PACIFIC RIM COMPLEX–SYMPLECTIC GEOMETRY CONFERENCE

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[Note: I do not take notes on slide talks]

1. July 31: Tomoyuki Hisamoto: Stability of a polarized manifold and coercivity of the K-energy functional

I want to start with some background for the so-called Yau–Tian–Donaldson conjecture. This is some conjecture for a general smooth projective manifold X over the complex numbers endowed with a [unintelligible]line bundle, (X, L), this is sometimes called a polarized complex manifold. We're interested in the existence of some standard, $\omega \in c_1(L)$, a Kähler metric.

We can take the curvature of a fiber metric $e^{-\varphi}$ and get the Kähler metric and in the other direction we can do something to get the fiber metric up to a constant. Let me explain a little more. For $s \in H^0(X, L)$, we get $|s|^2 e^{-\varphi}$, and if the line bundle is tensorized k times, then we get $|s|^2 e^{-k\varphi}$. This is the notation for the fiber metric today.

So what is the standard metric? Let me define the scalar curvature $S_\varphi \coloneqq \operatorname{Tr}_\omega \operatorname{Ric}_\omega$, the trace of the Ricci curvature, and we say that the metric has *constant scalar curvature* if this S_φ, which is a function on the space, is constant, if $S_\varphi = \widehat{S} = \frac{1}{v}\int_X S_\varphi \omega_\varphi^n$, we call $\omega_\varphi = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi$ a *constant scalar curvature Kähler metric* and this is what we're looking for.

Now let me briefly review the variational approach to the existence of this metric. There actually exists a canonical, well, let me define,

$$\mathcal{H} \coloneqq \{e^{-\varphi} \text{ fiber metrics on } L | \omega_\varphi = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi > 0\},$$

and this can be identified with the space of Kähler metrics, and there exists a canonical energy functional $M : \mathcal{H} \to \mathbb{R}$ characterized by

$$\delta M(\varphi) = -\int_X (S_\varphi - \widehat{S})(\delta\varphi)\omega_\varphi^n.$$

Some condition ensures that there exists a primitive of this gradient, this energy functional, which then has this constant scalar curvature metric as a critical point. We have a more explicit formula:

$$\begin{aligned} M(\varphi) = &\int_X \log \frac{\omega_\varphi^n}{\omega_\psi^n}\omega_\varphi^n \\ &+ \frac{1}{n+1}\sum_{i=0}^{n-1}\int_X (\varphi - \psi)\operatorname{Ric}\varphi \wedge \omega_\varphi^i \wedge \omega_\psi^{n-i-1} \\ &+ \frac{\widehat{S}}{n+1}\sum_{i=0}^{n}\int_X (\varphi-\psi)\omega_\varphi^i \wedge \omega_\psi^{n-i}. \end{aligned}$$

This is complicated, this is the K-energy functional. A remarkable fact is that M is geodesically convex with respect to the Riemannian metric $\int_X u^2\omega_\varphi^n$ for any smooth $u : X \to \mathbb{R}$. Again by the $d - \bar{d}$ lemma, the space of u can be identified with $T_\varphi\mathcal{H}$ and so this gives an infinite dimensional Riemannian structure on $\mathcal{H}$.

We want to know about the existence of critical points. Let me restrict to a fixed geodesic φ^t for $t \in [0,\infty)$. The K-stability condition introduced by Tian and Donaldson is that if

$$\lim_{t\to\infty} \frac{M(\varphi^t)}{t} > 0$$

for any "algebraic" ray (I'll explain the meaning later), then there exists a constant scalar curvature Kähler metric in $c_1(L)$. This is the condition.

So then what is the definition of this "algebraic" ray? To define it we pick some family $(\mathcal{X},\mathcal{L})$ of polarized manifolds, it's necessary actually to take a family of polarized schemes. We have not only the family but thy $\mathbb{C}^*$-action here. We say, take the family with an action which makes a $\mathbb{C}^*$-equivariant family, which makes the projection π to $\mathbb{C}$, the family, $\mathbb{C}^*$-equivariant. It means we have some $\lambda : \mathbb{C}^* \to \operatorname{Aut}(\mathcal{X},\mathcal{L})$. We also fix some metirc $e^{-\Phi}$ on $\mathcal{L}$ so $\varphi^t(x) \coloneqq \Phi(\lambda(e^{-t})xe^{-t})$, where e^{-t} is τ, the parameter for the underlying $\mathbb{C}$. This actually then defines a smooth ray (possibly not geodesic) but anyway, this has slope. And we can ask about the slope of the K-energy. This is a very special ray on the space, on $\mathcal{H}$.

All right. So what is the algebraic description of the slope at ∞? This is some part of our result, of our joint work with [unintelligible]and [unintelligible]. So let's take canonical (or trivial) compactification $(\overline{\mathcal{X}},\overline{\mathcal{L}})$ of $(\mathcal{X},\mathcal{L})$, this is over $\mathbb{P}^1$ but the $\mathbb{C}^*$-action around ∞ is very trivial. We have only one [unintelligible]configuration space. We define

Definition 1.1.

$$M^{NA}(\mathcal{X},\mathcal{L}) \coloneqq \underbrace{K_{\overline{\mathcal{X}}/\mathbb{P}^1}\overline{\mathcal{L}}^n}_{\text{main part}} + (n+1)^{-1}\widehat{S}\overline{\mathcal{L}}^{n+1} + \underbrace{(\mathcal{X}_{0,red} - \mathcal{X}_0)\overline{\mathcal{L}}^n}_{\text{error term from unreduced part}}$$

So

Theorem 1.1. *(Tian, Phong–Ross–Sturm, Dervan, Bouckson–H.–Jansson)*

$$M^{NA}(\mathcal{X},\mathcal{L}) = \lim_{t\to\infty} \frac{M(\varphi^t)}{t}$$

for any φ^t coming from $(\mathcal{X},\mathcal{L})$.

So this is still a bit, this is background. The conjecture says that if this M^{NA} is positive then there exists a constant scalar curvature metric. This is the precise statement. But note that we are still very far from the existence of the metric as you know.

[Question about stability versus poly-stability]

Yeah, I wanted to skip that case, the general case is a work in progress, I should have assumed at the beginning that $\operatorname{Aut}(X,L)$ was finite. I think it should be straightforward, the generalization.

So the question is what we can do to show the existence of the metric.

We'd like to propose some approach, which is about more, not only positivity of the slope but we will assume a more stronger growth condition for the K-energy, *coercivity*. This is a classical notion. The point is that there exists another canonical metric $J : \mathcal{H} \to \mathbb{R}$, I think this was first introduced by Aubin, we call this Aubin's J-functional, which is given by

$$J(\varphi) = \int_X (\varphi - \psi)\omega_\varphi^n - \frac{1}{n+1}\sum_{i=0}^{n}\int_X (\varphi - \psi)\omega_\varphi^i \wedge \omega\psi^{n-i}$$

and the sum is the main part and the left part is a normalization. This is different from the K-energy, this is more easier, in the following sense. First of all it's non-negative, and also it's exhaustive for a certain [unintelligible]on $\mathcal{H}$. This has rather a role of measuring the space $\mathcal{H}$ itself. In fact, this is equivalent to the Finsler metric $\int_X |u|\omega_\varphi^n$ for $u \in C^\infty(X, \mathbb{R}) = T_\varphi\mathcal{H}$. This J is equivalent to the L^1 norm.

One of the points is that this L^1 topology is the right topology for the growth condition of the K-energy functional. There exists no example, no manifold over which the K-energy has L^2 growth function. We put $\mathcal{E}^1$, if this is the space of singular metrics

$$\mathcal{E}^1 = \{e^{-\varphi} \notin C^\infty | J(\varphi) < \infty\}$$

This 1 stands for L^1, this finite energy class, this gives the completion of $\mathcal{H}$ with respect to the distance. This is, I think, a result of Dervas, recent. Then the condition on coercivity $M(\varphi) \ge \epsilon J(\varphi) - C$ on $\mathcal{H}$ implies the existence of a minimizer on $\mathcal{E}^1$. Not in $\mathcal{H}$. This is the point. So this stronger growth condition assures the existence of a minimizer on the space of singular fiber metrics. The regularity of the singular minimizers seems very hard. So the idea is to separate this into two parts. Existence of a weak solution is one part and regularization is the other part. This coercivity condition, the point is, can be translated into algebraic geometry as we explained for K-stability.

