PMI–POSTECH LECTURE SERIES
FACTORIZATION HOMOLOGY AND NUMBER THEORY OVER FUNCTION FIELDS

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Thank you very much for the invitation. This is my first time in Korea and I'm happy to be here.

I'll talk about factorization homology and applications to number theory over function fields. There are applications in many fields of mathematics, but factorization homology has its roots in physics. I want to translate the number theoretic problems to algebraic geometry and then use ideas from factorization homology.

Let me start with the classical motivation and then quickly move to a geometric picture and then formulate it in terms of moduli spaces of principal bundles and show how factorization homology can be used on it.

So let me start with the Tamagawa number 1 conjecture (Weil). Roughly it says the following. Let G be a semi-simple simply connected algebraic group over $\mathbb{Q}$. So most intents and purposes you can think of $G = SL_n$. Let me introduce some notation. Let $\mathcal{O}$ be the product $\prod \mathbb{Z}_p \times \mathbb{R}$, the integral adeles, so $\mathbb{Z}_p$ is called the "finite places" and $\mathbb{R}$ is ∞. Then $\mathbb{A}$ is defined as $\prod' \mathbb{Q}_p$, meaning (v_p) such that v_p is in $\mathbb{Z}_p$ for all but finitely many primes.

The Tamagawa measure on $G(\mathbb{A})$ is $\prod' G(\mathbb{Q}_p)$, and this is some canonical measure on this group μ_{Tam}, some kind of product of Haar measures, and the conjecture is that $\mu_{\mathrm{Tam}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is one.

So this is a theorem by Langlands, Kottwitz, Chernousov, over number fields using trace formulas (so analytic), but only recently proved by Gaitsgory–Lurie using factorization homology for function fields (this is geometric and categorical).

I'll restate what the conjecture says in the function field setting. I'll replace $\mathbb{Q}$ with rational functions on a curve over a finite field $\mathbb{F}_q$, so X will be smooth, projective, geometrically connected, and G will be an algebraic group over X. This is a bundle of groups where each fiber is an algebraic group. To make things simple we'll assume that G is a trivial fibration. What they prove is more general but for us G will be a constant family.

I'll also assume, again, that G_0 is semisimple and simply connected. So now $\mathcal{O}$ will be the product $\prod \mathcal{O}_x$, where $\mathcal{O}_x$ is non-canonically isomorphic to $k[[u]]$ where $k = k(x)$. The adeles will be $\prod' K_x$, where $K_x = X((u))$ and the prime means it's integral for all but finitely many and K is the rational functions.

So as in the number field setting we have a Tamagawa measure μ_{Tam} on $G(\mathbb{A})$ and again the conjecture is that $\mu_{\mathrm{Tam}}(G(K)\backslash G(\mathbb{A})) = 1$.

So roughly up to a normalization, μ_{Tam} is the product of μ_x.

The simplest picture, the local picture, is $\mathbb{A}^1(K_x) = K_x$. For this part, there is a canonical measure

$$\mu_x(\mathbb{A}^1(\mathcal{O}_x)) = \mu_x(\mathcal{O}_x) = 1$$

So we know $\mathcal{O}_x/m_x = k(x)$ so $\mu_x(m_x) = |k(x)|^{-1}$. So now suppose Y is some variety, smooth, over $\mathcal{O}_x$, then in particular you have, you want to associate, to get a measure on $Y(K_x)$? This is some p-adic variety, so just like in the case of manifold, to integrate on a submanifold first you have to pick a top form. So the input is a top form on Y, let me say Y_{K_x} is the generic fiber of Y over y. You need a top form on Y_{K_x} and using that you can get a measure.

But when Y is over $\mathcal{O}_x$ you can ask for something better, a top form on Y, and then we get a measure with very good normalization. Then $\operatorname{vol} Y(\mathcal{O})$ is the number of points of the residue field $|Y(k(x))|$ divided by $|k(x)|^{\dim Y/\mathcal{O}_x}$.

Let's specialize to the case where Y is $G/\mathcal{O}_x$. You don't want a random top form, you want one that's left-invariant with respect to the group G. You still have the equality

$$\mu(G(\mathcal{O}_x)) = \frac{|G(k(x))|}{|K(x)|^{\dim G/\mathcal{O}_x}}.$$

That's the local picture, so now let's define the global guy, try to piece them up. Some expressions may not converge in real life, so we'll have to say something to make sure this is well-defined.

So define the *unnormalized* Tamagawa measure, $\mu^{\mathrm{un}}_{\mathrm{Tam}}$ to be the product

$$\prod_{x\in|X|} \mu_x.$$

I'll use the notation that $\mathbb{G}_a$ is the additive group $\mathbb{A}^1$ and $\mathbb{G}_m$ is the multiplicative group $\mathbb{A}^1 \smallsetminus \{0\}$ with the product as the group structure. The first thing I want to compute is $\mu^{\mathrm{un}}_{\mathrm{Tam}}(\mathbb{G}_a(\mathcal{O}))$, and this is $\prod \mu_x(\mathcal{O}_x) = 1$.

Let me give a warning. This infinite product can diverge. So some care must be taken. If G is a constant finite group scheme over X, say $G = X \times \Gamma$, then over some factor this will be the order and then the product will be infinite. So in practice some care has to be taken.

So the thing that is more interesting is $\mu^{\mathrm{un}}_{\mathrm{Tam}}(G(K)G(\mathbb{A}))$ for $G = \mathbb{G}_a$. For this we have to use some algebra, so we have the following exact sequence:

$$0 \to H^0(X, \mathcal{O}_X) \to G(\mathcal{O}) \to G(K)\backslash G(\mathbb{A}) \to H^1() \to 0$$

This involves a resolution of $\mathcal{O}_X$ by *flasque* sheaves.

$$0 \to \mathcal{O}_x \to G_k \times G(\mathcal{O}) = toG(\mathbb{A}).$$

Then this gives you the following sequence

$$0 \to H^0(X, \mathcal{O}_x) \to \cdots \to \cdots \to H^1(X, \mathcal{O}_X) \to 0.$$

Now you do this computation and get by Riemann–Roch

$$\mu^{\mathrm{un}}_{\mathrm{Tam}}(G(K)\backslash G(\mathbb{A})) = \frac{|H^1(X, \mathcal{O}_X)}{f} ine|H^0(X, \mathcal{O}_X)| = q^{-1}.$$

Now we come to the normalized Tamagawa measure. Essentially [unintelligible].

So μ_{Tam} is defined to be $\frac{\mu^{\mathrm{un}}_{\mathrm{Tam}}}{q^{(g-1)d}}$, where d is the dimension of my group scheme G_0. The conjecture is that $\mu_{\mathrm{Tam}}(G(K))\backslash G(\mathbb{A})) = 1$, again.

So far there's not much geometry. How does geometry come in? This goes back to Weil. His observation is that there is a correspondence between the double coset $G(K)\backslash G(\mathbb{A})/G(\mathcal{O})$ and G-principal bundles on X. After we know how this works we can act in the bundle world, which is a very geometric place.

How do you construct these? You pick a small enough covering and glue together.

First think of $\mathbb{G}(\mathbb{A})$. This is typical, first think of things with extra structure and then quotient out.

So G principal bundles with generic trivialization. I have $\mathcal{P} \to X$ with a map $\operatorname{Spec}\mathcal{K} \to X$ and I have a trivialization over each $\mathcal{O}_x$ [pictures].

Suppose that $\mathcal{P}$ is such a guy, so from $\mathcal{P} \times_X \text{Spec } K$, this is (from generic trivialization isomorphic to $G \times \text{Spec } K$). So I get

$$G \times \text{Spec } K \times_{\text{Spec } K} \text{Spec } K_x.$$

[missed some].

So G_0 acts in two different ways and considering them together you get the correspondence.

So we can upgrade the statement from an isomorphism of sets to an equivalence of groupoids.

Let me quickly recall that a groupoid is a category in which all morphisms are invertible.

I have to tell you what I mean, what is the gorupoid structure on the right hand side?

So this is a groupoid whose morphisms are morphisms of G-bundles. The left hand side has as objects elements in $G(K)\backslash G(\mathcal{A})/G(\mathcal{O})$. So if $\gamma \neq \gamma'$ then $\text{Hom}(\gamma, \gamma') = 0$. Then $\text{Aut}(\gamma)$ is $\gamma(G(K))\gamma^{-1} \cap G(O)$.

The reason for the formula, let's do it after the break.

2. January 21, Part 2

So as promised we will upgrade the statement to one about groupoids. So I gave some formula to upgrade the left hand side, it's a natural thing to do. Let me recall the following general construction, called the groupoid quotient. In general let S be a set and let G be a group that acts on S. The usual way that we quotient S by G, we say it's the set of orbits of S by G. That's the usual way we define the quotient. We can do better and define the groupoid quotient. The resulting thing $[S/G]$ is a groupoid where the general philosophy is when you say that two things are equal you lose a lot of information. You are saying that two elements of S are *equal* if they are in the same orbit. We replace equality by morphism. So the objects are still the elements of S. The morphisms, $\text{Hom}(s,t)$ is the set of g in G such that $gs = t$. In particular, there may be multiple ways to make two points isomorphic. So in particular, the automorphisms of s is the stabilizer of s.

Whenever we talk about categories we can talk about equivalence of categories, so we consider $[S/G]$ with objects s in S/G and morphisms $\text{Hom}(s_1, s_2)$ empty if they are different, and otherwise the stabilizer. You pick one isomorphism class representative per orbit, and get an equivalent category. All of my quotients will be groupoid quotients and I will not write the brackets anymore.

So let's come back to $G(K)\backslash G(\mathbb{A})/G(\mathcal{O})$, and do it as a groupoid quotient. So let's do something easier first. So $G(\mathcal{O})$ acts freely on $G(\mathcal{A})$ with no stabilizer, so the naive quotient and the groupoid quotient coincide. So then we can take the quotient by $G(K)$. The objects are again the orbits but the morphisms, if γ is in $G(\mathbb{A})$, then $G(K)$ acts on $\gamma G(\mathcal{O})/G(\mathcal{O})$, and the stabilizer, when γ is 1 this is $G(\mathcal{O})$ and otherwise it's $\gamma G(\mathcal{O})\gamma^{-1}$ but we want it to be in $G(K)$ so it's $G(K) \cap \gamma G(\mathcal{O})\gamma^{-1}$. That's our automorphisms of γ.

Now we want to say that $G(K)\backslash G(\mathbb{A})/G(\mathcal{O})$ is equivalent to the G-bundles over X, and we want to say this now for automorphisms. The idea is, if you fix a bundle then $G(K)$ can be thought of as the automorphisms of the generic fiber of the bundle. The condition is that your object, your element, lives in there, is that the automorphism extends, and that's how you get the correspondence.

