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1. Dmitri Pavlov, Ordinary Differential cohomology via smooth topological field theories

I would like to express ordinary differential cohomology of a smooth manifold X as isomorphism classes of smooth n-dimensional field theories over X. This statement involves a lot of complicated machinery. It would be best to explain the easy one dimensional case first, discrete, smooth, and homotopy being three separate flavors, which correspond to different kinds of line bundles over X (flat, with connection, and arbitrary). In the second part we will discuss generalizing these three types to arbitrary dimensions. The goal is to categorify these. The smooth one will have to do with differential cohomology.

Maybe it would make sense to start with some motivation. The motivation is the cobordism hypothesis, which of course is now a theorem due to Hopkins and Lurie. I will only need the easiest version, with framed manifolds. If X is a space and C an (∞, n)-category, which has morphisms at all levels and only invertible morphisms above level n, and I assume that all objects and morphisms and so on are dualizable, (I should have said symmetric monoidal), for instance for vector spaces it is the finite dimensional spaces, we can compute the topological field theories over X, $Fun^{\otimes}(Bord_n^{fr}(X), C) \cong Maps(X, k(C))$, where k is the core of C, the infinity category you obtain by discarding non-invertible morphisms.

For example, we can take $n = 1$, and if the category is one object and morphisms $U(1)$, then you'll get flat line bundles.

This statement explicitly allows us to compute the discrete and homotopy case, but not the smooth case, because that requires the smooth structure of the manifold, whereas this version of the cobordism hypothesis involves only the topological structure. Most of the talk will be about explaining how to make this smooth. If you have things varying smoothly on one side, then they should also vary smoothly on the other; that's very imprecise. I'll say more later.

I have to say something non-rigorous which will help to describe what's going on.

Conjecture 1.1. *If X is a smooth manifold and C is the same as before, but now fibered over smooth manifolds MAN, then we would like to have some statement of the form:*

$$Fun^{\otimes}(Bord_n^{fr,smooth}(X), C) \cong \text{“differential refinement of } Maps_{smooth}(X, k(C))\text{”}$$

I cannot really define this object for you on the right hand. We can say what it is in some easy cases, for example refinements of categorifications of line bundles (bundle n-gerbes) should be bundle n-gerbes with connection.

Let's try to develop more or less rigorously some simple examples. Before I describe differential refinements, I would like to say a few words about the familiar case, the Atiyah-Siegel style topological field theories. So let me say some words about discrete 1-TFTs?

Definition 1.1. *If X is a space, then a* 1*-TFT over X (I should say from the start, my target will be always very simple category, with only one morphism at every level but an Abelian group at level* 1*, so $B^n(A)$ for some Abelian group A) is a functor $\pi_{\leq 1}(X) \to BU(1)^{disc}$. The fundamental groupoid has objects points and morphisms homotopy classes of paths. The category on the right has one object and $U(1)$ morphisms, but with no topology, viewed discretely.*

There are a couple of problems. The target here is not rich enough to give flat line bundles. I will make a simplifying assumption.

Remark 1.1. *All categories and functors and so on is a stack with respect to X, so we can assume that X is diffeomorphic to $\mathbb{R}^n$. Using the machinery of stacks we can get the statement for arbitrary X.*

Proposition 1.1. *Isomorphism classes of* 1*-TFTs are the same as isomorphism classes of flat line bundles over X.*

Why is this true actually? Let's define a map from right to left. The way you turn a flat line bundle into a field theory is, you have to send all of the objects to the single object in $BU(1)^{disc}$. We have a homotopy class of paths, but since the line bundle is flat, parallel transport is invariant under homotopy, and so parallel transport gives an element of $U(1)$.

Let's end the first part here, and let's talk about removing the flat condition. We'd expect that the definition would be pretty similar. You can still get a one dimensional TFT over X, you can take a smooth path to its parallel trasport. How can we make this precise so that the statement is true replacing discrete with smooth? This is well-known. The earliest reference I know is due to Dan Freed, 1992.

Let's follow the same pattern. If X is a manifold, then a smooth 1-dimensional TFT over X is a functor

$$1 - Bord_X^{smooth} \to BU(1)^{smooth}$$

What kinds of objects are these? They are not ordinary categories. Because of the smoothness condition, we have to do something more complicated, with Grothiendieck fibrations. I will not try to give a precise description of a Grothiendieck fibration. This category, $1 - Bord_X^{smooth}$, what does it mean to specify such a category? For every smooth manifold S in MAN, we have $1 - Bord_X^{smooth}(S)$ (this is an ordinary category) and for a map $S \to T$ we get a functor, a pullback functor $1 - Bord_X^{smooth}(T) \to 1 - Bord_X^{smooth}(S)$. This amounts to saying you have a functor $MAN^{op} \to CAT$. You have to use the appropriate notion of functor. Let's describe what those categories and functors are.

$1 - Bord_X^{smooth}(S)$ has as objects, you should think about them, over each point in S you have something that looks like $\pi_{\leq 1}X$, that is, $1 - Bord_X^{disc}$, and similarly for morhpisms. The objects are maps $S \to X$, smooth, think about it as, over every point of S you have an object, a map from a point to X. The morphisms, similarly, they are slightly more difficult, but not that difficult. A morphism between two maps f and g from S to X, a morphism $f \to g$ is the following data. Take a bundle over S with fibers intervals, and a map $E \to X$ such that if we restrict to one endpoint we get f and for the other g.