At this stage we again pick an algebraic family endowed with a $\mathbb{C}^*$ action and take a smooth ray and take

$$J^{NA}(\mathcal{X}, \mathcal{L}) \coloneqq \overline{\mathcal{L}}(\rho^* p_1^* L)^n - (n+1)^{-1}\overline{\mathcal{L}}^{n+1}$$

The point is that we have some domination [unintelligible]. So we can blow up the space, and take some [unintelligible]modification and get some $\rho : \mathcal{X} \to X \times \mathbb{C}$. We can pull back $p_1^* L$ by ρ to take its intersection number against $\overline{\mathcal{L}}$. These interseciton numbers precisely compare with the description I gave for the J functional which was very well-known.

So the conjecture is that

Conjecture 1.1. ("weak" Yau–Tian–Donaldson) $M^{NA}(\mathcal{X}, \mathcal{L}) \ge \epsilon J^{NA}(\mathcal{X}, \mathcal{L})$ for $\mathcal{H}^{NA}$, the space of algebraic families, is equivalent to $M(\varphi) \ge \epsilon J(\varphi) - C$ on $\mathcal{H}$.

What is known on this conjecture? The original Yau–Tian–Donaldson conjecture was proved for the Fano case around 2012, and you can see from the result that the weak version is also true for the Fano case, where $L = -K_X$. This is because the existence of a constant scalar curvature metric in fact implies the coercivity property of the energy by Dervas–Rubenstein. Then you can use the result of Chen–Donaldson–[unintelligible]–Tian to give the equivalence.

Also I'd like to note that there exists a direct proof of the weak version without Cheeger–Colding theory (work by Boukson–Berman–Jansson). It seems that this approach is somewhat effective. This is also okay for the toric case. In this case we have some automorphism group but we can have the modified formulation for the toric manifolds and then we have this type of equivalence between positivity of the intersection number and the growth condition for the K-energy functional.

So, I guess this terminology NA could be familiar to symplectic people, this NA stands for "non-Archimedean" and these intersection numbers can be identified with energy functionals on the space of non-Archimedean metrics. I have five minutes.

First, [unintelligible]can be identified with a Lie bundle on X^{an}, which is [unintelligible], and the definition of this metric is given by the following,

$$\varphi^{NA}(v) = G(v)$$

a value of the Gauss extension to the function field of the product space of the line bundle. The Gauss extension of the variation $G(v)(\sum f_\lambda \tau^\lambda)$ is defined by $\min_{\lambda\in\mathbb{Z}}\{\lambda + v(f_\lambda)\}$, there is some general picture that assures these energies can be non-Archimedean functionals on a certain space.

Last I'd like to give some sketch of the proof, I should say that, I skipped the statement. $J^{NA}(\mathcal{X},\mathcal{L}) = \lim_{t\to\infty}\frac{J(\varphi^t)}{t}$, I'd like to give some sketch of the proof.

The key to it is Deligne pairing. This is some line bundle which corresponds to the intersection number which appears in the coefficient of [unintelligible], for example

$$\dim H^0(X, L^{\otimes k}) = \frac{L^n}{n!}k^n \frac{-K_X L^{n-1}}{2(n-1)!}k^{n-1} + \cdots$$

and what we want is for

$$\det(\pi_* H^0(\mathcal{X}, \mathcal{L}^{\otimes k})) = \frac{\langle \mathcal{L}, \ldots \mathcal{L}\rangle}{(n+1)!}k^{n+1} + \frac{\langle -K_{\mathcal{X}}\mathcal{L}^n}{2n!}k^n + \cdots$$

So the $\langle \mathcal{L}, \ldots \mathcal{L}\rangle$ is the additive notation for the $\mathbb{Q}$-line bundle. So then, we didn't give the definition, but if this line bundle can be constructed, then this gives the self-intersection number of the line bundle, and we can prove the theorem by attaching some metric on this line bundle and applying the classical [unintelligible]. So what is the construction.

So $\mathcal{L}_0, \ldots, \mathcal{L}_n$ are $\mathbb{Q}$-line bundles on some family $\mathcal{X}$ on the base space B. Then we have the line bundle $\langle \mathcal{L}_0, \ldots, \mathcal{L}_n\rangle_{\mathcal{X}/B}$ over B, and this is constructed by the following inductive formula. We take some section s of $H^0(\mathcal{X}, \mathcal{L}_0)$ and $D \coloneqq \div(S)$, then the restriction of this bundle to D is equivalent,

$$\langle \mathcal{L}_0, \ldots \mathcal{L}_n\rangle_{\mathcal{X}/B} = \langle \mathcal{L}_1|_D \ldots \mathcal{L}_n|_D\rangle_{D/B}$$

Then we can also define the metric in a similar inductive formula.

Giver $\Phi_0, \ldots, \Phi_n$ on $\mathcal{L}_0, \ldots, \mathcal{L}_n$, then we have

$$\langle \Phi_0, \ldots \Phi_n\rangle_{\mathcal{X}/B} = \langle \Phi_1|_D \ldots \Phi_n|_D\rangle_{D/B} + \int_{\mathcal{X}/B} \log|s|^2 e^{-\Phi_0}\omega_{\Phi_1}\wedge\cdots\wedge\omega_{\Phi_n}$$

and this corresponds exactly to what I wrote before. For example,

$$J^{NA}(\mathcal{X},\mathcal{L}) = \mathcal{L}(\rho^* p_1^* L)^n - \frac{\mathcal{L}^{n+1}}{n+1} = \deg(\langle \mathcal{L}; \rho^* L \ldots \rho^* L\rangle - (n+1)^{-1}\langle \mathcal{L}\ldots\mathcal{L}\rangle)$$

and Aubin's J-functional is given (as a consequence of the inductive formula) as

$$J(\varphi) = \langle \varphi, \psi \ldots \psi\rangle_X - \langle \psi, \ldots, \psi\rangle_X - (n+1)^{-1}[\langle \varphi \ldots \varphi\rangle_X - \langle \psi \ldots \psi\rangle_X]$$

and we get

$$\langle \varphi_0^t \ldots \varphi_n^t\rangle - \langle \psi \ldots \psi\rangle = t(\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_n) + O(1)$$

but such equality is for a relation between the [unintelligible]with the intersection, the degree of the line bundle, so the left hand side is just the growth of the $\langle \Phi_0 \ldots \Phi_n\rangle$ on $\langle \mathcal{L}_0, \ldots \mathcal{L}_n\rangle$ over $\mathbb{C}$ and the right hand side is the degree of the same line bundle $\langle \mathcal{L}_0 \dots \mathcal{L}_n \rangle$ over $\mathbb{P}^1$, which is the same as the weight of the $\mathbb{C}^*$-action on $\langle \mathcal{L}_0 \dots \mathcal{L}_n \rangle$ on the point zero. Then we conclude by the Kempf–Ness argument from symplectic geometry.

2. August 2: Dmitry Tonkonog: The wall-crossing formula and Lagrangian mutations

This is joint work with James Pascaleff. Yesterday we heard from Yoosik about how to make monotone Lagrangian tori. Today I'll talk about how to turn such tori into new tori and calculate their potentials. The outline of the talk, I'll first state and prove a general wall-crossing formula in a really quite general setup. Then I'll talk about mutations of Lagrangian tori in del Pezzo surfaces in dimension 4, after Vianna as well as Galkin–Usnich–Cruz Morales and earlier work by Auroux. This brings together the geometry of Vianna with the algebra of the big group. Then I'll talk about higher dimensional toric mutations which give new Lagrangian tori in toric Fanos. This high dimensional story won't be as complete as the lower dimensional one and will pose interesting questions.

The main tool for the wall crossing formula is the LG potential. Recall if I start with a Lagrangian, monotone, $L \subset X$, I can fix a basis for its first homology, let's say integrally modulo torsion for simplicity, and I introduce formal variables $x_1, \dots, x_m$, where m is the rank. Then the Landau–Ginzburg potential W_L is a Laurent polynomial in these variables which is a sum over holomorphic Maslov two-disks, I compute tho boundary homology class, and write down the homology class in coordinates $(\ell_, \dots, \ell_m)$, and put in the polynomial $\pm x_1^{\ell_1} \cdots x_m^{\ell_m}$, where the sign comes from the orientation. Really we should put W_L in $\mathbb{Z}[\pi_1 L]$, but I'll be talking about C^*-local systems and there this Laurent polynomial is good enough.

