POSTECH SPECIAL LECTURE, SUMMER 2012

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1. Joey Hirsh: What is a homotopy [something]?

I never gave this talk before and have been thinking about these things for a while. There may be better people to talk to you about these things. There are many people who think only about categories, stop me if I'm not clear.

What is a homotopy something? What are the somethings I have in mind? People often say they want to do concept X up to homotopy or homotopy blah. We can make a list

- homotopy (co)limits
- homotopy mapping space (or derived mapping space)
- homotopy universal property
- homotopy adjunction
- homotopy Kan extensions
- homotopy theory

I heard people give talks in which all these concepts come up. Maybe I'll pick "homotopy mapping space." You write down a mapping space, a cofibrant replacement functor, and then you have a homotopy mapping space.

A derived mapping space, you replace $\mathcal{O}$ with $\mathcal{O}_\infty$ in Gabriel's talk and then you get a different kind of mapping space.

Homotopy limits and colimits, for example in a model category, you can move around a diagram and take a strict limit. Or you can fatten something up when you have an enrichment and use this to build a homotopy limit.

The answer to all of these questions come from homotopy theory. Once you make sense of what homotopy theory is abstractly, you can move all these concepts into homotopy theory from category theory in a coherent way. That's maybe what I'll focus on.

I was supposed to tell you in the beginning, I might not answer all of these questions in an hour.

So what is a homotopy theory? To answer this question, I need to ask first, what is a theory? Since I assume that people know what a category is:

Definition 1.1. *A* theory *is a category.*

A homotopy theory is a category like structure with objects and relations and maybe some extra stuff too. The main example of what we have at hand, we take this and try to abstract.

So let's look at the example of topological spaces. Maybe I'll ask the audience what makes this thing a homotopy theory? One way is to say that it's a model category. I don't really prefer these. Maybe it's a category with weak equivalences, a relative category. I want to think of homotopy theory as having more structure, some kind of category-like thing.

[Is a model category the classical answer?]

A qualified yes. Whatever I end up with should be a category-like structure, we would want, maybe, a tensor product, and so for example, spectra had no model categorical tensor product. It's wonderful to have a presentation as a model category but sometimes you only have something more primitive.

I'll say model category is to homotopy theory as a presentation is to an algebra. You'd better have generators and relations but this is not the right way to define things.

Presheaves on a model category with values in spaces have many model category structures.

So back to a relative category, which is a category with a subcategory of weak equivalences $(\mathcal{C}, W)$. A morphism between these is a functor which takes W to W. So for example you could take spaces with weak equivalences.

Maybe I should take this to be a category with homotopies. There are multiple ways to do this. One way to get the homotopy relation is to say $Map(X, Y)$ is a topological space, I can take π_0 of that. Then π_0 tells me which maps are homotopic. I can take a category enriched in spaces. You have a space of morphisms, not just a set.

An object is an object, a morphism is a relation between objects, there are homotopies between morphisms, and between homotopies and between higher homotopies. This then is some sort of ∞-category. These aren't arbitrary. You can always rewind a picture. Homotopies are somehow always invertible. A category with higher morphisms, all of which are invertible.

I have objects, morphisms, homotopies, higher homotopies, and so on, and starting with the homotopies everything is invertible.

Let's say I want to define a two-category, I want the morphisms and two morphisms to compose associatively, that's one option, or I could say that the one dimensional morphisms are associative only up to a two-morphism. I could say something weaker. So I could do the same thing with ∞-categories.

I could get a category out of these by collapsing the two-morphisms. That is the homotopy category of an $\infty - 1$ category.

Take the category of $\infty - 1$-categories. There is a functor from this to regular categories called "π_0" or "homotopy category." You could also take π_0 from a category enriched in Top. You replace a space of maps with π_0. That's like nothing.