If you have two paths and compose them in the obvious way, the composition does not appear to be smooth. This can be resolved in two ways. One can allow piecewise smooth morphisms, or require that points on the boundary of the interval have a small interval which extends a little past the point, and then the neighborhoods must agree, and the neighborhood is now part of the object. You require that the little germs agree, which gives you a smooth interval. You can also require them to be constant near the endpoints. Again, this is all pretty much irrelevant, so I'd rather stick with the simple version, but you can make it precise if only we had more time.

If we have a map $S \xrightarrow{f} T$, then a functor between such categories $1 - Bord_X^{smooth}(T) \to 1 - Bord_X^{smooth}(S)$ is just the pullback along f. One way to interpret informally is thatin this setting, the discrete setting, you can say that objects are maps from points to X, but you think of the objects as a manifold. Say that the objects form a smooth manifold and so do the morphisms. When you talk about functors, they are maps on objects and morphisms, and those also have to be smooth maps. Once we give a smooth topology, we'll get a very different answer than in the discrete case.

I guess I have to erase something.

[Are you trying to describe conditions on the holonomy given as a functor that will determine a smooth line bundle?]

Yes.

Once we have this definition, and strictly speaking I also have to define the target category, so what is $BU(1)^{smooth}$? It's pretty easy to describe. So $BU(1)^{smooth}$ has, fixing a smooth manifold S, there is only one object, and the morphisms are smooth maps $S \to U(1)$. Then we can say that a functor between such fibered categories, a functor between such categories is a natural transformation. For morphisms, for every manifold S, for any morphism we get a bundle of intervals over S, so you should get a smooth map $S \to U(1)$ in the target category. Intuitively, you think of the value of the morphism, the family of paths as a parallel transport functor, the requirement is that if you have a family of intervals varying smoothly over some base, then the parallel transport also varies smoothly over S. There is no reason why this would be true in the discrete case. You'd only get this for fibered field theories.

Theorem 1.1. *(Freed)*

$$\pi_0(1 - TFT^{smooth}(X) \cong \textit{isomorphism classes of line bundles with connection}$$

These are also called first differential cohomology of X.

Again, I suppress some technical details. Strictly speaking, I should be taking isomorphism classes of E modulo some equivalence relation, which again is not very important.

[Is that equivalent because parallel transport depends only on the path.]

Yes. Also, on constant paths you get trivial parallel transport.

I don't have much time left. I hope the essential idea of the approach is clear. Let me very briefly describe the rest.

Let me describe how you get line bundles without connection, which is the third part of the outline.

We know how to get flat line bundles and line bundles with connection. The remaining case is how to get line bundles without any connection. This is given by homotopy topological field theories. What is that? Again, I use the same structure

Definition 1.2. *Let X be a space. Then a one dimensional homotopy TFT over X is a functor $1-Bord_X^{homotopy} \to BU(1)^{homotopy}$, so let's describe those.*

Those are not ordinary categories but something slightly different. If you have two objects, morphisms between them are a set, but here we want morphisms to form a topological space. I don't want to get into the formalism of infinity categories, so I'll pretend these are categories enriched in topological spaces. The objects on the left are points in X, and morphisms are paths between the points. There is one object of $BU(1)^{homotopy}$, and the morphisms are $U(1)$, viewed as a topological space.

Proposition 1.2.

$$\pi_0(1-TFT_X^{homotopy}) = \textit{isomorphism classes of line bundles over } X$$

[Does such a thing exist at all? Holonomy on all continuous paths that composes?]

With the cobordism hypothesis, this can be computed as $[X, BU(1)] = H^1(X, U(1)) = H^2(X, \mathbb{Z})$.

I don't require that parallel transport be compatible with composition, but only up to homotopy.

Let me finish the talk by erasing another blackboard and then summarizing what we have done so far.

Summary: flat bundles are classified by $H^1(X, U(1)^{disc})$, and these give bundles with connection, and of course there is an obvious map that forgets the connection, so you get maps:

$$\begin{tikzcd}
 & H^1(X,U(1)^{disc}) \arrow[rr] \arrow[dr] & & H^1(X,U(1)^{top}) \arrow[dr] & \\
H^1(X,\mathbb{R}^{disc} \arrow[ur] \arrow[dr] & & \hat{H}^1 X \arrow[ur] \arrow[dr, "curv"] & & H^2(X,\mathbb{R}^{disc}) \\
 & \Omega^1 X/\Omega^1_{\mathbb{Z}} X \arrow[ur] & & \Omega^2_{\mathbb{Z}} X \arrow[ur] &
\end{tikzcd}$$

These diagonals are exact and everything commutes. A theorem due to Simons and Sullivan say that this diagram determines $\hat{H}^1$. The diagram can be generalised to arbitrary dimension. Set 1 to be equal to n and replace everything appropriately.

$$\begin{tikzcd}
 & H^n(X,U(1)^{disc}) \arrow[rr] \arrow[dr] & & H^n(X,U(1)^{top}) \arrow[dr] & \\
H^n(X,\mathbb{R}^{disc} \arrow[ur] \arrow[dr] & & \hat{H}^n X \arrow[ur] \arrow[dr, "curv"] & & H^{n+1}(X,\mathbb{R}^{disc}) \\
 & \Omega^n X/\Omega^n_{\mathbb{Z}} X \arrow[ur] & & \Omega^{n+1}_{\mathbb{Z}} X \arrow[ur] &
\end{tikzcd}$$

Theorem 1.2. *(Berwick-Evans, Pavlov,Stolz, Teichner)*
This diagram, for any n, admits an interpretation in terms of topological field theories of those three flavors with those definitions generalized, and provides a description of differential cohomology as described in the dimension one case.