This is the simplest Gromov–Witten invariant for a Lagrangian. For example, in $\mathbb{CP}^2$, I can take the Lagrangian torus and $W = x + y = \frac{1}{xy}$.

Recall that a local system on L is a map $\rho : \pi_1 L \to \mathbb{C}^*$, a homomorphism. This is the same (having specified a basis) as a point $\rho \in (\mathbb{C}^*)^n$, specifying its values on the basic loops.

Then I can equip L with a local system and it becomes an object of the monotone Fukaya category $\mathbb{L}$, and another interpretation of W_L is that $W_L(\rho)1$ becomes the curvature $m^0_{\mathbb{L}} \in HF(\mathbb{L}, \mathbb{L})$. This is a tautology.

Now I'm ready to state the general wall-crossing formula.

Theorem 2.1. *(Pascaleff–T.) Let X be monotone and Σ a smooth Donaldson divisor (that is, $[\Sigma]$ is kc_1 for $k \geq 1$). Let L_1 and L_2 be monotone Lagrangians in X which lie away from Σ are are exact in the complement $X \backslash \Sigma$. Equip them with any local systems, $\mathbb{L}_i = (L_i, \rho_i)$ and assume that $HF_{X\setminus\Sigma}(\mathbb{L}_1, \mathbb{L}_2)$, computed in $X \setminus \Sigma$, is nonzero.*

Then the curvature terms $m^0_{\mathbb{L}_1}$ and $m^0_{\mathbb{L}_2}$ are the same, so that $W_{L_1}(\rho_1)$ and $W_{L_2}(\rho_2)$ are the same.

The idea of how to apply this, we'll have many local systems on L_1 and many on L_2 and we'll assemble them.

So note that [unintelligible]is a Liouville manifold, so that $HF_{X\setminus\Sigma}(\mathbb{L}_1, \mathbb{L}_2) = HF_M(\mathbb{L}_1, \mathbb{L}_2)$ for any smaller Liouville domain M contained in $X \setminus \Sigma$ and containing both L_i. So the idea is to find an M with some nice structure and exploit that.

Originally the proof of this story was quite involved, used some Hamiltonian stretching procedures. But Seidel suggested an easy proof, which I'll give you now. The proof is, start with constant holomorphic strips between L_1 and L_2, and glue to L_1 a disk with a point on L_1 with an interior point going to Σ. For simplicity assume Σ is anticanonical and the point is the origin. The count of these configurations (weighted with a local system) gives me $m^0_{\mathbb{L}_1} \operatorname{Id}_{HF(\mathbb{L}_1,\mathbb{L}_2)}$. This is equivalent to the moduli space of strips with a point freely moving like this [picture] going to Σ. If I move it upward it goes to what I described. If I move it downward it becomes $m^0_{\mathbb{L}_2} \operatorname{Id}_{HF(\mathbb{L}_1,\mathbb{L}_2)}$ and this degeneration tells me that I have a homotopy between these two. There's further bubbling so this isn't true at the chain level but it's true in homology.

So now if $HF(\mathbb{L}_1,\mathbb{L}_2) \neq 0$, then the identity is nonzero, and multiplication of it by different numbers will never be the same, so the curvatures must be the same.

I'll explain Vianna's work from a slightly different point of view.

I start again with a Clifford torus in $\mathbb{CP}^2$, and I claim that it bounds three Lagrangian disks whose boundary Floer homology classes are orthoganal to $(-1,-1)$, $(2,-1)$, and $(-1,2)$. To see this, I draw [pictures].

Let's study a configuration consisting of a Lagrangian torus with one Lagrangian disk attached to it, $(T,D) \subset X_1$. Suppose I have two such configurations in different symplectic manifolds. There's a version of the Weinstein neighborhood theorem that says that the unions $T\cup D$ and $T_0\cup D_0$ have symplectomorphic neighborhoods. So we can look for a model of $T\cup D$, (I should say D is embedded and attached to T cleanly along a curve which is not contractible), and such a model is the space M which is an affine conic $M = \mathbb{C}^2 \smallsetminus \{xy = 1\}$ and from there I take a projection ont $\mathbb{C} \smallsetminus \{1\}$. [pictures]

So why is this model useful to me? In this model I can see a different Lagrangian torus. In this model, T_0 becomes exact and now the same model contains a different exact torus T_1, which using the same pictures as before can be drawn, [pictures]. Then this is the Clifford and Chekanov pair. Combining my local construction with the Weinstein neighborhood theorem, I can define what it means to mutate a Lagrangian torus along a Lagrangian disk. Let $(T,D) \subset X$ be a monotone Lagrangian torus and Lagrangian disk. Then there is another torus in X, in fact in a neighborhood of $T\cup D$ which is also monotone and we call this the mutation $M_D T$ the mutation of T along D. Locally this is passing from the Clifford to Chekanov torus or vice versa in the neighborhood.

Okay so starting from the Clifford torus in $\mathbb{CP}^2$, I can mutate in any of three directions, so I get three more tori, and I happen to get unlucky, these happen to be isotopic to each other, to the Chekanov torus in $\mathbb{CP}^2$. So how do I get infinitely many tori using this construction? The answer is essentially provided by Shende–Treumannn–Williams, and in a restated way it says the following. Suppose $T \subset X$ bounds Lagrangian disks D_i and I need to do some [unintelligible], denote by v_i the class $[\partial D_i]$ and these live in $\mathbb{Z}^2$, fixing a basis for $H_1(T)$. The boundaries of these disks have minimal intersections with each other. [pictures] Then, pick a $j \in \{1,\ldots N\}$, one of the disks along which I want to mutate. The torus mutated along D_j bounds the same number of Lagrangian disks, which I denote D'_i, again $i \in \{1,\ldots N\}$, and $[\partial D'_i] = v'_i \in H_1(T')$, where T' is the mutated torus and I've chosen some basis. So the second thing, that I'll write down now, is the computation (slightly nontrivial) of the formula for the boundary classes.

So $v_j' = -v_j$ and $v_i' = \mu_{v_j} v_i$ where $\mu_u v$ is $v + \max\{0, \langle u, v\rangle\}u$, this is called a tropical mutation.

Quite mysteriously this works for tori but not for higher dimensional Lagrangians. The proof I like would be like this (this is not the original proof. I'll only give a sketch). [pictures]

This theorem, together with mutation, gives me infinitely many tori in $\mathbb{CP}^2$, I get three neighbors in each case, these are the Vianna tori (pictures).

Now let's go to wall-crossing.

Theorem 2.2. *If I have $(T, D) \in X^4$ and T is monotone, then $W_{\mu_D T}$ is obtained from W_T by substitution where I send x to $x(1+x^{-u_2}y^{u_1})^{u_1}$ and $y \mapsto y(1+x^{-u_2}y^{u_1})^{u_2}$ this is where $[\partial D] = (-u_2, u_1) \in \mathbb{Z}^2$.*

For example, if $(u_1, u_2) = (0, 1)$ then $x \mapsto x$ and $y \mapsto y(1+x)^{-1}$.

The corollary is that know we know the potentials of all Vianna's tori? He computed only the Newton polytopes of these tori. [pictures]. These come from triangles with sides of length a^2, b^2, c^2 with sum $3abc$.

Some corollaries of this are

Corollary 2.1. *We have infinitely many tori in the blowup of $\mathbb{CP}^2$ at two points (This was the missing case after Vianna).*

Arguing a bit more, we can now compute (Hori–Vafa) potentials of tori in non-toric del Pezzos. For example, I can take the cubic surface, it has an almost toric fibration over the triangle whose potential is $\frac{(1+x+y)^3}{xy} - 6$. I don't think this was proved (it was expected). We can show that these are the only Laurent polynomials available after [unintelligible]. So I used the wall-crossing formula without the knowledge of the original potential. That is, you can prove a lemma, I know the torus bounds nine Lagrangian disks. I can write down nine mutations under which the potential stays Laurent. I'm cheating, because $\mu_{v_1} = \cdots = \mu_{v_3}$. The Lemma is that U Laurent, in $\mathbb{C}[x^{\pm 1} y^{\pm 1}]$, and the ring of Laurent polynomials staying Laurent is polynomial in one variable, generated by this polynomial. This follows from the work of [unintelligible]because the ring of functions, this is a well-known notion in cluster algebra. [something too fast]. Once I know that, I can explicitly check that this function stays in the ring, and then I have to check that this is the generator. Then Vianna proves that the potential for the actual torus has Newton polytope [unintelligible]. Then I check that the coefficients are just units.