So some relationships between these things, we have a functor $Model \to RelCat$ that I can call "forget." Another way to go from a topological category to a relative category would be to, take the the space of maps, I can forget to a set of maps, and then my weak equivalences will be, okay, if you give me a topological category I can take π_0, and I can look at the isomorphisms in $\pi_0(\mathcal{C})$, I have to say a few more words. You have a functor taking $Map(X, Y)$ to its components, and the preimage of isomorphisms, these are the weak equivalences.

Let's relate $\infty - 1$-categories to topological categories. The way I'll do this is the following. I'd like to justify an equivalence between $(\infty - 1)$-categories and topological categories. This is called the homotopy hypothesis, or fallows from it, so I thought I'd say a few words about that.

The homotopy hypothesis says that spaces are in some sense equivalent to weak ∞-groupoids. An $(n - k)$-category has n levels of morphisms, and above k everything is invertible. Weak here, I mean associativity is only true up to higher morphisms. Unital relations are also only weak. So I can rewrite this as an $(\infty, 0)$ category.

Strict means that associativity is true on the nose at each level. Weak means that associativity is true up to higher levels. If you only have one level, these are the same.

When I have a category like thing, the right hand side has morphisms, I can think about natural transformations. Categories, the collection of all of them have two categories. Secretly our objects have stuff inside so I can define natural transformations between functors. So Cat is a strict 2-category. You can have a formal definition that a strict n-category is a category enriched in strict $(n-1)$-categories.

So any time you take a category of (n, k)-categories, it'll be an $(n+1, k+1)$-category.

So this equivalence of spaces and $(\infty, 0)$-categories is an equivalence of $(\infty, 1)$-categories.

Spaces are an ∞-1 category where objects are spaces, morphisms are continuous maps, 2-morphisms are homotopies, and higher morphisms are homotopies of homotopies.

Let's say more about this homotopy hypothesis.

Definition 1.2. *A space X is a homotopy n-type if for every basepoint and every $k > n$, $\pi_k(X, x_0)$ is 0.*

This filtration is not exhaustive but every space is a homotopy ∞-type.

I can take the fundamental groupoid functor $\pi_{\leq 1}$ from spaces to groupoids. Objects are points of X and morphisms are homotopy classes of paths. If you pick an object in this category, the automorphisms are the fundamental group at that point.

We have another functor, we can try to undo this process in a universal way. Given a groupoid, we can construct a space BG which, I'll make a cell complex. Put down a point as a zero-cell for every object in G. Put a one-cell for every morphism in G, and then a 2-cell for every pair of composable morphisms, and so on. For every n composable morphisms, I attach an n-simplex.

A fact is that $\pi_{\leq 1} BG \cong G$ and BG is a homotopy one-type.

Then B factors through 1-types. And 1-types are equivalent to groupoids. Maybe I can show that 2-types are the same as 2-groupoids and so on. So the hope, naively, is that n-types are equivalent to strict n-groupoids by $\pi_{\leq n}$ of $\infty, 1$-categories or something.

That's not good enough, and here's the reason why. The first obstruction is, if we try to make $\pi_{\leq 2} X$, a two-groupoid, it's not a strict two-groupoid. Well, if you want to, what did we do for $\pi_{\leq 1}$? For a two-morphism to be a homotopy, you need morphisms to be paths, not homotopy classes of paths. So then the two-morphisms here are homotopy classes of homotopies.

Composition of paths is not a strict two-groupoid. This isn't that bad of an obstruction. You can still take 2-types, 2-groupoids, strict ones, and still write down B, and it's still an equivalence, you can do something here. There's a real obstruction at level three.

Fact 1. *3-groupoids and 3-types, B is not an equivalence.*

If you take, well, take S^2, every space has an associated n-type. So $(S^2)_3$ cannot be in the image of B for any groupoid. So this is not equivalent to BG for any strict 3-groupoid G. In $(S^2)_3$ you have the Hopf fibration, and BG should have a simple Postnikov tower. This is like maybe a product of Eilenberg MacLane spaces or maybe not quite that but certainly nothing like the Hopf fibration.