[A functor in the homotopy case is in a functor?] No, it's a homotopy functor. In the smooth case it's an ordinary category.

2. Daniel Pomerleano: Positive output two dimensional field theories

Definition 2.1. *A is a* smooth dga *over a field of characteristic zero if A is perfect as an A-A bimodule, that is, A is in the smallest idempotent closed triangulated category of A-A bimodules generated by $A \otimes A^{op}$.*

One thing to notice is that, if M is perfect as an A-A bimodule, then you have the following isomorphism (using A^e for $A \otimes A^{op}$):

$$A \otimes_{A^e} M = HH_*(A, M) \cong Hom_{A^e}(Hom(M, A^e), A)$$

In particular, if $M = A$, you'll get an element in the Hom complex, and well, Hochschild homology has a circle action given by the Connes operator B and the homotopy fixed points of that circle action $HH_*(A, A) \to HH_*(A, A)$ is, well, negative cyclic homology is the homotopy invariants.

$$HC^-(A) := (HH_*(A, A)[[u]], b + uB)$$

There's a map π_* from $HC^-(A)$ to $HH_*(A)$ (induced by $u = 0$)

Definition 2.2. *A has an n-Calabi-Yau structure if there exists an element in $HC^-(A)$, α, such that $\pi_*(\alpha)$ induces a quasiisomorphism from $Hom_{A^e}(A, A^e) \to A[u]$*

The theorem, which is not mine, is

Theorem 2.1. *(Lurie)*
Given such a structure, we get a two dimensional field theory over the PROP $C_*(\mathcal{M}_{m,n})$ where the number of outputs n is bigger than zero.

One example, possibly a richer structure, is string topology. If M is a compact oriented manifold, then $A = C_*(\Omega M, \mathbb{Q})$, and so $HH_*(A) \cong H_*(LM)$, and you have a map $H_*(M) \to H_*(LM)$, and you want the fundamental class $\pi_*(\alpha)$.

This is all background. What I want to talk about is, can we deform, this will be formulated vaguely right now, can we deform $C_*(\Omega M)$ so that we can obtain a field theory over $C_*(\mathcal{M}_{m,n})$ with no restrictions on n or m. We'll make the question more precise by restricting M to a nice class in terms of its rational homotopy type.

Definition 2.3. *The rational homotopy type of simply connected M is* pure *if $C^*(M, \mathbb{Q})$ has a model of the following form:*

$$C^*(M, \mathbb{Q}) = \mathbb{Q}[u_1, \ldots u_\ell] \otimes \Lambda[\epsilon_1, \ldots \epsilon_k]$$

where $d\epsilon_i = f_i(u_j)$ with degree at least two and $du_j = 0$.

If you do a little rational homotopy theory, you realize that the Homology of the chains on the based loop space is a Clifford algebra over $\mathbb{Q}[v_1, \ldots, v_k]$, and you can apply a perturbation lemma to see that the higher operations have the property that $m_n(w, \alpha_1, \ldots, \alpha_n) = 0$ if w is in $\mathbb{Q}[v_1, \ldots, v_k]$, which tells you that these functions wt satisfy the Maurer Cartan equation in $HH^*(A, A)[[t]]$.

So you get a deformation, a curved deformation, from it.

I was going to give an example that might be familiar. Look at $\Lambda(\epsilon_1, \ldots, \epsilon_k)$, then $C_*(\Omega M) \cong \mathbb{Q}[v_1, \ldots, v_k]$, and then you can consider any function, and you get a category which is the category of matrix factorizations $MF(\mathbb{Q}[v_1, \ldots, v_k], w)$. Here you have projective modules, P_0 and P_1 with maps in each direction, and composition in each direction is multiplication by w.

Theorem 2.2. *The following are equivalent:*

- *w has isolated singularities.*
- *HH^* is finite dimensional.*
- *$MF(\mathbb{Q}[v_1, \ldots, v_k], w)$ is smooth, proper, and Calabi-Yau.*

$HH^0(MF)$ is the Jacobian ring of W. The rest of Hochschild cohomology is a module over this guy.

You can take coherent sheaves on $\mathbb{Q}[v_1, \ldots, v_k]/w$ and mod out by perfect sheaves (in the sense we had earlier) on the same thing. This equivalent to matrix factorizations via the cokernel.

We will define a category $PreMF(A, w)$ as $(A[\epsilon], d\epsilon = w)$-modules which are perfect over A.

Definition 2.4. *Define $MF(A, w)$ as $PreMF(A, w)$ modulo $Perf(A[\epsilon], d\epsilon = w)$.*

The theorem for these guys is that

Theorem 2.3. *the Hochschild cohomology $HH^*(MF(A, w))$ is finite dimensional if and only if $MF(A, w)$ is smooth, proper, and Calabi-Yau.*

Showing the sufficiency is good because the left hand side can actually be computed. For example, if $M = \mathbb{CP}^n$, then your model is

$$\mathbb{Q}[u] \otimes \Lambda[\epsilon], d\epsilon = u^{n+1}$$

Then $C_*(\Omega M)$ is $\mathbb{Q}[v] \otimes \Lambda(\gamma)$ with $m_{n+1}(\gamma, \ldots, \gamma) = v$, and $w = v^d$. This is an example, you can check that for every d you geta TQFT which has an interpretation in terms of Floer theory and the Fukaya category, but I'll stop here.

[Can you explain how the deformation gives a TQFT?]

It satisfies all the conditions to be fully dualizable.

[Is there something to do with the fundamental chain of the moduli of holomorphic disks?]