I want to use the general wall-crossing formula, I want to avoid multiple covers.

Proof of 4D wall-crossing. I work with a local model M which is $\mathbb{C}^2 \setminus \{xy = 1\}$ that we discussed. I equip the tori with local systems. Let $\rho_0 = (x, y) \in (\mathbb{C}^*)^2$. Then Seidel has a lemma saying that

$$HF((T_0, \rho_0), (T_1, \rho_1)) \neq 0$$

if and only if $\rho_1 = (x, y(1 + x^{-1}))$.

This is explicit because you can compute explicitly in the model manifold. [pictures] □

You could have also proven that these have nonvanishing Floer homology et cetera with a isotopy and computation of the multiple covers, but this avoids that and gives a direct proof.

In a general monotone manifold, I use a version of the [unintelligible]theorem to say that if I have a torus for the disk, I have to compute away from a divisor, and then that's the same as computing in the local model.

In five minutes I'll try to mention something about the higher dimensional situation. The theorem is,

Theorem 2.3. *Let X be toric Fano and F a codimension $k \geq 2$ face of the moment polytope, and fix $v \in F$ an interior integral point of. Then X contains a monotone Lagrangian torus which I denote by $\mu_{F,v}T$, where T is the standard fiber, mutated along my datum, whose potential is obtained from the toric potential by performing the following substitutions. Let me write them down on monomials $x_1^{u_1} \cdots x_n^{u_n} =: \mathbf{x}^u$, and this goes to $\mathbf{x}^u(1 + \mathbf{x}^{u_1} + \cdots + \mathbf{x}^{u_n})^{\langle u,v\rangle}$. I have to tell you about u_i and w. The picture is this [picture]. So u_i form a basis for $\Pi^\perp$.*

The local model for this mutation, if I can use one more minute, if F is a vertex, then the local model is that of $\mathbb{C}^n \smallsetminus \{x_1 \cdots x_n = 1\}$ projected to $\mathbb{C} \smallsetminus \{1\}$. [pictures].

3. August 3: Young-Jun Choi: Fiberwise Ricci-flat metrics on Calabi–Yau fibrations

First of all I'd like to thank the organizers for the invitation. Let me introduce the setup. So $p : X \to \Delta$ is a family, a proper holomorphic map to Δ the unit disk in $\mathbb{C}$. Originally it is an arbitrarily complex manifold. I'll assume the base is the unit disk for simplicity. Let me start with the submersion case. Then every fiber $X_s = p^{-1}(s)$ is a complex manifold, assumed compact, and I'll assume it's Calabi–Yau (the first Chern class vanishes) and I'll give myself a polarization, (X, ω) is a Kähler manifold. Then by Yau's theorem there exists a unique Kähler–Einstein metric ω^{KE} on each fiber, Ricci flat, on $[\omega|_{X_s}]$. The fiberwise metric is defined this way.

Definition 3.1. A closed real $(1,1)$-form ρ is a *fiberwise Ricci-flat metric* if $\rho|_{X_s} = \omega_s^{\mathrm{KE}}$ on each X_s.

There are many of these, if I take a Kähler form on the base and pull back and add, this is also fiberwise Ricci-flat, so I need a normalization. But existence is easy.

These are well-studied, maybe since Vafa and Yau. For example, on each X_s there is a unique smooth ϕ_s such that

$$(\omega_s + dd^c|_{X^s}\phi_s)^n = e^{\eta_s}\omega_s^n$$

where $\mathrm{Ric}(\omega_s) = dd^c\eta_s$ and $\int_{X_s} e^{\eta_s}\omega^n = \int_{X_s}\omega^n$. This is not uniquely determined yet, we have to normalize, and then we do this via

$$\int_{X_s} \phi_s \omega_s^n = 0$$

and now if you glue these up you get a smooth function on the total space, and define

$$\rho = \omega + dd^c\phi$$

and this is a fiberwise Ricci-flat metric.

The longstanding conjecture about these fiberwise Ricci-flat metrics is:

Conjecture 3.1. $\rho \geq 0$ or $\rho > 0$ on X.

This is still open. One way to attack this might be by Schumacher's method.

Theorem 3.1 (Schumacher)**.** *Suppose $\tilde{p} : \tilde{X} \to \Delta$ is a family of compact Kähler manifolds with $c_1 < 0$. In this case, of course, the curvature is negative and not zero. There is a canonical way in this case to define a fiberwise Kähler–Einstein metric $\tilde{\rho}$, so that $\tilde{\rho} = \Theta_{\tilde{\rho}}(K_{\tilde{X}}/\Delta)$, the curvature of the relative canonical line bundle. In the polarized case we have a fiberwise Kähler–Einstein metric satsifying this, this is the most natural case. The theorem says that $\tilde{\rho} \geq 0$ on $\tilde{X}$ and $\rho > 0$ if the family is not locally trivial.*

I want to sketch the proof but let me give some definitions first

Definition 3.2. Let $p : X \to \Delta$ be a family of compact Kähler manifolds (no assumption on the Chern class). If we have τ a d-closed $(1,1)$-form on X such that $\tau|_{X_s} > 0$. Let me write $\frac{\partial}{\partial s}$ by V for simplicity. [pictures]. We want to lift V to X. So the *horizontal lift* V_τ with respect to τ is a vector field on X such that $dp(V_\tau) = V$ and $V_\tau \perp X_s$ with respect to τ. The *geodesic curvature* $c(\tau)$ is $\langle V_\tau, V_\tau \rangle_\tau$

The reason I introduced this is because

Proposition 3.1.

$$\tau^{n+1} = c(\tau)\tau^n \wedge \sqrt{-1} ds \wedge d\bar{s}.$$

,

We conclude that if and only if $c(\tau) > 0$ then and only then $\tau > 0$ and similarly $c(\tau) \geq 0$ if and only if $\tau \geq 0$.

This gives another proposition that proves Schumacher's theorem:

Proposition 3.2.

$$-\Delta c(\tilde{\rho}) + c(\tilde{\rho}) = |\bar{\partial} V_{\tilde{\rho}}|^2$$

on X_s.

This proves Schumacher's theorem. By compactness of the fiber, there is x_0 on X_s such that $\inf c(\tilde{\rho}) = c(\tilde{\rho})(x_0)$ which is in turn $|\bar{\partial} V_{\tilde{\rho}}|^2 + \Delta c(\tilde{\rho})(x_0)$ and both of these are non-negative, so $c(\tilde{\rho}) \geq 0$. Then [something about nontriviality].

This is all in the negatively curved case.

Now we go back to the Calabi–Yau family $p : X \to \Delta$ and ρ is any fiberwise Ricci-flat metric. Then we can compute the Laplacian of the geodesic curvature of this form, but unfortunately the shape is quite different:

Proposition 3.3.

$$-\Delta c(\rho) = |\bar{\partial} V_\rho - \Theta_{s\bar{s}}$$

Now we have no minimum anymore, Θ is constant, and we can't apply the same argument, both sides will be zero and that's the worst case for applying the maximum principle.

Corollary 3.1. $p^* \omega^{\mathrm{WP}} = \Theta_\rho$

The differences come from:

$$(\tilde{\omega} + dd^c \tilde{\phi})^n = e^{\tilde{\phi} + \tilde{\eta}} \tilde{\omega}^n; \tag{1}$$

$$(\omega + dd^c \phi)^n = e^{\eta} \omega^n \tag{2}$$

In the second case we have no zeroth order term in the exponent. And so

$$dd^c\tilde{\eta}\,\mathrm{Ric}([\mathit{unintelligible}]) + \omega \tag{3}$$

$$dd^c\eta = \mathrm{Ric}([\mathit{unintelligible}]). \tag{4}$$

So let me make an observation. Let (M, ω) be a compact Kähler maniflod. Then

$$(\omega + dd^c\phi)^n = e^{\epsilon\phi+\eta}\omega^n$$

for $0 < \epsilon < 1$ which implies that $\|\phi_\epsilon\|_{c^{k\alpha}} < C$, $\phi_\epsilon \to \phi$.