You can see why it should work in general. This failure tells you that you should replace strict with weak and it should work.

Let me show you this being tautological or not in the case $n = \infty$ This is the same as the equivalence of Top and $sSet$ (or Kan complexes).

2. Deformations with noncommutative parameters

I'm trying to let go of not having finished the last one. So in this talk I want to talk about moduli spaces, or think about them, and the reason why is that moduli spaces tell you things about objects that you're trying to study. You have some objects in a category, and you want to give them topology. You want to give them the structure of an algebro-geometric space. You want to think that this is the structure of a ring, more or less. Maybe a scheme. Suppose you're in that context, if you have a category of objects, it's hard to write down a moduli space. The best family that your object lives in, this could be hard because it's impossible. To address this problem, you use the following categorical yoga. To define an object in $\mathcal{C}$, you could write down an object. But you could also, given an object X in $\mathcal{C}$, you should know what $Hom(X, M)$ is.

The minimum you need to define M is to map into M.

The Yoneda lemma tells us that if M exists, it is determined by $F = Hom(\quad , M)$.

So to find your moduli space you can write a functor from the opposite category of algebraic gemoetric spaces, and that's like rings. Functors like this are determined by functors from unital commutative rings into sets.

This motivates the beginning of moduli space theory.

Definition 2.1. *A* commutative moduli problem *is a functor F from commutative k-algebras to sets.*

The problem is to find a space M so that F is isomorphic to $Map(\quad , M)$. This has two parts, existence and construction. There is a tradition of answering these representability problems, the first type, in category theory.

In the last talk the speaker promised to tell you what a homotopy thing is. I want to talk about proving some existence theorems. I want to modify things. I want to prove things only locally. I mean this to be formally around a point. If M is a space, pick a point x and working formally means you take smaller and smaller neighborhoods until you see jets, but infinitessimal neighborhoods basically.

The history of moduli spaces says to understand them, you can't hope to do it in this naive way, you need to work up to homotopy.

Theorem 2.1. *The moduli space of a structure with automorphisms, isomorphism classes is not representable.*

You might hear that the functor from rings to sets, isomorphism classes of elliptic curves over R is not representable, because some elliptic curve has an automorphism. Every bundle should be trivial, and an automorphism gives you a non-trivial bundle.

Historically the solution is to pass from functors from commutative algebras to sets to instead use functors from commutative algebras to groupoids. Then we can encode automorphisms. Then you could try to tell when these are representable, stacks or something, this is tricky or hard to, people do work to give theorems where stacks behave geometrically. It turns out that there is a coherent homotopy theory approach. So groupoids are a subcategory, they're the same as 1-types, which sit inside ∞-types, which are spaces. I can encode all higher obstructions in spaces. In the last talk you saw how to turn any adjective into a homotopy version. I can really translate all of these concepts. It's not clear what you're doing when you stop at *Gpd*. Taking it to spaces, you get expected results. I'll prove theorems about functors from commutative algebras into simplicial sets.

In classical geometry it's hard to work locally, but in algebraic geometry, you can work locally easily, using a subcategory called commutative Artin algebras or Artin rings. These are the things that only see locally. Then:

Definition 2.2. *A* derived commutative deformation or local moduli problem *is a functor from differential graded commutative Artin algebras to simplicial sets.*

The problem is to find a representing ring M.

We have the following definition:

Definition 2.3. *The* tangent space *of a derived commutative deformation problem F is defined to be the value it takes on $k[\epsilon]/\epsilon^2$.*

Proposition 2.1. *TF is a chain complex.*

I suppressed, I want functors that have one or two small conditions that would follow from representability, called Schlessinger's criteria. You need this for this proposition. It lands in simplicial Abelian groups, and these are actually vector spaces, and you can say:

Proposition 2.2. *(in characteristic zero) TF is naturally an L_∞ algebra.*

I've jumped a little bit. Let me say just a word about where this L_∞ structure comes from. If we look at the tangent space of a curve that intersects itself, it's two dimensional. The bracket, I have some directions that are off the space. The bracket measures whether there's second order, the triple bracket checks to third order.