Now we've moved to something more geometric, so we want to say it as something bundle-theoretic. Again, up to issues of convergence this is a very formal argument, so what is Tamagawa number on the geometric side? We're interested in

$$\begin{aligned}\mu_{\mathrm{Tam}}(G(K)\backslash G(\mathbb{A})) &= \sum_{\gamma \in G(K)\backslash G(\mathbb{A})/G(\mathcal{O})} \mu_{\mathrm{Tam}}(G(K)\backslash G(K)\gamma G(\mathcal{O})) \\ &= \sum_{\gamma \in G(K)\backslash G(\mathbb{A})/G(\mathcal{O})} \frac{\mu_{\mathrm{Tam}}(G(\mathcal{O}))}{|\gamma G(K)\gamma^{-1} \cap G(O)|} \\ &= \mu_{\mathrm{Tam}(G(\mathcal{O}))} \sum_{\mathcal{P} \text{ a } G\text{-bundle}/\sim} \frac{1}{|\operatorname{Aut}\mathcal{P}|}.\end{aligned}$$

Now suppose that, the Tamagawa conjecture says that $\mu_{\mathrm{Tam}}(G(K)\backslash G(\mathbb{A})) = 1$ but this we also have

$$\mu_{\mathrm{Tam}}(G(\mathcal{O})) = \prod \mu_x G(\mathcal{O}_x) = \prod \frac{|G(k(x))|}{|k(x)|^d} \frac{1}{q^{d(g-1)}}$$

and this then says our geometric version.

$$\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}\mathcal{P}|} = q^{d(g-1)} \prod_{x \in |X|} \frac{|k(x)|^d}{|G(k(x))|}. \tag{1}$$

Let me give a quick remark, that I was supposed to say before I wrote this down.

Suppose $\mathcal{C}$ is a groupoid. When you count a set, you count each element with value 1. The volume of a groupoid is supposed to be $\sum_{\mathcal{C}/\sim} \frac{1}{|\operatorname{Aut}(c)|}$.

So the Tamagawa number one conjecture is the statement that the count of G-principal bundles is this infinite product that looks like an L-function.

Let me say something quickly about classifying spaces of bundles. Really I mean stacks but for now let's keep it at a more heuristic level. So when you have G a group, say over, let G be a finite group, then you can form this thing called BG, the classifying space of G-bundles, at least topologically, and one way to construct this is to find a contractible space so that G acts freely and quotient out by G, so in the world of groupoids, you can view E as a point. So this is a good way to compute $*/G$. So in topology, homotopy classes of maps from X to BG is the same as G-bundles over X.

So back to the situation we had over there. So G is a group over X, and we can look at $BG(k(x))$ over each point, and we want to compute $|BG(k(x))|$. The only information is the automorphism, this is the whole group G, this is the denominator $\frac{1}{|G(k(x))|}$. Then if you push this far into stacks, the dimension of BG is, the relative dimension is the dimension of G, this is then $-d$. So you get

$$\frac{1}{q^{d(g-1)}} \sum_{\mathcal{P}} \frac{1}{\operatorname{Aut}\mathcal{P}} = \prod \frac{|B(G(k(x)))|}{|K(x)|^{\dim BG_x}}$$

and now both sides have something about G-bundles.

Okay enough motivation, how do we prove this. We're in algebraic geometry over a finite field, so we can do point counts cohomologically. Let me recall the Grothendieck–Lefschetz trace formula.

In general, we work with ℓ-adic sheaves, so let me quickly recall, the theory is complicated to set up. There are a small list of manipulations that you can do. For a scheme X, the derived category of ℓ-adic sheaves on X is $\mathrm{Shv}(X)$. What I mean, there are two complicated things here, one is ℓ-adic, there is a difficulty, but it's technical, and derived means chain complexes of such sheaves. The coefficients are in $\bar{\mathbb{Q}}_\ell$ where ℓ is a prime different from p. So now given $X \xrightarrow{f} Y$ we have pullback $f^* : \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$ and pushforward $f_* : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$. Everything is derived so I won't write L and R. These are adjoint, so $f^* \dashv f_*$ meaning

$$\mathrm{Hom}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}).$$

We also have the exceptional pushforward and pullback, and $f_! \dashv f^!$, extend your sheaf by zero to the compactification and then use normal pushforward, that's pushforward with compact support. Then the pullback is exceptional pullback.

Given two sheaves you can also tensor them, and another tensor product, the exceptional tensor, if you have $X \xrightarrow{\Delta} X \times X$, and then you have the projections p_i, and $\mathcal{F}_1 \otimes^! \mathcal{F}_2$ is

$$\mathcal{F}_1 \otimes^! \mathcal{F}_2 = \Delta^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = \Delta^!(p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2).$$

I also want to say something about Verdier duality, $D : \mathrm{Shv}(X)^{\mathrm{op}} \to \mathrm{Shv}(X)$ and $Df_! \cong f_* D$ and $Df^! = f^* D$.

There are two canonical sheaves, the constant sheaf $\mathbb{Q}_{\ell,X}$ and then the *dualizing sheaf* ω_X, its Verdier dual. One way to define the Verdier dual is hom into the dualizing sheaf.

Let me give you some properties, and these may look abstract but they're actually quite simple.

(1) When $X \xrightarrow{f} Y$ is *proper* then $f_* \cong f_!$.
(2) When $X \xrightarrow{f} Y$, the pullback of the constant sheaf is the constant sheaf, and the dual statement is that $f^!\omega_Y \cong \omega_X$.
(3) When the map is smooth of relative dimension n, then $f^!\mathbb{F} \cong f^*\mathcal{F}[2n](n)$, so this is the *Tate twist*. Cohomologically it's shifted to the left by $2n$. My differential goes up $i \to i+1 \to \cdots$ and you shift the complex to the left by $2n$.

Suppose X is defined over $\mathbb{F}_q$, then $\bar{X}$ is $X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. I'll use the following notation: $C^*(\bar{X}, \mathbb{F}) = \pi_*\mathcal{F}$ where $\pi : \bar{X} \to \mathrm{Spec}\,\bar{\mathbb{F}}_q$. Suppose $\mathcal{F}$ is coming from $\mathrm{Shv}(X)$ before base change. Then $C^*_{(c)}(\bar{X}, \mathbb{F})$ has an action of Frobenius. Then we can recall what Grothendieck said about point counting and cohomology.

Theorem 2.1 (Grothendieck–Lefschetz trace formula)**.** *Let $\mathcal{F}$ be a sheaf on X. On one hand you can form the following sum, where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at the point x:*

$$\sum_{x \in X(\mathbb{F}_q)} \mathrm{Tr}(\mathrm{Frob}_q, \mathcal{F}_x) = \mathrm{tr}(\mathrm{Frob}_q(C_c^*(\bar{X}, \bar{\mathcal{F}})))$$

where the bar means I base change everything to an algebraically closed field.

The one side is something happening over X and the right side is cohomological. By trace on the left side, I take the alternating sum of the trace acting on the chain complex, and likewise on the right side. This works as long as this thing converges.

Some remarks.

Remark 2.1. Say $\mathcal{F}$ is the constant sheaf, then this left hand side is $|X(\mathbb{F}_q)|$ and the right hand side is $\mathrm{tr}(\mathrm{Fr}_q(C_c^*(barX)))$, so this is a link between the point count and the cohomology.

All right so we can also dualize this statement using Verdier dual. Let me give one more property that I forgot to say, suppose X is smooth, then ω_X is very simple. The shriek pullback, for smooth maps, this is $\bar{\mathbb{Q}}_{\ell,X}[2\dim\ X](\dim X)$. So then the theorem is

Theorem 2.2 (Grothendieck–Lefschetz dual)**.** *This uses the* arithmetic *Frobenius instead of the geometric.*

$$\sum_{x\in X(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Fr}_q^{-1}, \iota_x^! \mathcal{F}) = \mathrm{tr}(\mathrm{Fr}_q^{-1}, C^*(X, \bar{\mathcal{F}})).$$

When X is smooth you see a very nice formula

Corollary 2.1. *If X is smooth,*

$$\frac{|X(\mathbb{F}_q)|}{q^{\dim\ X}} = \mathrm{tr}(\mathrm{Fr}_q^{-1}, C^*(\bar{X}, \bar{\mathbb{Q}}_\ell)).$$

I like this because you get a point counting over $q^{\dim\ X}$ that is something related to cohomology.

Let me say a word about how to construct $f^!$. At least étale-locally you can try to factor this to $X \to X' \to Y$ where $X \to X'$ is closed and $X' \to Y$ is smooth. For the smooth ones it's just a shift. For closed guys it's more difficult, but, if it's nice, well, suppose X is smooth of dimension n and x is a closed point in X. Suppose we have the constant sheaf $\mathbb{Q}_{\ell,X}$, then the upper shriek $i_x^! \mathbb{Q}_{\ell,X}$ is $\mathbb{Q}_\ell[-2n](-n)$.

Verdier duality, again, is a form of Poincaré duality. What does it mean to take the usual pullback? That's looking at a very small ball and taking a colimit. For this you look at sheaves with compact support around that point, and that's why the twist $(-n)$ shows up. The $2n$ is the real dimension of the variety that you consider. Everything is even dimension.

Here is the theorem for schemes. A stack is a more general kind of geometric objects that lets you construct these kinds of things. Suppose this holds for that world, then you could try to show this kind of statement, once you take the Frobenius trace you'll get this kind of formula, and the tool to do this will be factorization homology.

3. January 22

Thanks again, so let me start with where I left off last time, here is the formula, when X is a curve over a finite field and G is (for simplicity) a group scheme over X, $G = G_0 \times X$. Last time we said that the Tamagawa number 1 conjecture is equivalent to the statement

$$\sum_{\mathcal{P} \text{ a } G\text{-bundle}} \frac{1}{\mathrm{Aut}\,\mathcal{P}} = q^{d(g-1)} \prod_{x\in|X|} \frac{|BG(k(x))|}{|K(x)|^{\dim BG_x}}$$

It was a hope that we could prove something like this with étale cohomology. Let me quickly mention the objects, the geometric objects here, some kind of moduli spaces. Eventually the geometric objects we're going to consider will be certain moduli stacks. Let me give a quick overview of what I mean by moduli stacks. In a general, moduli problem, a moduli space is an object that solves a moduli problem. A moduli problem for us is a functor, a contravariant functor M from the category of schemes to the category of sets or maybe groupoids. Examples, I can consider $M(S)$ the groupoid of G-bundles over $X \times S$. So we can encode it in families. We should talk about how things vary in families and not just at a point.

You might wonder why I put in the groupoid condition. A moduli space $\mathcal{M}$ for the moduli problem M is such that $\mathrm{Hom}(S, \mathcal{M})$ is naturally equivalent to $M(S)$. So that's an object that represents this functor.

So what does this mean? The universal object $\mathcal{M}$, there, it will classify universally all the objects we're trying to classify, so that the collection of objects over S are the S-maps to $\mathcal{M}$ and the family we want is the pullback $\mathcal{U} \times_{\mathcal{M}} S$.

Usually what kind of space are we talking about? If $\mathcal{M}$ is a scheme, then $\mathcal{M}$ is said to be a *moduli scheme* for the moduli problem M. The natural question to ask is then, if instead of saying groupoids, I said the *set*, then [unintelligible]would solve the moduli problem.

So the remark is that this moduli problem is not representable by a scheme, if you use sets.

Suppose your moduli problem is representable by a scheme. That means your moduli problem is some kind of sheaf. So but how do you create a G-bundle usually? You look at an open cover, on each piece it's trivial, but you glue it in an interesting way. But thinking carefully, you're not taking two things on the open sets and saying they're the same on the intersection—you're remembering the witnessing isomorphism.

So we really need groupoid-valued functors, hence *stacks*.

Let me recall something very basic from geometry. Usually the object we consider is a presheaf. Say you have a scheme or space X. Consider the open sets of X, then a presheaf is just a contravariant functor from the open sets of X to Set. The sheaves are those presheaves such that satisfy a gluing property, what I mean by gluing is, suppose $\mathcal{F}$ is a sheaf and $\bigcup \mathcal{U}_i \to X$ is an open cover. Then $\mathcal{F}(X) \cong \lim (\mathcal{F}(\mathcal{U}) \rightrightarrows \mathcal{F}(\mathcal{U} \times_X \mathcal{U}))$.