[missed a little]

So ϕ_ϵ goes to one unique function ϕ given by the condition

$$\int_X \phi e^\eta \omega^n = 0$$

so we have to choose this second normalization condition to get this precise statement.

So let me go back to the equation:

$$(\omega_s + dd^c\phi_s)^n = e^{\epsilon\phi_\epsilon+\eta_s}\omega_s^n$$

with the normalization

$$\int \phi_s e^\eta \omega^n = 0$$

Define $\rho_\epsilon \coloneqq \omega + dd^c\phi_\epsilon$. This is not a fiberwise Ricci-flat metric. Assume that ρ_ϵ converges to ρ very well as $\epsilon \to 0$ (this is true but let me skip that argument), then it converges to a fiberwise Ricci-flat metric.

So we want to compute the geodesic curvature of ρ_ϵ, then the equation looks like this:

Proposition 3.4.

$$-\Delta c(\rho_\epsilon) + \epsilon c(\rho_\epsilon) = |\bar{\partial} V_{\rho_\epsilon}|^2 - \Theta_{s\bar{s}} + \epsilon\omega(V_{\rho_\epsilon}\bar{V}_{\rho_\epsilon})$$

and we have a zeroth order term but we can't get rid of the Θ which is topological, but we can obtain

$$\int c(\rho_\epsilon)\rho_\epsilon^n = \frac{1}{\epsilon}\int |\bar{\partial} V_{\rho_\epsilon}|^2 - \Theta_{s\bar{s}} + \int \omega(V_{\rho_\epsilon}\bar{V}_{\rho_\epsilon})\rho_\epsilon^n$$

and then as ϵ goes to zero the first term on the right side vanishes and we get

$$\int c(\rho)\rho^n = \int \omega(V_\rho, \bar{V}_\rho)\rho^n.$$

So then we get

$$\begin{aligned} p_*\rho^{n+1} &= \int_{X_s} \rho^{n+1} \\ &= \int c(\rho)\rho^n \wedge \sqrt{-1}ds \wedge d\bar{s} \\ &= \left(\int \omega(V_\rho, V_\rho)\right)\sqrt{-1}ds \wedge d\bar{s} > 0 \end{aligned}$$

So the fiberwise Ricci-flat metric satisfying my second normalization satisfies this condition, but there is bad news. Oh, this is joint work with Braun and Schumacher.

Theorem 3.2 (Braun–C.–Schumacher)**.** $p_*\rho^{n+1} > 0$.

Remark 3.1.
(1) ρ_1 (the original fiberwise Ricci-flat metric) and ρ_2 (this is our choice with this normalization) are not uniquely determined in $[\omega]$. So basically if you have the family and a Kähler form that gives a Kähler class. Maybe we want unique determination on each Kähler class, but changing the representative gives us a different fiberwise Ricci-flat metric.
(2) But there is a way to define the fiberwise Ricci-flat metric uniquely. There is a fiberwise Ricci-flat metric ρ_3 such that $\int_{X_s} \rho_3^{n+1} = 0$. But there is no chance to show positivity here. If you take the average, it's zero.

Corollary 3.2. *If you add a pullback, $\rho_3 + Cp^*\omega_{\mathrm{WP}} > 0$. But the constant C depends on the family so this isn't so satisfying. If the diameter of X_s is bounded by some constant then C can be chosen uniformly, it's like related to a Green kernel.*

I still have about twenty minutes. Let's consider extension of (the curvature of) this metric.

So now we go back to $\Theta_\rho \coloneqq \Theta_{h^\rho_{X/\Delta}}(K_{X/\Delta})$. Now we consider $p : X \to \Delta$ a surjective holomorphic map, maybe it has singularities now, but for now let's assume that $p : X \setminus X_0 \to \Delta^*$ is a submersion, [picture].

Theorem 3.3 (Schumacher, Păum)**.** *Let $\tilde{p} : \tilde{X} \to \Delta$ with the same assumptions. Then Θ_ρ can be extended to a positive $(1,1)$-current in $\tilde{X}$.*

So Schumacher used the embedding in some projective space. Păum generalized, also considering twisted Kähler–Einstein metrics, and showing that curvature in the twisted case can also be extended to the total space. In our case we also have this kind of theorem. We are going back to the Calabi–Yau fibration with singularity.

Theorem 3.4 (C.–Schumacher)**.** *Θ_ρ can be extended in this case.*

Let me introduce the sketch of the proof. Let ρ be a fiberwise Ricci-flat metric, so $\rho^n = e^\eta \omega^n$, So $dd^c \eta_s = \mathrm{Ric}(\omega_s)$ and $\int e^{\eta_s}\omega_s^n = \int \omega^n$, then the curvature satisfies

$$\Theta_\rho = dd^c \eta + \Theta_\omega.$$

The curvature, by definition,

$$\Theta_\rho = dd^c \log \rho^n \wedge dV_s.$$

outside X_0.

Now you want to extend Θ_ρ. You already know this is positive semi-definite on the total space. You want to show that the local potential is bounded from above by a uniform constant. For that let me write, first you have [pictures]. Take some coordinates $z^1, \cdots, z^{n+1}$ on the total space and an open U, and then U_s. So in U we have a local potential σ which is $\eta + \Psi_U$. And by Demailly's theorem, we have the approximation on each fiber:

$$\sigma_s(x) = \limsup_{m\to\infty, \|f\|_s^2 \le 1} \frac{1}{m} \log |f(x)|^2$$

Here we're considering the space $\mathcal{H}_m^2 = \{f \in \mathcal{O}(U) | \int |f|^{2/m} e^{-\sigma}(e^\eta \omega^n) < \infty\}$ So now we claim that if if $\|f\|_s^2 \le 1$, then we can extend this function around U by Ohsawa–Takegoshi extension, which says that when you extend a function, the constant of the extension is uniform in every case. This is a tedious computation but by the equations we have, you get

$$\int_{U_s} |f|^{2/m} \frac{dV_z}{p^* dV_s} < C$$

is uniformly bounded. You have to choose this particular norm, which blows up when you go to the singular fiber where the denominator vanishes. But by the condition we have, the function with respect to this norm, it's bounded from above. So C does not depend on s or m.

So now we apply the extension and get that there is a holomorphic F such that $F|_{X_s}$ is f and $\int_U |F|^{2/m} dV_z \le C \int |f|^{2/m} \frac{dV_z}{p^* dV_s}$. So now σ is bounded from above and so can be extended. Maybe I'll stop here.

4. Nuromur Hulya Arguz: Log geometric techniques for open invariants in mirror symmetry

Thank you very much, I'm glad to get the chance to give a talk here. I want to talk today about the mirror construction of Gross–Siebert (which is an algebro-geometric construction) and how this will fit into homological mirror symmetry (which is symplectic). On the algebraic geometry you have "log Gromov–Witten invariants" and on the other hand you have some sort of SH^*. I want to talk about this connection today.

What are log Gromov–Witten invariants? They are some kind of relative Gromov–Witten invariants. If you have a Calabi–Yau, X, and you want to construct its mirror, put it in a family $\mathcal{X}$ (for simplicity over $\mathbb{A}^1$), and they consider toric degenerations, where the central fiber is a union of toric varieties. For example, if you consider $x_0x_1x_2x_3 + txf_4 = 0$ in $\mathbb{P}_3 \times \mathbb{A}_1$, then you get a $K3$ surface, and the central fiber is a union of four copies of $\mathbb{P}^2$, intersecting each other along coordinate hyperplanes. The moment cones of each toric component patch together to give a tetrahedron B. This gives a way, if you consider a toric degeneration, you can come up with a dual complex, and put combinatorial data onto it to make it an integral affine manifold. You're looking, in the central fiber, at some structure, a log structure, on the singularities, Look at four singular points, which fiber over twenty-four points on B; I want to think of B as an integral affine manifold with singularities. I can cover B with affine open sets so that the transition functors away from the singular locus lie in $GL_2(\mathbb{Z}) \ltimes \mathbb{Z}_2$. The singular locus in general will be determined by a singular badly behaving part in the central fiber. The idea is to take the dual of B, $B^\vee$, the discrete log dual.

