E}$ is a map S_+ to the space, $\mu \mapsto t(e^{\frac{\mu}{t}} - 1)$.

This is just a map from $S+$ to this guy in the formal sense. We write this in this way because [unintelligible], the tangent map is identified with S_+ here. If you project this to S_+, you get a formal one to one correspondence to S_+.

This symplectic stuff, you think it must be related to a Lagrangian, and it's true.

Remark 2.3.
(1) The image of this period map $\mathcal{P}$ is a (formal) Lagrangian. It's the graph of dI_0, where I_0 is the local function on $\mathcal{E} = S_+$ via the classical theory.
(2) What is the meaning of the master equation? View Q as a linear operator on S, so viewed as a vector field on S, you can check that Q is tangent to this Lagrangian. This is a gauge symmetry, and this property is equivalent to the classical master equation for I_0. This is a way to show that this satisfies the classical master equation withotu a messy calculation.
(3) So you have a homotopy Lie algebra on S_+, that's the pushforward of the vector field,
$$\pi_{+*}(Q) = Q + \{I_0, \quad \}.$$

Now I want to discuss the quantization. First we have the Weyl space, we can do Weyl quantization naively. I describe a version of this in the toy model. Suppose I naively define W as
$$\prod_n (S_+^*)^{\otimes n}[[\hbar]]/\alpha \otimes \beta - \beta \otimes \alpha \sim t\langle \omega^{-1}, \alpha \otimes \beta \rangle$$
but here this is not well defined, you can't pair two distributions. This is essentially the same problem that happened yesterday.

We'll use the same method to regularize the Weyl algebra. We know that $\omega^{-1} = \delta \sum_k (-t)^k \otimes t^{-k-1}$. Then we can regularize
$$\omega_L^{-1} = e^{-L\Box} \sum_k (-t^k) \otimes t^{-k-1}.$$
Then $\omega^{-1} = \omega_L^{-1} + Q - \text{exact}$. This is not totally obvious.

Then I can define my effective Weyl algebra, analagously to what I did for the BV algebra. This is
$$W_L(S) = \prod_n (S^*)^{\otimes n}[[\hbar]]/\alpha \otimes \beta - \beta \otimes \alpha \sim \hbar\langle \omega_L^{-1}, \alpha \otimes \beta \rangle.$$
So then I can make a regularized Fock space
$$\mathrm{Fock}_L(S_+) = W(S)/W_L(S)\,\mathrm{Ann}(S_+)$$
modding out by the annihilator. Then Q applies to this space. How to describe the Fock space? One way is to use the creation operators. Using the splitting, $S = S_+ \oplus S_-$, you get a way to embed S_+^* on S^*, extension by zero, and this means the following. You can check, this is an isotropic subspace, it commutes in the Weyl algebra, $\mathcal{O}(S_+)$ maps to $\mathrm{Fock}_L(S_+)$ and this turns out to be an isomorphism. Then there's Q on the Fock space. So then Q induces a derivation on $\mathcal{O}(S_+)$, you might want to call it Q, but this is not the same as Q_L, because Q does not respect the splitting of S. You can check the discrepancy, and it's your homework, that $Q_L = Q + \hbar\Delta_L$, so this measures the failure of the splitting to preserve Q. This should remind you of a lot of stuff from Hodge theory.

Now what's a solution of the quantum master equation? I'ts a solution of
$$(Q + \hbar\Delta_L)e^{I[L]/\hbar} = 0$$
and you can think of $E^{I[L]/\hbar}$ giving like $|I\rangle$ in Fock_L, and $Q|I\rangle = 0$. This is a gauge invariant element in the Fock space. Then you can descend to cohomology, and find generating functions of higher genus.

The geometic picture is the following. From this point of view, another way of saying this, classically what's happening, you have S and a Lagrangian subspace $\mathcal{P}(\mathcal{E})$, and this is classical data. You do a noncommutative deformation of this and you get W and there you get Δ_L. This noncommutative Lagrangian is something like $|I\rangle$. So [pictures] and that's what's happening.

One last remark.

I want to say this can be computed. The simplest example is when X is an elliptic curve, a one-dimensional Calabi–Yau. You have a classical interaction I_0 and the quantization is given by a chiral deformation. If you quantize this this way and consider the generating functions, you get $g > 0$ mirror symmetry, and the chiral deformation can be computed precisely by $W_{1+\infty}$-algebra.

This completes the picture. Let me stop here.