Now for stacks, a stack is essentially a sheaf but where you have groupoids instead of sets. A prestack is a contravariant functor from opens of X to groupoids. A stack is a prestack that satisfies gluing. So you have something on each open set and they satisfy gluing.

Then it's immediate that $M(S)$, the moduli problem classfying G-bundles, is a stack. In fact it has a name, so let me, it's called Bun_G.

To actually study Bun_G is a little complicated, so let me talk about a simpler one. Let me talk about BG in this language. What is BG in the world of algebraic geometry.

First I have to define the moduli problem that BG solves. It's the following. It's the functor from $\mathrm{Sch}^{\mathrm{op}}$ to groupoids which sends a scheme S to the groupoids of G-bundles over S.

This is almost the same as the definition from topology, the classifying space of a group and it classifies bundles. So you can start with a moduli problem and try to see what this is.

So B_G and Bun_G are different, so $\mathrm{Bun}_G(S)$ is G-bundles on $X \times S$. So bundles you can also glue so it's a stack.

Then how do we say more about this stack? How do we present it in terms of schemes?

First, in addition to stacks we can talk about algebraic stacks, where you have an open cover by schemes. So roughly, this is a stack with a *smooth* covering by schemes.

This is now a geometric object, you can talk about the *dimension*, for example.

So how do we construct BG as an algebraic stack? One thing that you can do is to form the prestack first, and this thing is defined to be pt/G, and this quotient is the groupoid quotient that I mentioned yesterday.

I want a contravariant functor B_{pre} from Sch to groupoids, so to S I assign the S points of G, $\mathrm{pt}/G(S)$.

In particular, $B_{\mathrm{pre}}G(k) \cong \mathrm{pt}/G(k)$. In the stack world you can stackify and shuffle and move things together to get a stack. Then to form BG you stackify this thing.

So BG is obtained from $B_{\mathrm{pre}}G$ by stackififcation. The claim is that once you've done that you see that it's the moduli stack that represents moduli problems.

Let me recall a general construction, the pullback of a prestack, so let me talk about pullback of categories. Suppose you have a diagram of categories $\mathcal{C}' \to \mathcal{D} \leftarrow \mathcal{C}$ and I want to describe $\mathcal{C} \times_{\mathcal{D}} \mathcal{C}'$, then the objects are (c, c', γ) where $c \in \mathcal{C}$ and $c' \in \mathcal{C}'$ and $\gamma : p(c) \cong p(c')$. When you work with categories, saying equal is a bad sign, so you put in the witness for the isomorphism. That's the space of objects. Morphisms, you can guess, so essentially you have $c_1 \to c_1'$ by γ_1 and c_2 to c_2' by γ_2 and then the square commutes.

So now let's talk about pullback of prestacks. I have a diagram of prestacks, these are functors, so the morphisms are natural transformations. So I have a diagram $\mathcal{Y}_1 \to \mathcal{Y} \leftarrow \mathcal{Y}_2$ and I want to tell you the value of $\mathcal{Y}_1 \times_{\mathcal{Y}} \mathcal{Y}_2$ on schemes. So I define this as $\mathcal{Y}_1(S) \times_{\mathcal{Y}(S)} \mathcal{Y}_2(S)$ where this pullback is taken in categories.

So earlier I said something about smooth covers, that's geometric while stacks are formal, so how do I translate that to something geometric? A point maps to BG (and remember, a point is $\mathrm{Spec}\, \mathcal{F}_q$). [missed a little]

Let's try to compute the pullback of $\mathrm{pt} \to BG \leftarrow \mathrm{pt}$. So what is pt? It's the moduli space of trivial G-bundles. So you have two trivial G-bundles and an isomorphism between them, so that's an element of G. The point to BG is smooth, and this is now an atlas of the smooth covering we're looking for.

One more remark is that a point to BG is the universal G-bundle. Namely, any S, if you pull back $\mathrm{pt} \to BG$ then you get a bundle $\mathcal{P}$ over S.

Usually the way you compute dimension is by computing the tangent space if your sitution is smooth. So for that you do some deformation theory. If an algebraic stack is smooth, the dimension of $\mathcal{Y}$ is usually computed at y via the tangent space of $\mathcal{Y}$ at y. This tangent space is itself a stack, which in turn is computed by deformation theory.

Let me say why this shows up. How do we compute the dimension of Bun_G? Suppose we fix $\mathcal{P} \in \mathrm{Bun}_G(k)$. Suppose I want the tangent space of Bun_G at point $\mathcal{P}$?

So we want the maps from Spec $k[\epsilon]/\epsilon^2$ extending Spec $k \to \mathrm{Bun}_G$. So translating, these are bundles on $X \times \mathrm{Spec}\, k[\epsilon]/\epsilon^2$ which yields $\mathcal{P}$ when restricted to X. [pictures]

The space of all such deformations is given by $H^1(X, \mathrm{ad}(\mathcal{P}))$, where since $\mathcal{P}$ is a bundle over X, and G acts on $\mathfrak{g}$ by the adjoint action, and $\mathrm{ad}(\mathcal{P})$ is the associated bundle with the action, so $\mathrm{ad}(\mathcal{P})$ is $(\mathfrak{g} \times P)/G$, the quotient by the diagonal action.

The automorphisms are given by $H^0(X, \mathrm{ad}(\mathcal{P}))$. These are coherent cohomology. The same thing in general, in geometry coherent cohomology, and topological is ℓ-adic cohomology.

Then the dimension is the difference. So $\mathrm{ad}(\mathcal{P})$, the group is semisimple, and so by Riemann–Roch we have $\dim \mathrm{Bun}_G = d(g-1)$ where d is the dimension, and this looks like what we had in our equation for the conjecture.

So let's rewrite,

$$\frac{|\mathrm{Bun}_G(K)|}{q^{\dim \mathrm{Bun}_G}} = \prod_{x \in |X|} \frac{|BG(k(x))|}{q^{\dim BG_x}}.$$

Now we're in pretty good shape, when I ended the lecture, we had point counting divided by q to some dimension, so now both sides can be represented in terms of the Grothendiecke–Lefschetz trace formula.

So it will take a while to get to the cohomological formulation, but we can do a tentative fantasy one. Suppose Grothendieck–Lefschetz worked for stacks and so on. Then the left hand side should be about the cohomology of Bun_G, the ℓ-adic cohomology, and implicitly this is over $\bar{\mathbb{F}}_q$, and we want some equivalence, so suppose the right side is not an infinite product, you'd take the tensor of the two vector spaces with the action of Frobenius and then [unintelligible]. So since we're fantasizing, we want an infinite tensor product and we'd have

$$\text{``}\bigotimes_{x \in |X|}\text{''} C^*(BG_x).$$

So the next step is to make sense of the infinite tensor. Eventually factorization homology will let us make sense of this infinite tensor. This was first formulated in algebraic geometry by Beilinson and Drinfeld, but it wasn't until Lurie noticed that this thing had come up in topology a long time earlier that [unintelligible].

So now let's try to make the infinite tensor. Here is the first attempt. Consider the category of finite subsets of, the goal in the end is to prove the statement either ℓ-adically or about singular cohomology or whatever, the equivalence is just this cohomology. Only in the ℓ-adic situation we have Frobenius and recover our Tamagawa number one identification.

So anyway consider the category K of finite subsets of X, and for each $S \subset X$ we can form the tensor

$$\bigotimes_{x \in S} C^*(BG_x)$$

and so we could try to take

$$\operatorname*{colim}_{S \in K} \bigotimes_{x \in S} C^*(BG_x).$$

If we take the tensor in a naive way like this, you'll lose all information about the topology of X. The goal is to have some sort of colimit that incorporates the space X. So we want to have a version of colimit that takes topology of X into account. This is a natural place to stop, and in the next step I'll say what that is and continue with reformulation.

4. Part 4

So the second part is to try to make sense of the tensor product. We have to take some homotopy colimit instead of a colimit. So let me talk a little about $(\infty, 1)$-categories. We want to do homotopy colimits which will take into account the topology of X. Let me start with a classical story, the story of triangulated categories when working with chain complexes. So what does the theory do? Usually you start with an Abelian category, maybe the category of vector spaces, and then look at chain complexes in that Abelian category and mod out homotopy equivalence and invert quasi-isomorphisms and you end at a triangulated category.

In a triangulated category (think about chain complexes in vector spaces) there is this construction, the cone construction, that replaces or combines the concept of a kernel and a cokernel in an Abelian category. For any map $A \to B$ in a triangulated category, you can complete it with a map $B \to C$ where C is the "cone of f". Such a sequence is called a distinguished triangle, and you can take the cone of $B \to C$ and you get a shifted version of A, written $A[1]$, and you can go on. This cone construction is not functorial. It satisfies a property and there are some complicated axioms, but, well, what do I mean, there is a *functorial* kernel, you factor uniquely through a kernel if you satisfy the right property, but for the cone there is a map but it's not unique. It also satisfies some complicated axioms that are not very natural from the categorical point of view. Lastly, taking limits or colimits in a triangulated category usually doesn't give the correct answer. It's pretty ill-behaved.

The special cases, you have to do some weird thing instead of the natural thing of taking a limit or colimit. So the problem is that whenever you do chain complexes, you just kind of invert quasi-isomorphisms you lose a lot of information. If you take a quotient, we saw earlier, if you quotient as a set you lose information but if you take the groupoid quotient you get something more geometric.

So here again, we don't want to identify things, we want to do something more enhanced. So what it does is instead of hom sets, you should think of hom spaces. That's the upshot. Again, this is easy to say but to implement it, Lurie wrote this really long book. To use it is easier than to invent it. So what is ∞ and what is 1? The ∞ refers to the fact that we have 1-morphisms, 2-morphisms, et cetera, up to ∞, and 1 refers to the fact that morphisms of 2 and above are invertible. The Hom sets, it's, well, a space is like a category where objects are points, paths are morphisms, homotopies are two-morphisms, so all of them are invertible.

So in an $(\infty, 1)$-category, (usually people just say ∞-category) we can also talk about limits and colimits, so these are usually what people call "homotopy limits and colimits" but in this context these are the only notions that make sense. Let me say an example of what this really means.

The objects of topological spaces are spaces, the 1-morphisms are maps of spaces, and then 2-morphisms are homotopies and so on. So what does it mean to take a pullback in this setting? What if we take the pullback of $* \to X \leftarrow *$. If you take the usual pullback you get a point, which forgets almost everything about X. The way to take the homotopy pullback, you take a point in X and a point in X and then a path between them, and then this becomes the based loop space of X [sic].

So now let's look at a map from X to Y, in the most naive way, you contract the image to a point, so the cone of f is going to be a cone on X glued onto Y, and then you can do the same thing and go on and get ΣX, and continue.

Say you want to take the pushout of X with a point and a point then that's the same as the suspension of X, let me say that too. So this brings us to the theory of *stable ∞-categories*, the kind of linear version of ∞-categories. The definition is easy to state, so C an ∞ category is *stable* if two conditions,

(1) C should be pointed, there's a terminal object which is also initial, and
(2) all pullback squares exist and are also pushout squares.

Usually the point object is denoted by 0. So let's consider the category Vect of chain complexes, with objects chain complexes, maps chain maps, 2-morphisms chain homotopies, it's not so easy to actually construct this because you need a lot of data but manipulating it is not so bad. So in this world I define $X[1]$ to be the pushout of $0 \leftarrow X \rightarrow 0$. The other shift is the pullback of $0 \rightarrow Y \leftarrow 0$, and this says that $X[1][-1] \cong X$ and likewise in the other direction.