So the idea fo Gross–Siebert is to rebuild the mirror out of this dual. So the first step is to build the central fiber $\mathcal{X}_0^\vee$. This requires gluing data, if you want something like B to be the moment polytope of some toric varieties, you need a symplectic form, this requires gluing data. There's a moduli space of choices. If you have a torus orbit in two faces, you want to be able to glue them together. So that requires this choice of gluing data. Let me advertise that in the paper on the arXiv, we did this in some situation where we have real structures on each fiber. If you want a real generating set of real Lagrangians, then you need to restrict to some conditions. I don't want to talk about that today. I want to talk about how you start from this and get the mirror. You've made your choice of gluing data, and the main theorem of Gross–Siebert is that you have a procedure to construct $\mathcal{X}^\vee$, and then your mirror is the general fiber. This requires a wall-crossing like Kontsevich–Soibelmann.

So you can choose a piecwise linear function $\phi : B \to \mathbb{R}^n$ (this is giving an idea of a construction of the mirror). Around each vertex, you want to further subdivide and choose a polyhedral decomposition of $B^\vee$ so that each singularity lies between a vertex of the decomposition. The standard way is to take the barycentric subdivision. Each singular point lies between a vertex, and you do it in every face [picture]. You use the choice of a piecewise linear function to build a monoid around v_i. Around this it looks ilke this [picture] under ϕ_i and you're taking the cone around each vertex and taking the integral points inside this cone. This is the image of ϕ in a neighborhood, and then you obtain a monoid. Then you want to take $\operatorname{Spec} \mathbf{k}[M_{v_i}]$ and glue together over every vertex. Once we glue all these together this will be your total space $\mathcal{X}^\vee$. The difficult thing is to determine how to glue all these together.

To determine this, you need to ensure that you can glue compatibly, so you need a scattering algorithm, so this "gluing" requires *wall-crossing*.

So far the idea is based on a simple generalization of simple toric geometry. When you have a toric variety you can reconstruct in this way. In this case you degenerate your Calabi–Yau to something toric. Unlike in toric varieties, you have something degenerate, which requires some nontrivial wall-crossing. You put functions (wall-crosssing functions) on rays out of each singularity. The composition of your functions, going around, give the identity. So how do Gromov–Witten invariants enter? The log Gromov–Witten invariants give an enumerative interpretation of the formulas for wall-crossing.

So these are the main tools to construct. The correspondence was shown for Gross–Pandharipande–Sieber, and wasn't proven for more general cases when you have singularities. If you look at toric degenerations of toric varieties, you'll always have some singularities here. [picture]. You need to generalize log Gromov–Witten invariants to "punctured" log Gromov–Witten invariants, that's the right invariants to define symplectically the, to interpret the symplectic cohomology ring whenever you have a Calabi–Yau or whatever. There's some notes on Gross' webpage, and theres something else by Abramovich–Chen–Gross–Siebert.

So the difference between the punctured log Gromov–Witten invariants, whenever you have a singularity on B, you have a sum $1 + \sum a_i z_i$, and the a_i are in some monomial algebra. From the powers of those elements, these have to do with contact orders of your log Gromov–Witten curves, of log curves. The idea is when you have a singularity and you try to go around, if you have positive powers on one end you need negative powers on the other end, so you need to allow negative contact orders. So these negative contact orders are what they call "punctured."

Let me talk about simple log Gromov–Witten invariants, which are a special kind of relative Gromov–Witten invariants (Jun Li). You want to count curves with special tangency conditions to a divisor D. The idea of Gross–Ziebert is to put additional data on this. Li makes an accordian picture. But [unintelligible]define an additional approach by putting an example ofa a sheaf on D. If you define a morphism [picture] then this encodes the tangency.

The sheaves they're putting are given by log structures. Let me tell you what a log structure is.

Definition 4.1. A *log structure* on X is a structure homomorphism $\alpha : M_X \to \mathcal{O}_X$ so that $\alpha|_{\alpha^{-1}(\mathcal{O}_X^*)}$ is an isomorphism.

Let me give some examples. One is the *divisorial log structure*. You're defining, here, let me take $\mathbb{A}^2$, with divisor given by $xy = 0$. Then you can define M your sheaf of monoids $M_{\mathbb{A}^2,D}$ as the sheaf of regular functions on X with zeros on D. You can take α to be the inclusion. This will define a log structure.

[picture]

The monoids will look like this, the stalks will look like this, and how does this relate to the natural numbers from the other picture? One thing you can do is define another sheaf by getting rid of your invertible functions (we call this the ghost sheaf) and then map the monomials to their powers. So what you do is, a log structure has two parts, some discrete data, the ghost sheaf $\overline{M} = M/\mathcal{O}_X^*$ which encodes all the data of your tangency orders. So for instance you can take $y^a \mapsto a$ in this picture.

So the discrete data encodes the incidence convolutions of the Gromov–Witten invariants.

Whenever you have a tropical object, a tropical curve, this totally determines the discrete data. Whenever you want to define a count of log curves, the morphisms on the level of ghost sheaves lift to the discrete level. So you can include non-discrete data, whenever you have κ the quotient morphism $M \to \overline{M} = M/\mathcal{O}^*$, then you can look at $\kappa^{-1}(\overline{m})$ which is an $\mathcal{O}_U^*$-torsor. So you can look at lifts to normal log structures by counting these torsors.

So let me look at the Tate curve as an example of how we do these counts and how this relates to symplectic cohomology.

The Tate curve, how we construct this historically, B has no singularities, we take some decomposition like [picture] and then (this is a standard construction of Mumford) I take the cone of the integers, and then I get the unfolded Tate curve, and then when I fold I get the Tate curve. So this has a height function to $\mathbb{A}^1$, and this has a degeneration, which has general fiber, you read this off by looking at the generic preimage of a point in $\mathbb{A}^1$. [rationale]. So this has general fiber $\mathbb{C}^*$, and then you look at an arbitrary point in the ray and look at the preimage under the height function, and you see how they're related to each other, like this, and so the singular fiber is an infinite chain of $\mathbb{P}^1$. So this has a $\mathbb{Z}$-action, so that you obtain $T_t = \mathbb{C}^*/z \sim vz$. The translation in the unfolded Tate curve corresponds to [unintelligible]. The dual of each cone is of the form $\mathrm{Spec}[x,y]/x, y - u$. Then you can look at how things glue together here. If you shift, the shifting will correspond to multiplying by u. This corresponds to a $\mathbb{Z}$-action on the total space. You have to take the quotient This is obviously not properly discontinuous, and what you do is, you can do one of two things. You can either restrict to the unit disk and look at an analytic Tate curve, or take some formal completion around zero. This formal option is better if you want to relate to the Fukaya category. A good reference for this was written by Mark Gross' Clay book. So today I will talk only about the analytic Tate curve for simplicity. After the quotient is just an elliptic curve, and the central fiber, if you take the quotient, you get a nodal elliptic curve. If you took, say three shifts, you could get three copies of $\mathbb{P}^1$; if you shift only once you get a nodal elliptic curve.

Okay, so the general fiber elliptic curve T_0, the nodal elliptic curve T_t, then the aim is to understand the symplectic cohomlogy $T \setminus T_0$, which is the mapping cylinder of a Dehn twist τ. To understand the symplectic cohomology, our motivation was to look at the Fukaya category of the elliptic curve, and there's, if you [pictures]. Then there's a ring $HF(L, \tau^k L)$, where τ is the Dehn twist, this is supposed to give motivation, the symplectic cohomology is expected to be isomorphic as a ring to $\bigoplus HF(L, Z^k L)$, this is joint with Tonkonog and recently Pomerleano. The idea is to go to the wrapped Fukaya category.

This isomorphism will respect the ring structures, and we expect this to upgrade to an A_∞ quasi-isomorphism.