The relationship with Floer theory that I have in mind is that a compactification of the cotangent bundle gives something, and there are reasonable compactifications for the cotangent of $\mathbb{P}^n$.

3. Dennis Sullivan: Solved and unsolved puzzles in the FRG's focus on 2D field theories

I want to state four problems which are precise, although my presentation may not be. We may have to discuss it more. This meeting is using a lot the language of field theories. I wanted to concentrate on two dimensional field theories. There's always been this mixing of language that has confused and puzzled me. I'll start by making a comment about that.

Two dimensional field theories, I'm concentrating on those, can be caricatured, this talk will be a caricature of a math talk anways, by interactions of strings, if you have such a thing, even in its extended form, the boundaries are circles or arcs, and then you have algebraic structures on the surfaces with these boundaries. You can think of the surface as circles and arcs being swept out, and they interact and come out the other end. I remember fifteen years ago, physicists would draw pictures of string theory, where a string moves around in space, in this language, the vector space or thing that you attach to the circle, an element in that thing attaches a state to the circle. Think of the state as position in a manifold, it's moving in time according to some unknown evolution, you have more than one, and two come close together and maybe interact. Call that A. Or maybe you have a string that comes close to itself and breaks in two. Call that B.

Then you could also have non-closed strings, then I might change to open strings, and you could have them touch at the endpoints and come together, or they could break apart in a similar way. Call these C and D.

There are many more. You can mix these, have a non-closed [sic] string interacting with an open string and coming to an open string, or an open string breaking off a little closed string. These are E and F.

For example, let's talk about A. This one is very homogeneous. No matter where the interaction took place, there's no boundary.

Oh, I wanted to say, "Ho" can be completed to homotopic, homologous, or holomorphic. One can imagine discussions where these words are of primary importance. I don't want to say much about this, except to say the strings, in algebraic geometry, there's a version of talking about closed strings in a holomorphic context. Drinfel'd and Beilenson talk about it in dimension two and I think Kevin's extending it to higher dimensions. I"ll be mainly thinking in homotopic and homologous, and mostly in homologous.

So comments on A. It has no boundary. This is kind of an algebra, two things combining. This is not associative, it satisfies Jacobi, it's related to a Lie algebra structure. You might think, how does this satisfy Jacobi? Three pieces giving zero? You have to interact three strings to satisfy Jacobi. The third string interacts, and you get six possible terms which cancel in pairs. You should draw this picture and then try to make it concrete.

Okay, this structure is not literally in Daniel's statement of Lurie's theorem, but this is an equivariant version. By the nature of Lurie's theorem, if these strings are moving in a manifold—

Definition 3.1. *String topology is all conceivable operations where you interpret states as families of strings and perform classical intersection procedures to make things make sense, interpreted in algebraic topology.*

There are a couple of general things I forgot to say. For F, a non-closed theory gives you closed strings.

According to Lurie's machine, this A is homotopy invariant. That's good news in low dimensions, that's powerful in two dimensions and three dimensions, since most things in those dimensions is homotopy invariant. For example, Chas and Krongold showed that this Lie algebra solves the disjunction problem. Suppose you have two complicated curves on a surface. You can make them disjoint if the brackets of all powers of the first with all powers of the second are zero.

On a surface, curves automatically interact or not. This is a non-trivial theorem.

In three dimensions, you can use the structure of this Lie algebra (this is Chas-Gadgil), interacting with geometrization, you can read the number of pieces and decomposition sites using this Lie algebra.

B has a boundary. You can imagine tiny interactions, and then you get no interaction. One piece gets smaller and then disappears. This is a kind of boundary in the middle of the process. We'll call this boundary an "anomaly."

Let's remember it has a boundary. The operation B has a boundary. It satisfies a coJacobi identity dual to the other thing. There's a way to get rid of the boundary by modding out by the small loops. Then algebraically you repress the boundary and get something defined modulo fixed points of the circle action. The two operations satisfy the Drinfel'd Lie bialgebra identity and another, called the involutive identity. These identities correspond to studying the four surfaces of Euler characteristic -2. The point I'm dealing now is irrelevant, but when you deal with the anomaly, you can add pieces at infinity, and you get these four relations.

The first problem can be stated now.

Problem 3.1. *Is this combined structure* not *a homotopy invariant?*

This is an involutive Lie bialgebra. For surfaces, this was done (without involutive) by Goldman and Turaev.

One piece of this is described carefully in Sullivan, more is in Kate Poirier's thesis, and there's a natural compactification germane to this problem, and joint work of Kate Poirier with Nathaniel Rounds and Gabriel C. Drummond-Cole is describing this.

Problem 3.2. *Describe the relevant compactification of moduli spaces that describes (universal) string interactions.*

You'd like to hide as much boundary as you can. I'm stating this for closed strings, you could also do it for open strings. What can we say about that? For C I didn't say enough before, it's natural to think of the open strings as running between two things.

The compactification will have limit points where you go toward Deligne-Mumford points while still keeping a harmonic function. You can intuitively see that it acts in string topology.

So C. C has no boundary, no new boundary. It's associative. If you get technical, you might have to do an ∞ version of that, and maybe you have to do a category version, let's call it associative. There have been many papers written, Graeme Segal had this idea that non-closed strings should be morphisms in a category, this leads to categorical language, then many people have done many things about this category. In particular, Kevin (Costello) explained how when you have non-closed strings, where does this operation come from, if you take a Hochschild construction, you get the closed strings. If you take the associative algebra and take a Hochschild, you get a Lie algebra, and so A comes in directly out of C, thanks to Kevin. Did you think of that yourself?