Let's say what the cone is. The cone is the pushout of $Y \leftarrow X \rightarrow 0$. So you can extend in both directions

$$\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & \mathrm{Cone}(f) \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & X[1] \arrow[r] & Y[1].
\end{tikzcd}$$

Now I want to do an illuminating computation, not the infinite tensor but we're gotting there. So X is a topological space which we view as an ∞-groupoid. What do I mean? Even the 1-morphisms are invertible, the objects are points, the 1-morphisms are paths, and so on. Now pick a simple chain complex $\mathbb{C}$, the constant diagram $X \rightarrow \mathrm{Vect}$, we can take the colimit over x. In the classical world, if X is connected, you just get $\mathbb{C}$. In this world you get the homology of X with values in $\mathbb{C}$, you get $C_*(X, \mathbb{C})$. You can do this by seeing that you get the answer you expect for the ball and then gluing together is a kind of Mayer–Vietoris, it's a kind of colimit statement.

Okay, let's see one more formulation of this. Let me recall the construction called left Kan extension. So we're in the following setting. You can think of this as integration or pushforward. So think we have $\mathcal{C} \xrightarrow{F} \mathcal{E}$ and we also have $\mathcal{C} \xrightarrow{\alpha} \mathcal{D}$, we want to extend and get a universal map $\mathcal{D} \rightarrow \mathcal{E}$, the *left Kan extension of F along α*, and so we get an extension by α from $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ to $\mathrm{Fun}(\mathcal{D}, \mathcal{E})$ and left Kan extension is left adjoint to pullback along α.

Suppose that $\mathcal{E}$ is nice, it has all colimits, then you can construct this in a pretty explicit way, say you want to know $\mathrm{LKE}_\alpha F(d)$. Then you can take the fiber product $\mathcal{C}_{/d}$ which is the fiber product of $\mathcal{D}_{/d}$ with $\mathcal{C}$ over $\mathcal{D}$, so this has objects c and then a map from $\alpha(c) \rightarrow d$, then the Kan extension is

$$\operatorname*{colim}_{c \in \mathcal{C}_{/d}} \widetilde{F}(c)$$

where $\widetilde{F}$ is F composed with the canonical map $\mathcal{C}_{/d} \rightarrow \mathcal{C}$.

Okay, so now let's get homology using this. So say we have

$$\begin{tikzcd} * \arrow[r, "V"] \arrow[d, "\alpha"] & \mathrm{Vect} \\ \mathrm{Top} \arrow[ur, dashed, "\mathrm{LKE}_\alpha V"] & \end{tikzcd}$$

so we form the comma category,

$$\begin{tikzcd} */X \arrow[r] \arrow[d] & * \arrow[r, "V"] \arrow[d, "\alpha"] & \mathrm{Vect} \\ \mathrm{Top}_{/X} \arrow[r] & \mathrm{Top} \arrow[ur, dashed, "\mathrm{LKE}_\alpha V"] & \end{tikzcd}$$

which has objects maps from a point to X and then morphisms paths in X, so this is the same as the ∞-category we had so we get

$$\mathrm{LKE}_\alpha V(X) \cong C_*(X, V)$$

like we said.

Let me go back to the thing we're interested in, so the infinite tensor product (take two). So we use the topology. So now let $E \xrightarrow{\pi} M$ be some kind of fibration with base an n-dimensional manifold. Consider the category $\mathcal{U}_0(M)$, where the objects are open subsets of M that are homeomorphic to $\mathbb{R}^n$. We can consider, this is now (intuitively) an $\infty, 1$-category, and we can consider the functor $\mathrm{Sect}_\pi : \mathcal{U}_0(M)^{\mathrm{op}} \to \mathrm{Top}$ which sends U to $\mathrm{Maps}_M(U, E)$, the space of sections.

Now taking cohomology, which is a contravariant functor, we see the following,

$$C^* \circ \mathrm{Sect}_\pi : \mathcal{U}_0(M) \to \mathrm{Vect}$$

I have this diagram in Vect, and for each U in $\mathcal{U}_0(M)$, there's a natural map from $C^*(\mathrm{Sect}_\pi(U)) \to C^*(\mathrm{Sect}(M))$ which is $C^*(\mathrm{Maps}(M, E))$.

Then the remark is that when M is an algebraic curve over $\mathbb{C}$ then we can take E to be $BG \times X$, so topologically we think of this as the classifying space of G-bundles, and Bun_G is $\mathrm{Maps}(X, BG)$, or equivalently, the space of sections $\mathrm{Maps}_X(X, X \times BG)$.

So the right hand side will be about cohomology of Bun_G. The left hand side, because U is contractible this will just be about BG. So now the last thing we want to do is to take colimits, all the maps are compatible with the diagram in Vect, and so we get a map

$$\operatorname*{colim}_{U \in \mathcal{U}_0(M)} C^*(\mathrm{Sect}_\pi(U)) \to C^*(\mathrm{Maps}(M, E))$$

or

$$\operatorname*{colim}_{U \in U_G(M)} C^*(BG) \to C^*(\mathrm{Bun}_G).$$

There's no tensor right now, though. We have to modify it as follows.

Note that Vect has a tensor structure, if you have two chain complexes you can tensor them. So one can talk about commutative algebras in Vect, the category of commutative algebras. So it looks innocent, but we have to be careful about what we mean. If we talk about commutative algebra, we have $xy = yx$. Here we can't say that, we have a homotopy between xy and yx and so on for higher things. So this can be pretty complicated.

But then our functor $C^* \circ \mathrm{Sect}_\pi$ has values from $\mathcal{U}_0(M) \to \mathrm{Com\,Alg(Vect)}$ and so this colimit should be taken in commutative algebras rather than just vector spaces.

Let me say quickly why this is the right thing to do. The remark is that, say you have two algebras A and B, then what is, so, let's say everything is over $\mathbb{C}$, if we want to take the pushout in the category of commutative algebras over $\mathbb{C}$, that's the tensor product. We'll get a quotient of the direct sum if we work in vector spaces, which is not what we want.

So here is the theorem that one can prove. If M is n-dimensional and the fibers of E are n-connected (meaning that all homotopy groups up to and including n are trivial) then the map $\mathrm{colim}_{U\in U(M)} C^*(\mathrm{Sect}_\pi(U)) \to C^*(\mathrm{Maps}(M,E))$ is a weak equivalence.

For us, our group is simply connected so BG is 2-connected and our curve is 2-dimensional, so we have

$$\operatorname*{colim}_{U\in\mathcal{U}_0(M)} C^*(BG) \xrightarrow{\sim} C^*(\mathrm{Bun}_G).$$

So let me say one word about how this is proved. The formulation in algebraic geometry is a little harder, but let me say something about what goes into this proof. The ingredients, the first thing is that when M is $\mathbb{R}^n$, there's nothing to prove. Then you use a gluing procedure, supposing that the statement is true for U, V, and $U\cap V$, then we want it to be true for $U\cup V$ as well. At the level of spaces we have

$$\begin{tikzcd}
\mathrm{Sect}_\pi(U\cup V) \arrow[r] \arrow[d] & \mathrm{Sect}_\pi(V) \arrow[d] \\
\mathrm{Sect}_\pi(U) \arrow[r] & \mathrm{Sect}_\pi(U\cap V)
\end{tikzcd}$$

and the claim is that when you apply cohomology to this guy, you get some sort of multiplicative Künneth, which means that the value at $U\cup V$ is the pushout in commutative algebra.

This is a theorem in algebraic topology, called the Eilenberg–Moore spectral sequence, for the square

$$\begin{tikzcd}
C^*(\mathrm{Sect}_\pi(U\cup V)) & C^*(\mathrm{Sect}_\pi(V)) \arrow[l] \\
C^*(\mathrm{Sect}_\pi(U)) \arrow[u] & C^*(\mathrm{Sect}_\pi(U\cap V)) \arrow[l] \arrow[u]
\end{tikzcd}$$

and this lets you glue together and complete the theorem. That's the kind of end of the discussion for taking the infinite tensor product in topology.

You may complain that this looks too big and you can't use it to compute. I claim that it's not so bad. How do we compute this colimit? It boils down to something abstract and quite formal.

So you have from Vect to Com Alg(Vect) a free functor which is taking the tensors and modding out by the relations for commutativity in a homotopically correct way. This is sometimes called the symmetric algebra.

So a classical thing is that $C^*(BG) \cong \mathrm{Sym}\, V$, formally it has this shape, it's Sym of something. Now any left adjoint commutes with colimits.

So let's plug in and see.

$$\begin{aligned}\operatorname*{colim}_{x\in X} C^*(BG) &\cong \operatorname*{colim}_{x\in X} \operatorname{Sym} V\\ &\cong \operatorname{Sym}(\operatorname*{colim}_{x\in X} V)\\ &\cong \operatorname{Sym}(C_*(X)\otimes V)\end{aligned}$$

and this is known classically as the Atiyah–Bott formula for the cohomology of the moduli space of principal G-bundles over X.

5. January 24: Part five

Last time I formulated the infinite tensor construction and how it's related to the Tamagawa number one conjecture. Today I'll start with the algebro-geometric formulation of the same thing. In a lot of the things I talk about, first I'll do the topological picture and then the algebro-geometric version. While logically they're independent, it's good to see the topological version first. Recall from topology that the infinite tensor product is defined in the same way as for homology except that we take a certain colimit in the category of commutative algebras rather than just vector spaces.

In algebraic geometry we can do the same thing. What is homology in algebraic geometry? Suppose I have a scheme X and the map π to a point. I have adjoint functors between $\operatorname{Shv}(X)$ (I mean ℓ-adic sheaves) and vector spaces, I have $\pi_!$ and $\pi^!$, and these are adjoint functors, and one way to define ℓ-adic homology is to say that $C_*(X) = \pi_!\pi^!\bar{\mathbb{Q}}_\ell$, i.e., it is $C^*_c(X,\omega_X)$. In general, what we're really after is $C^*_c(X,\mathcal{F})$, i.e., $\pi_!\mathcal{F}$, where $\mathcal{F}$ is in $\operatorname{Shv}(X)$.

The special multiplicative homology is in commutative algebras so we want to replace in this way. The category Vect (the ∞-category) has a natural monoidal structure, and $\operatorname{Shv}(X)$ also has, well, we use the exceptional tensor product $\otimes^!$, which is $\boxtimes$ and then pull back via the shriek pullback:

$$\mathcal{F}\otimes^!\mathcal{G} = \Delta^!(\mathcal{F}\boxtimes\mathcal{G})_{X\times X}.$$

So we should be working with commutative algebra objects in Vect, and instead of $\operatorname{Shv}(X)$, we want to use $\operatorname{Com\,Alg}(\operatorname{Shv}(X)^{\otimes^!})$.

So $\pi_!$ is determined uniquely, abstractly, via its adjointness property. And $\pi^!$ is monoidal, if you have two chain complexes, tensor and then pull back, it's the same as pulling back and doing the tensor, so if you set up the theory correctly, then the left adjoint of $\pi^!$, I'll denote it $\pi_?$, goes from commutative algebras in sheaves to commutative algebras in vector spaces. Another way to see this, is, we want to do a kind of colimit, so the left adjoint is what we want to work with.