What I'd like to talk about is how log Gromov–Witten invariants appear on $HF(L, \tau^k L)$. There's work of Abouzaid and Gross and Seibert in terms of tropical Morse trees. Their tropical Morse category has objects integers, hom spaces $\frac{1}{n_i - n_j}\mathbb{Z}$-points of S^1. If you look at $S^1 \times S^1$ then the Lagrangians can be characterized by their slopes, and we look at the Lagrangians which are not good enough to determine the Fukaya category. We can just take the integers as the objects. The hom spaces are then given by the [unintelligible]of the Lagrangians. So the hom spaces, if you have a Lagrangian of slope 2 and 4, this corresponds to, $\ldots \frac{-1}{2}, 0, \frac{1}{2}, 1, \ldots$, all the intersections. The structure coefficients and product structure is given by tropical Morse trees. To determine structure coefficients you want to count things bounding Lagrangians. How do you define such tropical objects? You take some ribbon graph, a tropical Morse tree is a map from your ribbon graph to S^1, let me call it ϕ. The conditions, you'll put integers on the areas bounded by the ribbon graph. You put differences on the edges. Then you put conditions to make this a a triangle. You need to have positive acceleration on the leaves. You need negative acceleration on the root. [pictures]

So now you want to use these to write more ordinary tropical objects corresponding to these. What we did so far in trying to understand SH^* of the mapping cylinder, that we have the same product structure on $\mathrm{Fuk}(E)$, where the product structure coefficients is given by tropical Morse trees. I lift these to *tropical corals*, which are standard curves, and wrote down a relationship to certain punctured log Gromov–Witten invariants, that's my thesis works.

Maybe in the last ten minutes I can say how this works.

You have tropical objects. You lift tropical Morse trees in radial directions [picture]. At the end, the correspondence is not one to one, there's a one parameter family, obtaining a tropical object. In general you can have many stops. How do you write the correspondence, these live on the central fiber of the degeneration of the Tate curve, because, the whole reason, [missed some], a further degeneration of the total space, take the truncated cone, and define [unintelligible]by putting this in $\mathbb{R}^3$ and take the cone of the truncated tone, this will define $\Sigma_{\tilde{T}} \to \mathbb{A}^2$. The general fiber $\tilde{T}_t$ will be the Tate curve. The central fiber $\tilde{T}_0$ is $T_0 \times \mathbb{A}^1$. When you define tropical objects at height one, the main idea is to define some tropical objects at height one, the height one part determines the central fiber and the tropical objects deform to the general fiber. The corals give some log curves here. These degenerate to give solutions [unintelligible]Tate curve itself. These components, these vertices, correspond to, if you have this tropical curve [picture] and want to find all log curves with this as the tropicalization, for these you put $\mathbb{P}^1$ and for the stops the $\mathbb{A}^1$, these correspond to marked points. Then to define a map to your target [pictures] and the $\mathbb{A}^1$ components map to constants and don't carry additional information, so you can contract them and put some extra data, the punctured points. These are the "punctures" of the Gromov–Witten invariants where you carry negative powers. If you deform it you have cylinders with positive and negative ends. My main theorem was to define these invariants for the Tate curve and show that you can define a count which is finite and can be given in terms similar to the [unintelligible]count.

Theorem 4.1. *(A.) The number of log corals = the number of tropical corals.*

[missed some.]

5. Bin Zhou: Optimal regularity of plurisubharmonic envelopes on compact Hermitian manifolds

It's a pleasure to speak here and I'd like to thank the organizers for the invitation. This is a joint work with Jianchun Cho. Essentially it's a PDE problem. Let me start with the notion, let me start from the linear theory. The first envelope, the Perron–Bremerman envelope. If you have Ω a domain in $\mathbb{R}^n$ and a function φ then you can define the envelope as the supremum

$$u_\varphi \coloneqq \sup\{u \in \mathrm{SH}(\Omega) | u \leq \varphi \text{ on } \partial\Omega\}.$$

It's well known that this satisfies the equations

$$\begin{cases} \Delta u_\varphi = 0 & \Omega \\ u_\varphi = \varphi & \partial\Omega. \end{cases}$$

You can also consider the *subharmonic envelope*

$$u_\varphi \coloneqq \sup\{u \in \mathrm{SH}(\Omega) | u \leq \varphi \text{ on } \Omega\}.$$

Now we can do the nonlinear version. Let $F(D^2 u) = f$, and we can define

$$u_\varphi \coloneqq \sup\{u \text{ is } F-\text{subharmonic in viscosity sense} | u \leq \varphi \text{ on } \Omega\}$$

and the convex envelope

$$u_\varphi \coloneqq \sup\{u \text{ convex}, \det D^2 u \geq f(x) | u \leq \varphi\}$$

For the convex envelope, Caffanelli proved that if $\varphi \in C^{1,1}(\overline{\Omega})$ and $f \in C^0(\Omega)$ then $u_\varphi \in C^{1,1}_{\mathrm{loc}}(\Omega)$. This result plays an important part in the regularity theory for the real [unintelligible]equation. Here for the envelopes the optimal regularity cannot be better than $C^{1,1}$. Let me give an example.

Example 5.1. Let $\varphi(x) = (x^2-1)^2$ for $x \in [-2,2]$. The graph looks like this: [picture]

The envelope looks like this, and it's not differentiable at the points $x = \pm 1$. You can take $\varphi'(x) = 2(x^2-1)x$ and $\varphi''(x) = 6x^2 - 2$. So this u_φ is not C^2.

De philipippis–Figali in 2015, they considered boundary regularity. When Ω is uniformly convex. The closure of the boundary is [unintelligible]positive and at least C^2. If you have a boundary function which satisfies $\varphi \in C^{3,1}(\overline{\Omega})$ and $f = 0$, then you can prove that $u_\varphi = C^{1,1}(\overline{\Omega})$. This is for the convex envelope.

In this talk we are mainly concerned with teh *pleurisubharmonic envelope*. Let (M,ω) be a Hermitian manifold, and f is a given function. Here I will change a little the symbols, for Kähler geometry we usually use φ for potential functions.

$$\varphi_f \coloneqq \sup\{varphi \in \mathrm{PSH}(M,\omega) | \varphi \leq f\}.$$

Here you can of course study the regularity problems.

(1) The first result is from Berman–Demailly, when ω is Kähler and $f \in C^\infty$, then φ_f is in $C^{1,\alpha}(M)$ for some $0 < \alpha < 1$, strictly less than one. A natural conjecture

Conjecture 5.1. $\varphi_f \in C^{1,1}(M)$.

(2) (Berman) When $[\omega]$ is an integral class, φ_f is in $C^{1,1}(M)$. Then Ross–Nystrom extended this, you can consider envelopes with prescribed singularities, and also have $C^{1,1}$-regularity outside the singular points. They needed the integral assumption because of the Bergman kernel which they used to do this.

You can also treat this problem as a PDE problem, it is an obstacle problem. To deal with obstacle problems, you have the "penalty method." A good reference is given by K. A. Lee, who is from Korea. You can construct a family $\beta_\epsilon(t)$ a smooth family of functions which satisfies $\beta_\epsilon(0) = 1$, it is always positive, and $\beta_\epsilon(t) \to 0$ as $\epsilon \to 0$ for negative t while $\beta_\epsilon(t) \to \infty$ for positive t. Also $\beta_\epsilon'(t) \geq 0$ and $\beta_\epsilon''(t) \geq 0$.

Why is this the "penalty method?" We can consider $F(D^2 u_\epsilon) = \beta_\epsilon u_\epsilon - \varphi$ on Ω and $u = \varphi$ on $\partial\Omega$. This is the "penalized problem" (equation). The idea is if you can get a good estimate for this problem, you should get $u_\epsilon \to u_\varphi$ as $\epsilon \to 0$.

This becomes a PDE problem now. Berman used this $\beta_\epsilon(t) = e^{\frac{1}{\epsilon}t}$.

So if you consider the equation

$$(w + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{\frac{1}{\epsilon}(\varphi_\epsilon - f)}, \qquad \epsilon > 0 \tag{5}$$

(1) Equation (5) is solvable for any $\epsilon > 0$ (Yau–Aubin in the Kähler case and [unintelligible]in the Hermitian case).
(2) $\varphi_\epsilon \xrightarrow{C^{1,\alpha}(M)} \varphi_f$ $(\Delta\varphi_\epsilon \leq C)$

Theorem 5.1 (Z.–Chu)**.** *(M, ω) is Hermitian and $f \in C^{1,1}(M)$ then $\varphi_\epsilon \xrightarrow{C^{1,1}(M)} \varphi_f \in C^{1,1}(M)$ as $\epsilon \to 0$. This regularity is "optimal."*

Remark 5.1. Tossati proved this at the same time, but just for the Kähler case. The main ingredient is the a priori estimate for the second derivative. Dinew, earlier this year, considerd the domain case. This was withdrawn, so maybe there were problems. So that case may still be open.