[Kevin: I learned a lot from Kontsevich.]

[The cobracket has an anomaly? Does that have an interpretation?] I think that's work to be done. Let's say I don't know, I have some things I could say, but, [unintelligible].

Kevin says that the category, if you take the non-closed strings, that forms a category, and [missed].

I have another list of names here. There's a solved problem, if you put yourself in the appropriate setting, and assume that the equivariance of the circle action can be deformed to the identity, then you can see that the theory extends to the full, Deligne-Mumford compactification. So this leads to Hochschild, et cetera.

Now, D, let's discuss D. I've always wondered, you get a category here. Do you guys talk about cocategories? D is a coalgebra, maybe sometimes you don't have to say the co word if you use duality. So D is coassociative. It's like a coalgebra, this was a Lie algebra, Lie coalgebra, associative algebra, and now we have the associative coalgebra, but now it has boundary. Whatever made this string cut, it could be the end of your surface was like this, your cut may happen very near the boundary. There's some special circumstance you have a cut too near the boundary. I've been puzzling about this for years in the very simple setting, where the based loop space is a non-closed string where the two ends agree. So you should think of this as part of non-closed theory. Just this case is interesting, it would be a free loop that intersects itself near the ends. You can't do the trick of modding out by short things, because they are the unit.

Now the anomaly of D has been treated, after a long interlude, using transversal boundary conditions. You say, if they touch the boundary condition again, they're transversal. So I do these operations in the new space of strings, cutting out the infinite but finite codimension subset that you don't want. The algebraic topology changes, and you can still do operations. I'm hoping that when you do these operations on geometrically restricted strings, you get invariants, the dream is to get four-manifold invariants.

Basically, three manifolds are determined by their homotopy types, except for lens spaces and formal things made up out of them. These have played an interesting role in topology, so for example $L(7,1)$ and $L(7,2)$ have the same homotopy type ($L(p,q)$ and $L(p',q')$, the fundamental group gives you p and if q and q' differ by something with a square root, you get isomorphic cohomology rings). So these are homotopy equivalent but not homeomorphic.

This was a problem we posed, is string topology a homotopy invariant? This modified version is not. What Somnath does, you take the space of all these transversal strings, several intersections, you can't remove a transversal intersection.

You want to get this coalgebra structure. You take the chains on this space, you have the coalgebra Δ, and on a chain, it's the alternating sum of its splittings. You make a differential which is the sum of the internal differential on the chains, plus the alternating sum of resolution, and you take the sum over all ways of avoiding each bad point. You get a differential coalgebra out of the open strings, and then step two, you have to take a fiber construction and you take the cobar construction on this, and it turns out you kind of find the Pontrjagn ring of the based loop space of the compliment, twisted by α, the element of H_1 in the loop space. You use rational homotopy theory ideas here to find Massey product structures in covers of $L(7,1)$ and $L(7,2)$, and one of these has trivial Massey products and the other doesn't. This is based on the work of Salvatore and Longoni showing that these have different Massey product structures.

There's an interesting codimension two example, I'll do this fast so I can state the other two problems, this is joint with Michael Sullivan, using Somnath and Jason McGibbon. If you look at all the spheres, the top homology of the zero sphere is $\mathbb{Z} \oplus \mathbb{Z}$. You'll add these two ways of resolving a codimension two singularity. You take the cobar as Somnath does, but with this notion of resolution, getting a differential graded coalgebra, and then you get an algebra, it doesn't have a unit because of the twisting, and you get the twisted group ring of the complement, applying this to knots in S^3, and then (that was paper one) you add a unit and put the meridian in the center, adding a relation, an algebraic construction out of D, and this is isomorphic to the Lenny Ng coalgebra, produced out of the knot complement, and we get this chord algebra out of a functorial operation in the string theory. He was able to distinguish all knots up to 11 crossings. It's also isomorphic to degree zero Legendrian knot homology, [lost]. So that's a powerful knot invariant.

For the second paper, the Costello idoa is that somehow the open strings determine the Lie algebra of closed strings, and applying that to this type of algebra, using C, resolving, and one finds the Lie algebra of the complement, and then by Chas-Gadgil argument, you can read off the geometric decomposition of the complement. One can read that off by applying Hochschild to the combined structure. That leads me to:

Problem 3.3. *Describe the algebraic structure of this transversal open string theory. You can get rid of the anomaly and the answer is different.*

When you state a problem like this, the problem is to formulate the problem. What Jacob Lurie did with this category theory, he found the structure of fully dualizable objects. To describe what these structures might be, as a conjecture, there are many instances of this thing I just mentioned, the Ng chord algebra which is something you get out of looking at the cotangent to S^3 and J-holomorphic curves, you have a general situation where taking all the possible constructions, this is some sort of real analogue of J-holomorphic curves. When you study open sypmlectic manifolds with contact boundary, and look at the J-holomorphic curves there, and the compactification is almost the same, here there is a function which is superharmonic. There are minima but no maxima. You use the symplectic filling of the contact manifold. I don't know the analogue of the transversality.

Problem 3.4. *Formulate conjectures relating the naive real analogue in string topology corresponding to J-holomorphic curves.*

Is symplectic playing the role of Poincaré duality in this realm? You don't really have anything until you have a symplectic structure. Latshev and [illegible] did this in the case of [lost].