So let's try to say something about this functor. It looks unsatisfactory. You need a better way to handle this functor. But we're not too bad. In the topological setting, since we got lucky and the thing we're taking homology of is a polynomial algebra. The same thing will happen here. The fact that this is a left adjoint—

Lemma 5.1. *Let $\mathcal{F}$ be a sheaf on X. Then I can form the polynomial algebra $\operatorname{Sym}\mathcal{F}$, which is a commutative algebra in sheaves. Since $\pi_?$ is a left adjoint, one can show that there is a natural equivalence between $\pi_?\operatorname{Sym}\mathcal{F}$ and $\operatorname{Sym}(\pi_!\mathcal{F})$, so some mysterious functor applied to the symmetric algebra is the symmetric algebra on the value on an easier functor.*

The proof is the Yoneda lemma plus the two adjunctions $\pi_? \dashv \pi^!$ and $\pi_! \dashv \pi^!$. So Gaitsgory proposes that this should be the right candidate for the infinite tensor product. So let me do the following construction. Suppose you are mapping Y to a scheme X by f (eventually Y will be BG). Then we define $[Y]_X$ as $f_* f^* \omega_X$. Suppose X is smooth (it's a curve in our case), then this is ω_X up to a shift. We want to do it this way because we're adapted to the functors we're taking and we'll just get [unintelligible].

The first remark is that this construction is contravariant with respect to Y. If Y_1 maps to Y_2 over X then at this level you get a map the other way, it's just like cohomology.

The second remark is that $[Y]_X$ is in the category of commutative algebra objects in sheaves on X, essentially using the cup product.

So now let's put this together. We have $\mathrm{Bun}_G \times X \to X \leftarrow BG \times X$ and there's a map $\mathrm{Bun}_G \times X \to BG \times X$, which is the identity on X and is given on $\mathrm{Bun}_G \times X \to BG$ by the universal property of BG, since BG is universal for G-bundles and Bun_G for bundles with base X.

Now suppose you have $X \times Z$ where Z is a stack and X is a scheme, mapping to X, then we can show that $[X \times Z]_X$ is the same as $\pi^! C^*(Z)$.

Applying this observation, first using contravariance there's a map $[BG \times X]_X \to [\mathrm{Bun}_G \times X]_X$, and then using the observation this is the same as a map of commutative algebras

$$\pi^! C^*(BG) \to \pi^! C^*(\mathrm{Bun}_G),$$

which induces a map like so:

$$\pi_? \pi^! C^*(BG) \to C^*(\mathrm{Bun}_G).$$

So now the cohomological formulation of the product formula in algebraic geometry is the following. The natural map

$$\pi_? \pi^! C^*(BG) \to C^*(\mathrm{Bun}_G)$$

is an equivalence as commutative algebras, this is the same as the topological statement.

So last time I used something that looked like that and got something called the Atiyah–Bott formula. Let's see that I get the same thing. For simplicity assume X is a complete curve. Recall that $C^*(BG)$ is a symmetric power of some vector space V. Then

$$\begin{aligned} C^*(\mathrm{Bun}_G) &\cong \pi_? \pi^! C^*(BG) \\ &\cong \pi_? \operatorname{Sym} \pi^! V \\ &\cong \operatorname{Sym} \pi^! \pi^! V \\ &\cong \operatorname{Sym}(C_*(X) \otimes V) \end{aligned}$$

and let's see that we can get the numerical statement, the product formula, from this.

So we need the following ingredients.

- the Grothendieck–Lefschetz trace formula for Bun_G. For constant group scheme this is Behrend, for nonconstant by Gaitsgory–Lurie.
- let's be careful about what converges and what doesn't, we need some convergence of some infinite products, which boils down to estimating the size of the eigenvalues of Frobenius.

Up to these things, the derivation from the cohomological product formula is essentially formal. The first observation is that, after having those two things, if F is an operator acting on V, then we have the following statement, F acts on the symmetric powers of V, and if we want the trace of F on $\operatorname{Sym} V$, it's the same as taking the exponential:

$$\operatorname{tr}(F, \operatorname{Sym} V) = \exp(\sum_{n>0} \frac{1}{n} \operatorname{tr}(F^n, V)),$$

as long as we have convergence. Now replace F, now $F = \mathrm{Frob}^{-1}$ acting on V, where $\operatorname{Sym} V$ is the cohomology of BG.

The exponential is the thing that turns the multiplicativity of the product formula to the additivity of the trace formula,

$$\begin{aligned}
\operatorname{tr}(\mathrm{Frob}^{-1}, C^*(\mathrm{Bun}_G)) &= \operatorname{tr}(\mathrm{Frob}^{-1}, \operatorname{Sym}(C^*(X, \pi^! V))) \\
&= \exp\left(\sum_{n>0} \frac{1}{n} \operatorname{tr}(\mathrm{Frob}^{-n}, C^*(X, \pi^! V))\right) \\
&= \exp\left(\sum_{n>0} \sum_{x \in |X|(\mathbb{F}_q)} \frac{1}{n}\right) \qquad \text{by G.–L. trace formula} \\
&= \exp\left(\sum_{x \in |X|} \sum_{n} \frac{1}{n} \operatorname{tr}(\mathrm{Frob}_x^{-n}, V_x)\right) \\
&= \prod_{x \in |X|} \operatorname{tr}(\mathrm{Frob}_x^{-1}, C^*(BG)) \\
&= \prod_{x \in |X|} \frac{|BG(K(x))}{q^{\dim \ BG_x}}
\end{aligned}$$

So this recovers the numerical product formula.

The goal now is to sketch the cohomological statement of the formula

$$\pi_? \pi^! C^*(BG) \cong C(\mathrm{Bun}_G).$$

It's quite different from the topological setting, where it's straightforward. Let me give a kind of a quick introduction to the subject of topological factorization homology. So before doing this, let me give the main ideas for the proof. So this is essentially two steps. The first step is non-Abelian Poincaré duality. Instead of proving something about the cohomology of Bun_G and cohomology of BG, we'll say something to link the homology of Bun_G to the homology of the double loop space of BG, i.e., the loop space of G. Then we'll do some Verdier or E_2-Koszul duality, which turns the C_* to C^* and then does something similar to the $C_*\Omega^2$.

This part showing up is a bit mysterious, and this is the relation to factorization homology.

The colimit I discussed last time is the simplest kind of factorization homology, with coefficients in a commutative algebra. Homology of $\Omega^2 BG$ is not a commutative algebra, so we have to capture what kind of commutativity this has, so the coefficients is some kind of E_n algebra. The heuristic or intuition is that an E_n-algebra interpolates between associative and commutative algebras. When $n = 1$, an E_1 algebra is an associative algebra. An E_∞ algebra is a commutative algebra. How do we formulate what an E_n algebra is?

So consider the following category $\mathrm{Disk}_n^{\sqcup}$. The objects are disjoint unions of $\mathbb{R}^n$. The morphisms are embeddings. A morphism from $(\mathbb{R}^n)^{\sqcup k}$ to $(\mathbb{R}^n)^{\sqcup \ell}$ is the space of embeddings between these spaces. So $\sqcup$ is a symmetric monoidal structure on this category.

Pick $C^{\otimes}$ to be a symmetric monoidal category, maybe either $\mathrm{Vect}^{\otimes}$ or $\mathrm{Spaces}^{\times}$. Then a definition is that an E_n-algebra in C is a (strong) symmetric monoidal functor from $\mathrm{Disk}_n^{\sqcup}$ to $C^{\otimes}$.

This looks a bit abstract so let me give some intuition about this definition. Let's start with $n = 1$. Then embeddings from two copies of $\mathbb{R}^1$ to one copy. So A of two copies becomes A of one copy squared. Then the map becomes $A(-)^{\otimes 2} \to A(-)$. So let me write $A = A(-)$ by abuse of multiplication. You see that for each embedding you get a multiplication map.

Now suppose you have three guys, you can [pictures]. This thing commutes, so that means that

$$\begin{tikzcd} & A^{\otimes 2} \arrow[dr] & \\ A^{\otimes 3} \arrow[ur] \arrow[dr] & & A \\ & A^{\otimes 2} \arrow[ur] & \end{tikzcd}$$

commutes. Then the homotopy structure encoded shows that you have associativity here. Because we're working with $(\infty, 1)$-categories. When we say $(ab)c = a(bc)$ classically, we say there's an invertible morphism between them that satisfies coherence.

Now let's look at E_2 algebras. The picture is that embeddings are of two-disks into two-disks. You have relative position of points (essentially) in a plane instead of a line. If you have two guys, for each configuration you get a multiplication. If we fix one guy, then the multiplication is parameterized by, essentially, a circle, homotopically speaking, the second disk can be anywhere except at the first disk, so we have a circle's worth of multiplications.

Then when $n = \infty$, because S^{∞} is contractible, essentially, this gives you the fact that multiplication is commutative. Fix a point, and then you get $\mathbb{R}^{\infty} - \{\mathrm{pt}\}$. Let's take a break here.

6. Part 6

So first I'll talk about factorization homology in topology. So I sometimes write Spaces and sometimes Top. Let me try to be consistent, if you do a left Kan extension along a vector space V

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \mathrm{Vect} \\ \mathrm{Top} \arrow[ur, dashed] & \end{tikzcd}$$

then you get $C_*(X, V)$. We can do a similar thing for factorization homology, you have a fully faithful embedding of $\mathrm{Disk}_n^{\sqcup}$ into $\mathrm{Mfd}_n^{\sqcup}$, the category of n-manifolds and embeddings and you can do a Kan extension

$$\begin{tikzcd} \mathrm{Disk}_n^{\sqcup} \arrow[r, "A"] \arrow[d, "\alpha"'] & C^{\otimes} \\ \mathrm{Mfd}_n^{\sqcup} \arrow[ur, dashed, "\mathrm{LKE}_\alpha A"] \end{tikzcd}$$

This is the *factorization homology* of M with coefficients in A, denoted $\int_M A$.

That looks abstract but there are cases where you say something about it. The first remark is that any E_∞ algebra is also an E_n-algebra, and so suppose we have A a commutative algebra object in some category $\mathcal{C}$, then we can forget to E_n algebras, and compute $\int_M A$. The theorem says that

$$\int_M A = \operatorname*{colim}_{U \in U_0(M)} A$$

This is where you have only one disk. This is a theorem of Ayala and Francis. This is the thing that we use to formulate the topological version of the infinite tensor product. So in particular, if you start with V then you can form the free commutative (polynomial) algebra generated by it, and we see that

$$\int_M \operatorname{Sym} V \cong \operatorname{Sym} C_*(M, V)$$

in vector spaces. The second case I want to mention is free E_n-algebras. So note that if, if you start with a theory of algebras, you want the free objects, if you have $E_n - \mathrm{alg}(\mathcal{C})$ you can forget by oblv to $\mathcal{C}$ and then the left adjoint is Free_{E_n}. So one thing you want to understand is what is the free E_n-algebra. So the general way to think about free objects, if you think about free associative algebras, you take all the tensors and add them up. So you take the things that are not there and add them, step by step.

Intuitively, we can do the following. First, we want to add how to multiply k things together, but in, say, E_2 algebras, you have a circle. You have something with topology. It looks like the topology of configurations in the plane. So what you do is you take

$$\bigoplus_{k=0}^{\infty} \left(C_*(P \operatorname{Conf}_k \mathbb{R}^n) \otimes V^{\otimes k} \right) / \Sigma_k$$

where $P \operatorname{Conf}_k X$ is X^k minus the fat diagonal where any x_i is x_j.