Before I give the proof of the theorem, let me first show you some examples to indicate why it is optimal.

First, you can consider a local example. This is very similar to the real case. Consider a ball $B_2(o) \subset K$. Then $f = (|z|^2 - 1)^2 \in C^\infty(B_2)$. Then $\varphi_f \in C^{1,1}(B_2(0)) \setminus C^2(B_2(0))$.

We can also get a global example on $\mathbb{P}^1$ with the Fubini–Study metric. So let $[z_0, z_1]$ be homogeneous coordinates and let U and V be the coordinate charts, $U = \{[1, z_1]\}$ and $V = \{[z_0, 1]\}$. We just need to extend the local example to the global manifold. Here we write

$$f = \begin{cases} (|z_1|^2 - 1)^2 - \log(|1 + z_1|^2) & U_0/\{[1, z_1]||z_1| \leq \frac{5}{4}\} \\ \tilde{h}(|z_0|^2) - \log(1 + |z_0|^2) \end{cases}$$

If you write the first function, the graph looks like this [picture]. You cut this to get h and modify it to get something smooth. You need to choose the domain appropriately so that you can glue the two parts together.

Okay, with this construction

Proposition 5.1. $\varphi_f \in C^{1,1}(\mathbb{P}^2) \setminus C^2(\mathbb{P}^2)$

because the envelope looks the same as in the local example.

Now we go to the proof.

In other words, we just need to consider Equation (5). You need the a priori estimate independent of ϵ. For the first step it will be a C^0-estimate. I'll just talk about the Kähler case. The Hermitian case the idea is the same but the computation is more complicated.

Proposition 5.2.

$$\max\{\varphi_\epsilon - f\} \leq C\epsilon$$
$$\min\{\varphi_\epsilon\} \geq \min f.$$

Proof. Easy. Look at $\varphi_\epsilon - f$. The maximum is at at p means that

$$\sqrt{-1}\partial\bar\partial\varphi_\epsilon(p) - \sqrt{-1}\partial\bar\partial f(p) \leq 0$$

so that

$$\varphi - f \leq \epsilon \log \frac{\omega + \sqrt{-1}\partial\bar\partial f}{\omega^n} \leq C\epsilon.$$

For the minimum,

$$\varphi_\epsilon - f = \epsilon \log \frac{(\omega + \sqrt{-1}\partial\bar\partial\varphi_\epsilon)^n}{\omega^n} \geq 0.$$

□

Corollary 5.1. *Then* $\lim_{\epsilon\to 0}\varphi_\epsilon = \varphi_f$

Proof. For any pleurisubharmonic $u \leq f$, define u_ϵ as $(1-\epsilon)u + \epsilon(\log \epsilon^n + \min f)$. Then

$$(\omega + \sqrt{-1}\partial\bar\partial u_\epsilon)^n \geq e^{\frac{1}{\epsilon}(u_\epsilon - f)\omega^n}$$

so

$$u_\epsilon \leq \varphi_\epsilon \xrightarrow{\epsilon\to 0} \lim_{\epsilon\to 0}\varphi_\epsilon \geq \varphi_f.$$

□

Okay, now for blowup.

Theorem 5.2 (Approximation)**.** *For* (M,ω) *Hermitian, for any* φ *pleurisubharmonic there are* $\{\varphi_i\}$ *smooth and pleurisubharmonic so that* $\varphi_i \searrow \varphi$.

Let me give names, Demailly, Blocki–Kolodziej, Berman, Kolodziej–Nguyen.

This PDE approach, this idea can also be used for $F(D^2u, Du) = 0$, so if you have an elliptical operator, then for subharmonics for this operator, you can always smooth it by this approach. For example,

$$H_k(u) = S_k[D\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)]$$

the k-curvature, this is a good place to use it.

Okay, now the second derivative estimate. Write $g \sim \omega$ if $\tilde g \sim \omega + \sqrt{-1}\partial\bar\partial\varphi_\epsilon$.

So first Berman gets a Laplacian estimate $\sup|\Delta\varphi_\epsilon| \leq C$. We'll go further and show that the real Hessian is bounded.

Proposition 5.3.

$$\sup|\nabla^2\varphi_\epsilon| \leq C$$

with C *independent of* ϵ.

With this estimate you can get $C^{1,1}$. Okay, the proof is complicated so let me focus on some major points.

Major points. From the Laplacian bound $\Delta\varphi_\epsilon \leq C$, we get $|\nabla^2\varphi_\epsilon| \leq C\lambda_1(\nabla^2\varphi) + C$ where λ_1 is the largest eigenvalue of $\nabla^2\varphi$.

So now the proof is to do the computation with $Q = \log\lambda_1(\nabla^2\varphi) + h_D(|\partial\varphi|^2 g) + e^{-A\varphi_\epsilon}$. Here A is a constant to be determined and $h_D^{(S)}$ is $-\frac{1}{2}\log(D+\sup|\partial\varphi|^2 g - s)$.

Then there is a problem when you want to differentiate Q, because λ_1 is the largest eigenvalue of the Hessian, but it may not be smooth. If the eigenvalue is multiplied by 2, then this is not smooth. You need some more trick. The real eigenvalues are decreasing

$$\lambda_1(\nabla^2\varphi) \geq \cdots \geq \lambda_{2n}(\nabla^2\varphi)$$

with eigenvectors $V_1, \ldots, V_n$. Assume Q attains its maximum at p. Choose coordinates near p. Then you can extend $\{V_\alpha\}$, the eigenvectors, in the coordinates, by taking the components to be constant.

I should say that I have to take this at the point p, and the V_α is also with respect to the eigenvalue at p.

Define Φ^α_β by $g^{\alpha\gamma}\nabla^2_{\gamma\beta}\varphi - g^{\alpha\gamma}B_{\gamma\beta}$ where $B_{\alpha\beta} = \delta_{\alpha\beta - V_1^\alpha V_1^\beta}$.

Write Φ for the matrix (Φ^α_β), and then $\lambda_1(\Phi)$ is strictly larger than $\lambda_2(\Phi)$ (although this does not stay the same for smaller eigenvalues). Moreover V_1 is still an eigenvector. We also have $\lambda_1(\Phi) = \lambda(\nabla^2\varphi_\epsilon)$ at p and $\lambda_1(\Phi) \leq \lambda_1(\nabla^2\varphi_\epsilon)$ near p.

We actually do this computation with respect to this Φ.

Now you can replace Q by $\tilde{Q}$, which is $\log\lambda_1(\Phi) + h_D(|\partial\varphi_\epsilon|^2) + e^{-A\varphi_\epsilon}$ and you just need to compute the Laplacian of $\tilde{Q}$ at this point p and you get

$$\begin{aligned} 0 &\geq \Delta_{\tilde{g}}\tilde{Q} \\ &= \frac{\Delta_{\tilde{g}}\lambda_1}{\lambda_1} - \frac{\tilde{g}^{ii}|\partial_i(\lambda_1)|^2}{\lambda_1^2} + h'_D(\Delta_{\tilde{g}}|\partial\varphi|^2 g) \\ &\quad + h''_D\tilde{g}^{ii}|\partial_i|\partial\varphi||^2 - Ae^{-A\varphi}\Delta_{\tilde{g}}\varphi \\ &\quad + A^2 j^{-A\varphi}\tilde{g}^{ii}|\varphi_i|^2 \end{aligned}$$

So then you get $V_1V_1(\frac{1}{\epsilon}(\varphi_\epsilon - f))$ greater than or equal to $\frac{1}{\epsilon}(\lambda_1 - ||f||c^2)$ and then

$$2\Re(\frac{1}{\epsilon}(\varphi_i - f_i)\varphi) \geq -\frac{1}{\epsilon}(3\sup|\partial\varphi_\epsilon|^2_g + ||f||^2_{c'})$$

and for D large enough we get

$$\frac{1}{\epsilon}(\lambda_1 - ||f||_{c'}) \geq \frac{h'_D}{\epsilon}(3\sup|\partial\varphi_\epsilon|^2_g + ||f||^2_{c'}).$$

□

I'd better stop here.