[What about chain level]

[Some discussion]

There's a fifth question. Let me say some things. We learned about calculus, we took duals, obstruction theory, Steenrod algebra, infinity algebra, and someone said, if you make the simplices tiny, there's the Grassman algebra, and that models these things. So I was imagining that all of this algebraic topology in the loop space, there would be some de Rham theory that would be defined by a limiting procedure, and some beautiful structure of an algebraic nature. You can imagine doing that, that's problem five.

[Kevin: Gabriel has a nice paper he wrote last year, proving that result.] I was reading it last night, it's in genus zero. I couldn't remember a lot of this stuff, I got back some wonderful responses from people that I asked for references, Gabriel's paper is there, some of your papers, [unintelligble]'s papers, Teleman's papers, I decided I couldn't talk about it, just write some names down.

4. Kevin Costello I, Supersymmetric field theory and derived geometry

I'll apologize in advance, there will be no algebraic topology, these will be far from what the people here are thinking about. So physicists have gotten a lot of mileage out of supersymmetric field theory, have shown some good results in mirror symmetry, in S-duality (this is that $N = 4$ gauge theory for G is the same as for the Langlands dual group L_G).

It's long been a dream of mine to formulate and maybe prove some of these duality results.

People know something about these, they've done work in mirror symmetry. Almost all math work on this kind of thing involves topologically twisted theories, like Donaldson theory, Gromov-Witten invariants, and the A and B models of mirror symmetry. I say almost all, there have been some nice work about two dimensional theories that are not twisted, like Peter and Stephan.

I'll be interested in things between the physical supersymmetry and the topologically twisted theories, something called *minimally twisted* field theories. At a heuristic, big picture level, the physical is concerned roughly with the Riemannian or Kähler geometry of moduli spaces. The minimally twisted is concerned with the holomorphic geometry of moduli spaces, and the topologically twisted theories are about the algebraic topology of moduli spaces.

[Why do physical theories take so much data?] In quantum mechanics you can get the moduli of the equations of motions on the target M, this target will be a moduli space coming from some other field theory. Tdis will involve spinors and a Dirac operator, there will be heat flow on M. A field theory on $X \times \mathbb{R}$, we reduce to a field theory on $\mathbb{R}$, then we're studying, roughly, quantum mechanics on the space of solutions to the equations of motion on X. One reason quantum field theory is hard is because the solutions to the equations of motion may be infinite dimensional.

Topologically twisted	Minimally twisted	Physical
algebraic topology	holomorphic geometry	Riemannian/Kähler

The goal of the lectures are:

(1) define a classical field theory
(2) discuss supersymmetry in dimension four, with twisting
(3) describe minimally twisted theories
(4) see what the theories are at the quantum level.

The diagram gets easier to the left and there are inclusions, the physical theory knows about all of these. The minimally twisted theory gives us some but not all of the information of the physical theory.

So, what is a classical theory? I'll give a fancy definition with no Lagrangians. I never got anywhere near where I wanted to go. That time I got to defining a classical theory, so this time I'll go faster.

Suppose we have M a manifold, a "space-time." I want to give a fancy definition first that's a little hard to make precise, at least for me.

Definition 4.1. *(Ideal definition)*
A classical field theory on M is a sheaf of "derived stacks" of some kind on M equipped with a Poisson structure (bracket of degree one).

The derived stacks should be the space of solutions to the equations of motion, that might be infinite dimensional, that's hard.

[Why is there a bracket?]

This is a derived critical locus, so as part of the BV formalism you get this structure, that's how you know it's not a random sheaf of spaces.

To make things easier, I'll define a perturbative classical field theory. To make this definition, I'll need to recall deformation theory. If X is a derived stack and x is a point in it, then there is a differential graded Lie (or L_∞) structure on the shifted tangent space $T_xX[-1]$, and this structure describes completely a formal neighborhood of x. I think Joey is going to talk about this this week. I'll cite Lurie's DAG X and Hirsh's thesis. Suppose we work near the given solution to the equations of motion, then a sheaf of derived stacks is described by a sheaf of differential graded Lie algebras.

Definition 4.2. *An elliptic differential graded Lie algebra on M is a sheaf of topological differential graded Lie algebras $\mathcal{L}$ such that:*

1. *there's a graded vector bundle L on M with $\mathcal{L}$ the sheaf of sections $\mathcal{L}(U) = \Gamma(U, L)$*
2. *d is a differential operator and $[\,,\,]$ is bidifferential*
3. *(technical condition) $(\mathcal{L}, d)$ is an elliptic complex*

As an example, let X be a complex manifold and $P \to X$ a holomorphic G-bundle, then $\Omega^{0,x}(X, \mathfrak{g}_P)$, the Dolbeaut complex with coefficients in the adjoint Lie algebra, is an elliptic differential graded Lie algebra corresponding to moduli of holomorphic G-bundles near P.

We've successfully simplified the first part, and now we can ask what the Poisson structure is. I always seem to get tangled up in definitions. Suppose X is a derived stack which is symplectic. Then T_xX will have an antisymmetric pairing. Then the shifted tangent space $T_xX[-1]$ will have a symmetric pairing which is invariant for the Lie bracket. When we translate from a derived stack to a sheaf of Lie algebras, the Poisson structure becomes a kind of invariant pairing.

An example with a pairing is: M is a three-manifold, $P \to M$ a flat G-bundle, then $\Omega^*(M, \mathfrak{g}_P)$ has a pairing from the usual Poicaré pairing and the Killing form (G is semisimple).