This is at the intuitive level but it's not very conceptual. Suppose we have, on the one hand we have, let's consider the following category $\mathrm{Disk}_n^{\mathrm{iso}}$. The objects are integers at least 1, well, finite sets of size n (let's do the non-unital version), and the morphisms are isomorphisms, and the symmetric product is $\sqcup$.

Note that (strong) symmetric monoidal functors from $\mathrm{Disk}_n^{\sqcup}$ to $\mathcal{C}^{\otimes}$ is the same thing as E_n-algebras in $\mathcal{C}$. We also have (strong) symmetric monoidal functors from $\mathrm{Disk}_n^{\mathrm{iso},\sqcup}$ to $\mathcal{C}^{\otimes}$ and because this has no interesting maps this is just $\mathcal{C}$ by evaluating at the singleton set. Suppose you have an object in $\mathcal{C}$. The category $\mathrm{Disk}_n^{\mathrm{iso},\sqcup}$ embeds into $\mathrm{Disk}_n^{\sqcup}$ with k points becoming k copies of $\mathbb{R}^n$. Now the pullback along this is the forgetful functor and our task is to produce the left adjoint, the left Kan extension

$$\begin{tikzcd} \mathrm{Disk}_n^{\sqcup} \arrow[dr, dashed] & \\ \mathrm{Disk}_n^{\mathrm{iso},\sqcup} \arrow[u] \arrow[r] & \mathcal{C} \end{tikzcd}$$

and using this kind of reasoning, with $\mathrm{colim}_{x\in X} V \cong C_*(X,V)$, you recall the definition of left Kan extension, compute a colimit, you see that the configuration spaces show up, and so say you want to compute

$$\int_M \mathrm{Free}_{E_n}\mathbb{C}$$

The factorization homology is left Kan extension and the free thing is the left Kan extension and by abstract arguments left Kan extensions compose, and you get

$$\bigoplus_{k\geq 1} C_*(\mathrm{Conf}_k M, \mathbb{C})$$

Knudsen used this to give a concrete formula for the cohomology of configurations and with Gabriel did computations explicitly in many cases.

Now I want to move on to the non-Abelian Poincaré duality and then apply this to Bun_G. All of these have the feature that I do something to make an E_n algebra and then take factorization homology with coefficients there.

Suppose X is a based topological space then you can take $\Omega^n X$ which is the maps from S^n to X, this is maps with compact support from $\mathbb{R}^n$ to X. Then $\Omega^n X$ has the natural structure of an E_n-algebra. How so? Suppose you have a map with compact support from two disks to X, then you can extend by zero to a bigger disk holding them.

So is there anything you can say about factorization homology of a manifold with coefficients in such an algebra?

Theorem 6.1 (Non-Abelian Poincaré duality— Lurie; Ayala–Francis)**.** *Suppose M is an n-dimensional manifold and X is $(n-1)$-connected. Then*

$$\int_M \Omega^n X \cong \mathrm{Maps}_c(M,X).$$

When $X = BG$ and M is a curve, an algebraic curve, if you think of a manifold C, say, compact, then you have the statement that $\int_C \Omega^2 BG \cong \mathrm{Maps}(C,BG) \cong \mathrm{Bun}_G$.

So the goal is to implement non-Abelian Poincaré duality in algebraic geometry, so the steps will be

(1) what is an E_n-algebra in algebraic geometry?
(2) factorization homology
(3) realize $\Omega^2 BG$
(4) prove non-Abelian Poincaré duality

So how do we get E_n algebras? The diagonal of X in $X\times X$ is a copy of X, and the complement is two points floating in X, so in each way to move to the diagonal, that's like encoding multiplication. We want a sheaf $\mathcal{F}$ on X equipped with the following piece of data:

$$i^! j_! j^* \mathcal{F}^{\boxtimes 2} \to \mathcal{F}$$

where

$$\mathring{X}^2 \xrightarrow{j} X^2 \xleftarrow{i} X$$

You also need similar conditions for higher powers, you need higher powers in the towers of X. This is a picture in X^2. Most of what I want to talk about is in X^2. We want to use this multiplication to produce a sheaf on X^2. The way to do it is to look at the following, $i_! i^! j_! j^! \mathcal{F}^{\boxtimes 2} \to i_! \mathcal{F}$ by applying $i_!$ and look at the diagram of sheaves in X^2 and get a pushout

$$\begin{tikzcd} i_! i^! j_! j^! \mathcal{F}^{\boxtimes 2} \arrow[r] \arrow[d] & i_! \mathcal{F} \arrow[d] \\ j_! j^! \mathcal{F}^{\boxtimes 2} \arrow[r] & \mathcal{F}^{(2)} \end{tikzcd}$$

and then if you do $i^! \mathcal{F}^{(2)}$, you just get $\mathcal{F}$, and if you restrict by $j^!$ you get $j^! \mathcal{F}^{\boxtimes 2}$ because this is supported on the diagonal.

This tells us to maybe reformulate E_n-algebras as follows. Let's say, for every non-empty finite set I, we want a sheaf $\mathcal{F}^I$ on X^I. When I is two elements, we get this $\mathcal{F}^{(2)}$. The properties are

- If $\alpha : I \twoheadrightarrow J$ then $\Delta_\alpha X^J \to X^I$ and we want, for each such α, then I want $\Delta_\alpha^! \mathcal{F}^I \cong \mathcal{F}^J$, this is a general version of the first point
- So $\mathcal{F}^I$, when you restrict it to $\mathring{X}^I$, you should get $\mathcal{F}^{\boxtimes I}|_{\mathring{X}^I}$.

There's also an Σ_I-action that I didn't say explicitly.

So let me say one thing about how to do this more generally and then we can formulate factorization homology in algebraic geometry. Eventually we want a geometric object, a prestack with a sheaf that captures these properties. So suppose Y is a prestack, a contravariant functor from schemes to groupoids, so I want the sheaves on Y. This is the limit over $S \to Y$ of $\mathrm{Shv}(S)$. Essentially this is the usual trick. So Y is some object that you are making up, it doesn't really exist, but if it exists, then for each map from a scheme to it, you can pull back, so we turn around and use this as a definition. Then the transition maps [unintelligible]. I ran out of time, can I have five more minutes?

So automatically, we know how to pull back between sheaves on two stacks, suppose f is a morphism of prestacks $Y_1 \to Y_2$, then $f^!$ is a functor from $\mathrm{Shv}(Y_2) \to \mathrm{Shv}(Y_1)$. Then one can essentially by abstract nonsense, say that $f^!$ admits a left adjoint $f_!$. Then the prestack we're interested in is the *Ran prestack*. Suppose X is a scheme, then $\mathrm{Ran}\, X$ is a prestack, in this case, it's just a presheaf valued in groupoids, so

$$(\mathrm{Ran}\, X)(S) = \{\text{nonempty finite subsets of } X(S)\}.$$

[pictures]

One can show that, fact, $\mathrm{Ran}\, X$ is the colimit of X^I, where I runs over finite sets, nonempty, with surjections.

This colimit is in prestacks. It kind of looks like my X^I glued all together. This formulation says that sheaves on $\mathrm{Ran}(X)$ are the same thing as the limit of sheaves on X^I where I runs over all finite sets.

Let's unwind this definition. For each $\alpha : J \twoheadrightarrow I$, we get

$$\begin{tikzcd} X^I \arrow[rr, "\Delta_\alpha"] \arrow[dr] & & X^J \arrow[dl] \\ & \mathrm{Ran}\, X & \end{tikzcd}$$

It's compatible with the diagonal pullbacks. The upshot is that a sheaf on the Ran space (prestack) is the same as sheaves on powers of X satisfying the first condition, the Ran condition. With some more work (Beilinson–Drinfeld; Francis–Gaitsgory) you can capture the factorizable condition or fatcorization condition. So the output is that you get $\mathrm{Fact}(X)$, the factorizable sheaves on X. One last thing, we know what a factorizable sheaf is, what is factorization homology?

Now it's not mysterious. Now $\mathrm{Ran}(X)$, the structure map π to a point, for any sheaf $\mathcal{F}$ on $\mathrm{Ran}(X)$, you can define $C_c^*(\mathrm{Ran}\,X, \mathcal{F})$ and so in particular, when $\mathcal{F}$ is a factorizable sheaf, then $C_c^*(\mathrm{Ran}(X), \mathcal{F})$ is the algebro-geometric analog of factorization homology.

Tomorrow I'll define the algebraic analogue of the loop space and say precisely what it is that we are doing.

7. January 25: Part 7

Proposition 7.1. *If X is connected, then $C_*(\mathrm{Ran}\,X) = C_c^*(\mathrm{Ran}\,X, \omega_{\mathrm{Ran}\,X}) \cong \bar{\mathbb{Q}}_\ell$.*

I didn't talk about the dualizing sheaf of a prestack. Given $\mathcal{Y} \xrightarrow{\pi} \mathrm{pt}$ induces $\pi_!$ and $\pi^!$ functors between sheaves on $\mathcal{Y}$ and vector spaces, and then $\omega_{\mathcal{Y}} = \pi^! \bar{\mathbb{Q}}_\ell$.

So the proof is simple. We start with

$$H_0(\mathrm{Ran}\,X) = H_0(C_*(\mathrm{Ran}\,X)) = \bar{\mathbb{Q}}_\ell$$

since X is connected. Let n be the first integer where $H_n(\mathrm{Ran}\,X)$ is zero. Then by Künneth,

$$H_n(\mathrm{Ran}\,X \times \mathrm{Ran}\,X) \cong H_n(\mathrm{Ran}\,X) \oplus H_n(\mathrm{Ran}\,X).$$

We have a union map $\mathrm{Ran}\,X \times \mathrm{Ran}\,X \to \mathrm{Ran}\,X$ which induces a map on the homology level

$$H_n(\mathrm{Ran}\,X) \oplus H_n(\mathrm{Ran}\,X) \to H_n(\mathrm{Ran}\,X)$$

and essentially by symmetry I can tell that this map has to be $u \oplus u$ for some u.

On the other hand I can consider the diagonal map

$$\mathrm{Ran}\,X \xrightarrow{\Delta} \mathrm{Ran}\,X \times \mathrm{Ran}\,X$$

If I postcompose the union I get the identity map, so at the level of homology, $2u = \mathrm{id}$ as maps from $H_n(\mathrm{Ran}\,X) \to H_n(\mathrm{Ran}\,X)$. Now you do the same game for three factors, and get $3u = \mathrm{id}$. That means $u = 0$ so $\mathrm{id} = 0$ and the only space is the 0 vector space.

This is a fundamental result in this theory, contractibility of the Ran space. In this algebro-geometric setting I think it was first considered by Beilinson and Drinfeld.

Last time I outlined factorization homology in algebraic geometry. One thing I had to do was alegbrize the double loop space $\Omega^2 BG$. We want non-Abelian algebraic geometry, $\int_X \Omega^2 BG = \mathrm{Bun}_G$.

So we want to look at $\mathrm{Maps}_c(\mathbb{D}, BG)$, so what does compact support mean? There's a distinguished point of BG, and everything at ∞ maps there. So let's interpret this. That's the same as a G-bundle on $\mathbb{D}$ but everything at ∞ maps to the distinguished point. So we want a trivialization of the bundle $\mathcal{P}$ at the boundary of the disk.

How do we do something like this in algebraic geometry? Vell $\mathbb{D}$ is $\mathrm{Spec}\,k[[t]]$ and the boundary should not have the special point, $\mathrm{Spec}\,k((t))$. That means we want to consider the following moduli problem. We want G-bundles plus a trivialization. For each ring R we want to associate the following data, $\mathcal{P}$ is a G-bundle on $\operatorname{Spec} R[[t]]$ and γ is a trivialization on $\operatorname{Spec} R((t))$. Here R is a K-algebra.