So I have a short exact sequence of rings:

$$ts \to k[t,s]/(t^2,s^2) \to k[t]/t^2 \oplus k[s]/s^2$$

and there's some sort of obstruction map that gives the bracket.

[Is there more structure from being a simplicial set rather than just a set?]

Theorem 2.2. *(Schlessinger, Deligne, Manetti)*
There is a functor "Maurer Cartan" from L_∞-algebras to derived commutative deformation problems, which takes L to the functor which takes R to the set of Maurer-Cartan elements in $L \otimes R$. You might say this isn't a simiplicial set, so take for your n-simplices the Maurer-Cartan solutions in $L \otimes R \otimes \Omega\Delta^n$

For any deformation problem F, that MC_{TF} and F are equivalent on π_0, isomorphic as functors from differential graded Artin rings to sets.

If you know a little higher category theory and Manetti's theorem, you can rephrase it as:

Theorem 2.3. *(Manetti)*
MC is an equivalence of $\infty - 1$ categories, with the construction above as an ∞-inverse.

The L_∞ structure on TF is amorphous. The construction takes a differential graded Lie algebra and gives a deformation problem, and going back is like giving you an L_∞ structure on homology. This isn't functorial but passing to the homotopy category it is.

Essentially this says that L_∞ algebras are representing, every functor is equivalent to one of these coming from an L_∞ algebra. Also, you might complain, you're taking Maurer-Cartan, you're not representing. Let me say why Maurer-Cartan is like representing a functor.

A Maurer-Cartan element is a map from k to sL. Given L, there's a natural object associated to sL, called BL, the cofree cocommutative coalgebra on sL, with a differential from the bracket of L. Because this is cofree, and k is a coalgebra, we can lift this to a coalgebra map e^γ to BL. We'd like this to be a dg coalgebra map. This is $1 + \gamma + \frac{1}{2}\gamma \wedge \gamma + \cdots$

So e^γ is a differential graded map if $D(e^\gamma(1)) = 0$, which is true if and only if γ is a Maurer-Cartan element. So $MC_L(R)$ is equivalent to differential graded coalgebra maps from $R^\vee$ to BL. So this is really a representability result.

Now I guess sometimes it makes sense to work over commutative rings. But maybe if you're a physicist you want your moduli space to have more structure, maybe it's a quantum field theory. This should apply, the Yoneda lemma, to more categories or more situations. The moduli space, if it has symmetry or structure, you'd expect the representing object to have features that reflect that. You have differential graded commutative Artin rings, you could look at this inside differential graded Artin rings. Suppose your functor is robust enough to extend to this more complicated setting in a complicated way. So what if you have P-Artinian algebras for an operad P, with commutative Artinian algebras inside of them.

The answer is yes, and here is a theorem.

Theorem 2.4. *(Hirsh) There is a Maurer-Cartan functor from $P^!_\infty$ algebras to derived P-deformation problems. This functor is an equivalence of $\infty, 1$-categories, with the tangent space construction as an inverse in the homotopy category. So you can interpret an extension of the L_∞ structure to a $P^!_\infty$ structure.*

An application is to give Deligne's conjecture. I don't have time to say this. An associative algebra, its deformations are an L_∞ algebra, it's actually an E_2-algebra, a G_∞-algebra, and it's the universal such thing that acts here on the associative algebra A. So you plug in E_2 which is self-dual. So send an E_2 algebra to all the ways it can act on A. You can show that these commute. That shows that the Hochschild L_∞ structure extends to the E_2 one. I think of string topology operations on the based loop space, as coming from these things. One more word, I keep getting distracted, we needed characteristic zero for Com and Lie. But for many P you don't need characteristic zero. So associative deformations have representing objects in characteristic p.