This motivates the definition:

Definition 4.3. *If $\mathcal{L}$ is an elliptic differential graded Lie algebra, then $\mathcal{L}^!(U) = \Gamma(U, L^* \otimes Dens_m)$, the Verdier dual of $\mathcal{L}$.*

An invariant pairing of degree k on $\mathcal{L}$ is an isomorphism of $\mathcal{L}$-modules from $\mathcal{L}$ to $\mathcal{L}^![k]$ which is symmetric.

A classical field theory *is an $\mathcal{L}$ with an invariant pairing of degree -3.*

So let's look at Chern-Simons theory. If $\mathcal{L} = \Omega^*(M, \mathfrak{g}_P)$ for M a three-manifold, so it turns out that $\mathcal{L}^! = \mathcal{L}[3]$, so we have a pairing of correct degree.

[some discussion of various points]

So far we have a definition of a classical field theory. I could give more examples. A field theory is some sort of symplectic thing. One easy way to get those is as cotongent bundles.

Definition 4.4. *If $\mathcal{L}$ is an elliptic differential graded Lie algebra. Then the cotangent field theory is the elliptic differential graded Lie algebra $\mathcal{L} \oplus \mathcal{L}^![-3]$ with the natural action.*

Why would you do this? You should think of $\mathcal{L}$ as a sheaf of spaces. This corresponds in the language of derived stacks to replacing X by $T^*[-1]X$, which automatically has a symplectic form of the correct degree.

Just as symplectic cotangent bundles are the simplest class of symplectic manifolds, cotangent field theories are the simples examples of field theories of this sort.

Here's one example. Suppose that X is a complex surface, and $\mathcal{L} = \Omega^{0,*}(X, \mathfrak{g}_P)$. Then the cotangent theory is

$$\Omega^{0,*}(X, \mathfrak{g}_P) \oplus \Omega^{0,*}(X, \mathfrak{g}_P \otimes K_X)[-1]$$

so the corresponding moduli space is G-bundles on X with adjoint valued sections in $H^0(X, K_X \otimes \mathfrak{g}_P)$.

4.1. **supersymmetry in dimension four.** A supersymmetric field theory on $\mathbb{R}^4$ is a classical field theory in the sense that we defined with an action of $Spin(4) \ltimes \mathbb{R}^4$ and an extentsion at the Lie algebra level to a certain $\mathbb{Z}/2$-graded Lie algebra.

Recall, $Spin(4) = SU(2) \times SU(2)$. Let S_+ and S_- be the fundamental 2-complex dimensional representations of the two copies. Let $V_{\mathbb{R}}$ be the fundamental representation of $Spin(4)$. When I tensor with $\mathbb{C}$, there is an isomorphism $V_{\mathbb{C}} \cong S_+ \otimes S_-$ as $Spin(4)$-representations.

This allows us to define the $N = 1$ translation algebra:

$T^{N=1}$ is the $\mathbb{Z}/2$-graded Lie algebra with $V_{\mathbb{C}}$ in the even part, and the parity shift of $S^+ \oplus S^-$ in the even part. The bracket is the map $S^+ \otimes S^- \to V_{\mathbb{C}}$ which we wrote down before.

Evidently, $Spin(4)$ acts on this guy, so a supersymmetric field theory is one where $\mathfrak{spin}(4, \mathbb{C}) \ltimes T^{N=1}$ acts, extending the action of $Spin(4) \ltimes \mathbb{R}^4$.

For this to make sense, our classical field theory must be defined over $\mathbb{C}$. It also, well, we were thinking of $\mathbb{Z}$-graded objects classically. So we also need $\mathcal{L}$ to be a cochain complex of $\mathbb{Z}/2$-graded vector spaces. The differential respects the supersymmetric grading. So we'll have degree $(1, 0)$.

Next time I wanted to talk about twisting. People should make sure to remember the supersymmetry Lie algebra because I'll be using that.

5. Ryan Grady, algebraic index theory

Thanks. So, uh, yeah, I also want to talk about observables, so let's talk about observables and the algebraic index theorem, okay, so, ah, sort of, the algebraic index theorem is an approach using ideas from quantization to discuss index theory and index theory in noncommutative geometry. The theorem, due to Nest-Tsygan with an alternative proof due to myself and my collaborator Owen Gwilliam is:

Theorem 5.1. *Let X be a manifold. Then $HC(Diff_X^{\hbar}) \cong (\Omega^{-*}(T^*X)[[u]][[\hbar]]ud + \hbar L_\pi + \hbar\{log\hat{A}_u(X), \quad\})$. Here $|u|$ is 2. So putting u in with d makes it degree $+1$ again. L_π is the Poisson homology differential, the Lie derivative with respect to π. The interesting part is the last one. This gets adjusted a little bit because of the u but it's like something from index theory.*

This comes out of a BV construction, and this is the BV bracket.

A remark is in order. Why might this be related to index theory? Any trace will factor through Hochschild homology. Nest and Tsygan give you the Atiyah-Singer index theorem out of this. That's sort of, Owen and I are on that path, there's work to be done.

So what's a sketch of the proof?

[What's the left hand side?]

I have my algebra elements $\sum \hbar^i F^i Diff_X$, then I want the circle invariants, maybe it would be easier to write Hochschild homology but that wouldn't give what I want.

The right hand side will be the global quantum observables. $Diff_X^\hbar$ will be the local quantum observables. Then the left hand side is the factorization homology of the local quantum observables. Now I'll discuss the various pieces in here.

Before I get to constructing the field theory, let me discuss observables and the structures that they possess. I'd like to discuss factorization algebras. These are precosheaves and then they have some properties, let M be a space, a factorization algebra on M is an assignment $F(u)$ of a chain complex for any open set $U \subset M$. We satisfy some conditions.