This is classical in representation theory. In fact, it's represented by an ind-scheme, namely, an inductive limit of schemes, called the affine Grasmannian. Gr_G.

Remark 7.1. Classically, this space also appears in number theory, the k points

$$Gr_G(k) = G(K)/G(\mathcal{O}) = G(k((t)))/G(k[[t]])$$

By Lang's theorem, any bundle over [unintelligible]is trivial. We have a bundle over $k[[t]]$, and the [unintelligible]is smooth, so if the special fiber is trivial then the whole thing is trivial. A trivialization at the boundary is a point in G. You mod out the different trivializations on the whole disk.

So we have some algebro-geometric object that plays the role of $\Omega^2 BG$. We want to be able to say that Gr_G is a factorizable scheme. Before doing that we want a more concrete description of Gr_G.

We have the following, given R we want a G-bundle $\mathcal{P}$ on $X_R = X \times \operatorname{Spec} R$, so fix $x \in X$. Then γ is a trivialization of $\mathcal{P}$ on the complement of $x \times \operatorname{Spec} R$.

These are the same. A trivialization on the punctured disk, you glue to get a bundle on the whole thing, or just restrict back. So the two moduli problems are the same, this is a result by Beauville–Laszlo. In the picture here I've fixed x and to say something about the Ran space we have to let the point x move as well.

The new moduli problem associates to R first a point $x \in X(R)$ and $\mathcal{P}$ a bundle on X_R and then γ is a trivialization of $\mathcal{P}$ outside of x.

[picture]

Let me name this $\mathrm{Gr}_{G,X}$, and you can (by the same idea) show this is representable by an ind-scheme. You can map to X by forgetting everything except the point. So if x is a k point of X, then the fiber is Gr_G. So this is a family of Gr_G where you let the point x move.

Now we have one point, we're getting there. Let me do multiple points and formulate the Ran version of the affine Grasmannian. We want to say in what sense this double loop space is an E_2 algebra. Consider the moduli problem

$$R \mapsto \begin{cases} I \subset X(S), \text{ i.e. } I \in (\operatorname{Ran} X)(S) \\ \mathcal{P} \text{ a } G\text{-bundle on } X_R \\ \gamma \text{ a trivialization of } \mathcal{P} \text{ outside } \Gamma_I. \end{cases}$$

We want a bundle with a trivialization outside of the graph. So I'll write $\mathrm{Gr}_{\mathrm{Ran}}$ fixing a group G. There's a map to $\operatorname{Ran} X$ by forgetting everything but I. This is sometimes called the Beilinson–Drinfeld affine Grassmannian. Let' say in what sense this is a factorizable scheme over $\operatorname{Ran} X$. Suppose you have $X^2 \to \operatorname{Ran} X$, this is the canonical map, and you can pull back $\mathrm{Gr}_{\mathrm{Ran}}$ and get Gr_{G,X^2}, and pull back to the complement of the diagonal to get $\mathrm{Gr}_{G,\mathring{X}^2}$. So $(\mathrm{Gr}_{G,\mathring{X}^2})(R)$ is pairs of disjoint points x_1 and x_2 in $X(R)$ and P is a G-bundle on X_R and γ is a trivialization outside x_1 and x_2.

Now note that because the two are disjoint, you can construct a G-bundle on the formal disk around each and a trivialization on the boundary. So that's the same as $\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}$ pulled back to $\mathring{X}^2$.

This is also by Beauville–Lazslo. This is the precise sense in which the Ran version of the affine Grassmanian is a [unintelligible]version of [unintelligible].

Okay now there is a morphism from $\mathrm{Gr}_{\mathrm{Ran}}$ to Bun_G which induces an equivalence on the level of homology.

A couple of remarks.

(1) One can prove that the affine Grassmannian is proper, and so one can form $f_!\omega_{\mathrm{Gr}_{\mathrm{Ran}}}$, and because of the factorizable property, we get factorizable sheaf on $\mathrm{Ran}\,X$, i.e., an object in $\mathrm{Fact}(X)$.
(2) Remember the adelic interpretation $G(K)\backslash G(\mathbb{A})/G(O)$. For $\mathrm{Gr}_{\mathrm{Ran}}$ it's $G(\mathbb{A})/G(O)$. It's trivialized outside a finite number of points. Then the map is the quotient map by $G(K)$.

Okay, so let me call this object $\mathcal{A}$, this $f_!\omega_{\mathrm{Gr}_{\mathrm{Ran}}}$. And let me formulate this à la non-Abelian Poincaré duality. So

$$\begin{aligned} C_*(\mathrm{Gr}_{\mathrm{Ran}}) &\cong C_c^*(\mathrm{Gr}_{\mathrm{Ran}}, \omega_{\mathrm{Gr}_{\mathrm{Ran}}}) \\ &\cong C_c^*(\mathrm{Ran}\,X, f_!\omega_{\mathrm{Gr}_{\mathrm{Ran}}}) \\ &\cong C_c^*(\mathrm{Ran}\,X, \mathcal{A}). \end{aligned}$$

So this is the factorization homology of $\mathrm{Ran}\,X$ with coefficients in $\mathcal{A}$.

So the map $\mathrm{Gr}_{\mathrm{Ran}} \to \mathrm{Bun}_G$, this map is an equivalence if and only if factorization homology with coefficients in $\mathcal{A}$ is $C_*(\mathrm{Bun}_G)$. So this is non-Abelian Poincaré duality in this case.

Let me briefly say how to try to say that $\mathrm{Gr}_{\mathrm{Ran}} \to \mathrm{Bun}_G$ is an equivalence. Then the strategy for the proof of non-Abelian Poincaré duality is as follows. So $\mathrm{Gr}_{\mathrm{Ran}} \to \mathrm{Bun}_G$, we have [pictures]

The first step is a result of Drinfeld and Simpson which says that locally on R, any G-bundle on X_R is geometrically trivial. This says that if you look close enough, then you can find a divisor and make this trivial outside of it. From this result one cane reduce to the case when P is trivial.

So what is the problem and how do we solve it when the bundel itself is trivial? Right, so let Bun_G get its map from $\mathrm{Gr}_{\mathrm{Ran}}$ and also from $\mathrm{Spec}\,R$, and we want to look at the pullback. We want, the problem becomes comparing the cohomology of Bun_G and $\mathrm{Spec}\,R$. The pullback—to make things simpler let me assume $R = k$. The pullback has thy following moduli interpretion. It takes R to sets consisting of a subset of $R(S)$, a genereic trvialization (i.e., a map from $X_R\backslash\pi$) to G.

So suppose we can solve the problem. Let Y be an affine scheme, then

$$\mathrm{Maps}(X, Y)_{\mathrm{Ran}}$$

contains all of the data, $I \subset X(S)$ and γ with Y this time. Suppose Y can be covered by open affine subschemes, they're isomorphic to open subschemes of $\mathbb{A}^n$, then this thing has trivial homoloogy.

It's conjectured that it's anything smooth and birational to $\mathbb{A}^n$ but it's not known. So these $\mathrm{Maps}(X, Y)_{\mathrm{Ran}}^{\mathrm{rat}}$ should be considered as $Y(k(X))$. It's not outrageous, it's not a proof, but $\mathbb{G}_m(k(X)))$, this is kind of like $\mathbb{R}^\infty \smallsetminus \mathrm{pt}$. This is known to be contractible.

Let me give a sketch of how to prove this thing. The first step is to show that

$$\mathrm{Maps}(X, U)_{\mathrm{Ran}}^{\mathrm{rat}} \to \mathrm{Maps}(X, U \subset_{\mathrm{gen}} Y)_{\mathrm{Ran}}^{\mathrm{rat}}$$

(where in the codomain, the maps are those maps $X \to Y$ that go generically to U) induces an isomorphism on homology when U is open in Y. Then the next step is to show that

$$\mathrm{Maps}(X, U \subset_{\mathrm{gen}} Y)^{\mathrm{rat}}_{\mathrm{Ran}} \to \mathrm{Maps}(X, Y)^{\mathrm{rat}}_{\mathrm{Ran}}$$

[unintelligible]$Y = \mathbb{A}^n$.

So then for each U we have an open subprestack

$$\mathrm{Maps}(X, U \subset_{\mathrm{gen}} Y)^{\mathrm{rat}}_{\mathrm{Ran}} \to \mathrm{Maps}(X, Y)^{\mathrm{rat}}_{\mathrm{Ran}}$$

so you can cover by these and it suffices to figure out here. So we can look at

$$\begin{tikzcd}
 & \mathrm{Maps}(X, U \subset_{\mathrm{gen}} Y)^{\mathrm{rat}}_{\mathrm{Ran}} \arrow[dr] & \\
\mathrm{Maps}(X, U)^{\mathrm{rat}}_{\mathrm{Ran}} \arrow[ur] \arrow[dr] & & \mathrm{Maps}(X, Y)^{\mathrm{rat}}_{\mathrm{Ran}} \\
 & \mathrm{Maps}(X, U \subset_{\mathrm{gen}} \mathbb{A}^n)^{\mathrm{rat}}_{\mathrm{Ran}} \arrow[r] & \mathrm{Maps}(X, \mathbb{A}^n)^{\mathrm{rat}}_{\mathrm{Ran}}.
\end{tikzcd}$$

and then some of these maps are equivalences, and then we're done since this is an affine space.

The second step, the first step was

(1) To link $C_*(\mathrm{Bun}_G) \cong C_*(\mathrm{Gr}_{\mathrm{Ran}})$.
(2) Then use Verdier duality on the Ran space—this doesn't behave well since $\mathrm{Ran}\, X$ is infinite dimensional, e.g., $D\omega_{\mathrm{Ran}} = 0$, so apply $D\mathcal{A}$, denote this B, and we show that the fiber of B is the same as the cohomology of BG.

The cohomology of BG is close to the cohomology of G, and it's also close to the cohomology of ΩG, which is $C^*(\Omega^2 BG)$. The link is E_2-Koszul duality, and then Verdier duality switches homology and cohomology and all together we have the following statement, $C^*(\mathrm{Bun}_G)$ is the same as $C^*_c(\mathrm{Ran}\, X, B)$, where B comes from $C^*(BG)$.

This part is a complicated exercise in homological algebra. Sorry for the overtime.

8. Final part

I am going to say something about my work now. One motivation for me is that factorization homology has some nice product formula. So now if you see an infinite product you can ask whether factorization homology gives you a cohomological version of it.

Let me start with a number theory observation. Suppose you are interested in the density of square-free integers over all numbers,

$$\lim_{n\to\infty} \frac{|\{d \in [1,n] : d \text{ squarefree}\}}{n} = \zeta(2)^{-1}.$$

This is classical and well-known. Less well-known is

$$\lim_{d\to\infty} \frac{|\{(m,n\} \in [1,d]^2 : m,n \text{ relatively prime}}{d^2} = \zeta(2)^{-1}$$

So the thing that unifies these, for m and n integers, look at a_1 through a_m inside $[1,d]^m$ that are relative n-prime (I'll explain this) and divide by d^m, the limit as $d \to \infty$ is $\zeta(mn)^{-1}$.

Relative n-prime means that $p^n \nmid \gcd(a_1, \dots, a_m)$ for any p.

So $(1,2)$ is the first version and $(2,1)$ the second. This is number theory, you can do the function field analogue. We're counting points.

So let X_0 be a scheme over $\mathbb{F}_q$, if you let X be $X_0 \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$.

So the points on X_0 should be thought of as primes. Then we're thinking about symmetric powers. So fix m, n, and we want to consider $\underline{d} = (d_1, \ldots, d_m)$ and

$$\mathrm{Sym}^{\underline{d}} = \prod_{i=1}^{m} \mathrm{Sym}^{d_i} X_0.$$

Okay, and the thing about this, we'll take an open subscheme and remove a certain locus, disallowing certain configurations. So we will call this $Z_\infty^{\underline{d}}(X)$, and

$$Z_n^{\underline{d}}(X_0)$$

will be the open subscheme of $\mathrm{Sym}^{\underline{d}}(X_0)$, these are particles in X_0, colored, such that no point on X_0 appears with multiplicity at least n for all colors.

Let me give an example. When $m = 1$ and $n = \infty$ then this is $\mathrm{Sym}^d X_0$, the symmetric powers, When $m = 1$ and $n = 2$ then this is ordered configuration space of X, then you have $\mathrm{Conf}_d X_0$, so no point can have multiplicity 2.

When $m = 2$ and $n = 1$ then this is $Z_1^{(d_1,d_2)}(X)$ so $(\mathrm{Sym}^{d_1} X \times \mathrm{Sym}^{d_2} X)_{\mathrm{disj}}$. So you can take the limit as $d \to \infty$

$$\lim_{d\to\infty} \frac{|Z_n^{\underline{d}}(X_0)(\mathbb{F}_q)|}{|Z_\infty^{\underline{d}}(X_0)(\mathbb{F}_q)|} = \zeta_X((\dim X)mn)^{-1}.$$

So it's natural to ask a question, is this a shadow of some cohomological coincidence?

Let me phrase the question more precisely. Let me introduce some more notation,

$$Z_n^m(X_0) = \coprod_{\underline{d}} Z_m^{\underline{d}}(X_0)$$
$$Z_\infty^m(X_0) = \coprod_{\underline{d}} Z_\infty^{\underline{d}}(X_0)$$

Let me refine the discussion. I want to consider the cohomology as a whole,

$$A_{m,n}(X) = \bigoplus A_{\underline{d},n}(X) = \bigoplus C^*(Z_n^{\underline{d}}(X)).$$

That's the first step, and we we can do either n or ∞, and the next step, I have an asymptotic statement, so we want an asymptotic statement in the world of cohomology, so we want to make sense of

$$\lim_{d\to\infty} A_{\underline{d},n}(X).$$

This is a limit in the world of chain complexes, you should have maps to take some sort of colimit, we want a map $A_{\underline{d},n}(X) \to A_{\underline{d}+1_k,n}(X)$. So having such a map is enough to compute, to define the colimit, so the question is, is there homological stability? Suppose you fix a degree i, do you have a map

$$H^i(A_{\underline{d},n}(X)) \to H^i(A_{\underline{d}+1_k,n}(X))$$

and you'd like this to be an equivalence when n is big compared to i.

So we'd like to make sense of the quotient homologically, want something that looks like

$$\frac{\bar{A}_{m,n}(X)}{\bar{A}_{m,\infty}(X)}$$

where $\bar{A}_{m,n}$ is our colimit of $A_{\underline{d},n}$, so that after Frobenius trace we recover the numerical density.

So you can phrase this in singular or de Rham cohomology until the end. So in some sense we use arithmetic to guess what the cohomology thing should be.

So this question was considered by Farb–Wolfson–Wood. Let me say what is known about this problem.

- in the topological steting, there is a paper of Knudsen, who proved homological stability in the case $(1,2)$, so for configuration spaces.
- There was work by Kupers–Miller, who proved homological stability for $(1,n)$,
- Farb–Wolfson–Wood proved stability for (m,n) arbitrary but X smooth and coincidences in quotients between Poincaré series for Hodge–Deligne polynomials.

What do I mean? Even though we don't know the quotient, what they prove is that the Poincaré polynomials, if you divide them as series, you get this kind of thing with the same kind of coincidence as in the numerical version.

So we want to be able to do the non-smooth case and do this at algebra. So they do some Leray spectral sequence and combinatorics with the strata and computations with the spectral sequence. It's geometric but we don't see why the coincidences appear. So a question is how to see these quotient coincidences conceptually.

We can handle the non-smooth case as well, but anyway, let me assume smooth in this talk. What is the idea of this quotient? The idea is very simple. The upshot is that derived tensors of commutative algebras give you the quotients. Let me give a very simple example of why it is true. In the category of graded commutative algebras, Λ is a base field, usually it's field coefficients, either $\mathbb{C}$ or $\bar{\mathbb{F}}_q$. So t has cohomological degree 0 and internal degree 1. So if I take $\chi^{\mathrm{gr}}(\Lambda[t],u) = 1+u+u^2+\cdots = \frac{1}{1-u}$.

So now let's consider $\Lambda\otimes^L_{\Lambda[t]}\Lambda$. I'll always derive so I'll leave out L, and so this is $\Lambda\oplus\Lambda[1]$, let's compute $\chi^{\mathrm{gr}}(\Lambda\otimes_{\Lambda[t]}\Lambda)$, this is $1-u$.

This suggests that we should equip A^m_n with the structure of a graded commutative algebra, graded by m colors. Note that when X is geometrically connected, then $H^0(A^m_n(X)) = \Lambda[x_1,\dots,x_m]$, keeping track of connected components of $Z^m_n(X)$. If this is a commutative algebra, then we can act on it by x_i, which will give a map from $A^m_{\underline{d}}(X)$ to $A^m_{\underline{d}+1_k}$. So then we get these maps to formulate cohomlogical stability.

The second game that we get is to compute, show that the same kind of coincidences happen for $A^m_n(X)\otimes_{A^m_\infty(X)}\Lambda$ and then show that in the $\mathbb{F}_q$-setting, taking Frobenius trace recovers numerical density. Essentially this boils down to equipping this space with a commutative algebra structure.

Not surprisingly the tool that we'll use is factorization homology. We'll use the following, recall that X, we have $\mathrm{Shv}(X)$, we want to do $\mathrm{Com\,Alg}(\mathrm{Shv}(X))$, so now it's natural to use graded commutative algebra objects, graded by $\mathbb{Z}^m_{>0}$. We have the adjoint functors $\pi_?$ and $\pi^!$ between this and $\mathrm{Com\,Alg}(\mathrm{Vect}^{\mathbb{Z}^m_{>0}})$.

One can show (up to some details) that

$$\chi^{\mathrm{gr}}_{\mathrm{Frob}_q^{-1}}(\pi_? A,t) = \prod_{x\in|X|}\chi^{\mathrm{gr}}_{\mathrm{Frob}_x^{-1}}(i^!_x A, t^{\deg x}).$$

This sort of multiplicativity allows us to recover the zeta values, and the proof is similar to the Tamagawa number one case.

Now let's take the link between conifiguration spaces and factorization homology. In the graded setting, we have a graded version of the Ran space, which is simpler, because here the different degrees are separated. The graded version, it's $Z^m_\infty(X)$, so this has all these powers, glued together. Now let me say what is a commutative factorization algebra on X via Ran. Essentially, $\operatorname{Ran} X \times \operatorname{Ran} X \to \operatorname{Ran} X$, the union map, using that you can construct the star monoidal structure on sheaves, namely

$$\mathcal{F} \overset{\bigstar}{\otimes} \mathcal{G} = \cup_*(\mathcal{F} \boxtimes \mathcal{G})$$

and then you can talk about commutative algera objects on sheaves on $\operatorname{Ran} X$ under $\overset{\bigstar}{\otimes}$, and let $\operatorname{Com}\operatorname{Fact}(X)$ be the ones in their with an appropriate restriction off of the diagonal. Then the proposition of Gaitsgory–Lurie is that $\operatorname{Com}\operatorname{Alg}^!(X) \cong \operatorname{Com}\operatorname{Fact}(X)$. If I start with a commutative factorazation algebra, I can pull back along the map $X \to \operatorname{Ran} X$. and get something in the appropriate category. This $\delta^!$ has a left adjoint $\delta_?$, and in fact this is an equivalence of categories.

Let me give you an idea of how this functor δ_+ can be thought of. Start with A a commutative algebra object under $\otimes^!$, on the first power of X I get A itself. Then next I get X^2, the flavor is similar, and now for commutative algebras it's a bit simpler. An object there has $A^{\otimes^! 2} \to A$, I can rewrite this

$$\Delta^!(A^{\boxtimes 2}) \to A$$

and I can postcompose wiht $\Delta_!$ and then $A(2)$ is the pushout as follows

$$\begin{array}{ccc} \Delta_!\Delta^!(A^{\boxtimes 2}) & \longrightarrow & \Delta_! A \\ \downarrow & & \downarrow \\ A^{\boxtimes 2} & \longrightarrow & A^{(2)}. \end{array}$$

It's the same as before but simpler. Now we can use this construction to find which A to plug in. The reference for higher powers of X, which are more complicated, this is in the paper of Gaitsgory, semi-infinite IC something Ran. He produced a precise diagram that one can take.

All right so using this one can guess what commutative algebra object to put in to prove this.

Remark 8.1. Suppose $A \in \operatorname{Com}\operatorname{Alg}^!(X)$. On the one hand you can do $\pi_? A$ which is a left adjoint to the pullback. There's a way to compute it, you can do $\delta_?(A)$ and then compute the cohomology $C^*_c(\operatorname{Ran} X, \delta_? A)$, and that's $\pi_? A$.

The upshot after thinking about these diagrams is the following, the objects in $\operatorname{Com}\operatorname{Alg}^!(X)$ that we're interested in are pullbacks from Vect of $A_{m,\infty} = \Lambda[x_1, \dots, x_m]$ where x_i is in degree 1_i and cohomological degree $2d$, with Tate twist d. This is essentially just a polynomial algebra, so the factorization homology is a symmetric thing and recovers Z^m_∞, so

$$\pi_? \pi^! A_{m,\infty} \cong A_{m,\infty}(x).$$

Then $A_{m,n}$ is the same guy but with the relation $(x_1 \cdots x_m)^n$. Intuitively, we want the stalk at the disallowed locus to be zero. So densities, then the ? is a left adjoint and preserves pushouts, and here that's relative tensors.

So

$$A_{m,n} \otimes_{A_{m,\infty}} \Lambda \cong \Lambda \oplus \Lambda[1-2(\dim X)mn](-(\dim\ X)mn)$$

with this latter in degre $(n, \ldots, n)$. And you see the ζ values appearing.

There are two things remaining:

- homological stability and
- the multiplication formula

The idea is to use Koszul duality between commutative algebras and Lie algebras. The real version should be with Lie coalgebras so there's some shift, and one of them is the coChevalley complex. This has a filtration whose associated graded is the symmetric algebra, and we know how to handle these.

The ? is kind of exponential in nature. The nature is this quotient, and what I want to say is that $\pi_? A$, if $A = \operatorname{co}\operatorname{Chev}\mathfrak{a}$ then $\pi_? A = \operatorname{co}\operatorname{Chev}(\pi_! \mathfrak{a})$, and this is a generalization of $\pi_? \operatorname{Sym} = \operatorname{Sym} \pi_!$. Then we can prove the product formula in this case.

The associated Lie algebra is just two steps, it's very simple, and essentially from, it exchanges trivial and free. The objects are not free but are close enough, and then one can formulate a theorem about homological stability of factorization homology. I think I'm way over time.