(1) If $U \subset V$, then we get a map $F(U) \to F(V)$, compatible with double inclusions.
(2) Suppose I have pairwise disjoint open sets in V, I get a map $\bigotimes F(U) \to F(V)$
(3) Locality: "F is determined by its values on small balls."

Let me give an example, with $M = \mathbb{R}$, coming from an associative algebra A. Define $F_A(U) = A$ for any open set A, and that satisfies the first condition, and then I use the algebra multiplication. By the inclusion of two smaller open subsets into a bigger one, I can use multiplication on the algebra A.

Theorem 5.2. *(Lurie, Gwilliam)*
There is an equivalence of categories between associative algebras and factorization algebras on $\mathbb{R}$ which are translation invariant. This should have some other strictness

You can lose the strictness and then work on $\mathbb{R}^n$ and then you replace associative algebras by E_n algebras.

Now, it's a fact that factorization algebras satisfy descent for group actions. What do I mean? If I have a G-equivariant factorization algebra on M then I get a factorization algebra on M/G.

Suppose we let $S^1 = \mathbb{R}/\mathbb{Z}$, then F_A is $\mathbb{Z}$-equivariant and so descends to a factorization algebra on S^1, and I can compute its global sections. I can ask what it associates to all of S^1. I might call this taking factorization homology. You can identify this with the Hochschild complex $A \otimes^{\mathbb{L}}_{A\otimes A^{op}} A$.

So I hinted at this earlier, but these things show up naturally when you consider observables in quantum field theory. What is the data of a classical field theory? Kevin gave a nice definition this morning but let me give a new definition. What Kevin called $\mathcal{L}$ I call $\mathcal{E}$, which are sections of $E \to X$, I have a degree 1 differential Q, a local functional I, and a pairing $\langle\, ,\, \rangle$. The locality depends only on the jets.

If I let φ denote a section then $S(\varphi) = \langle \varphi, (Q + I)\varphi\rangle$. In Kevin's example this morning of Chern-Simons theory (in three dimensions), we had $\mathcal{E} = \Omega^*(M) \otimes \mathfrak{g}[1]$,

$Q = d$, and what should I be? It just is given by the Lie bracket $[\,,\,]$. So up to some constants that I'll neglect, it's $S(\alpha) = \frac{1}{2}\langle \alpha, d\alpha\rangle + \frac{1}{6}\langle \alpha, [\alpha, \alpha]\rangle$.

Let me say what a quantization of a classical field theory is. This consists of the following. It is a collection of interaction terms $\{I[i]\} \in \mathcal{O}(\mathcal{E})[[\hbar]] = \widehat{Sym}(\mathcal{E}^\vee)[[\hbar]]$. This needs to satisfy some conditions, let me write them down and then explain them verbally. These satisfy:

(1) RGE (renormalization group equation), things depend on l nicely.
(2) QME (imposes some naturality)
(3) Locality
(4) $\lim_{i\to 0} I[i] \cong I \mod \hbar$

I'd be happy to talk more about any of these later, but maybe after the talk.

I told you all of this factorization algebra business, but let me write down a chain complex that comes from the global observables. The global observables, well, we have the classical ones and then we have the quantum ones. The chain complex here is $Obs^{\varphi} = (\widehat{Sym}(\mathcal{E}^\vee), Q + \{I, \quad\})$. That's the classical observables. The quantum observables at scale L are $(\widehat{Sym}(\mathcal{E}^\vee)[[\hbar]], Q + \hbar\Delta_L, \{I[i], \quad\}_L)$.

So the classical structure has a Poisson bracket of degree $+1$, I'll call that a P_0 algebra, whereas the right hand side isa BD algebra. I make this remark, I start with a P_0 algebra, and I want you to give me a natural quantization which is a BD algebra.

We can also define local observables and they form a factorization algebra. I want to give an example. Let's take a Riemannian manifold. I want my fields $\mathcal{E}$ to be C^∞_M, I trivialize and can think of it as top forms $C^\infty(M) \to \Omega^n(M)[-1]$. Suppose we have typical elements φ and ψ, I want to discuss some classical observables. If I give you a pair, then I could do $(\varphi, \psi) \mapsto \varphi(x)$, that's degree 0. I could do $(\varphi, \psi) \mapsto \psi(x)$, which has degree -1, or I could do more interesting combinations, say, $(\varphi, \psi) \mapsto \varphi(x)\varphi(y)\psi^2(x)$, that's degree -2. What is the differential? $(Q\psi)(x) = \Delta\varphi x$, and $Q\varphi = 0$.

You can compute the homology of local or global observables. I have $H^*(Obs^{cl}(U)) \cong \mathbb{C}[q_1, \ldots, q_n, p_1, \ldots, p_n]$. The quantum observables are more interesting, I still have my q and p variables, and I adjoin $\hbar$, modulo $[q_i, p_i] = \hbar$.

What is the main theorem? It is,

Theorem 5.3. *(Grady, Gwilliam)*
For X a smooth manifold, there exists a one-dimensional Chern-Simons theory such that $Obs^q(U) \cong Diff^{\hbar}_X$, and such that the global observables is this funky $\Omega^- * (T^*X)[[u]][[\hbar]], ud + \hbar L_\pi + \{\log \hat{A}_u(X), \quad\})$.

The theory is, it studies the moduli of maps $S^1 \to X$ but in a neighborhood of constant maps.

After telling you what the theory is, this is a good place to stop.

6. Kate Poirier, compactifying universal string topology

Thanks a lot, Stephan, I'm happy to speak and see everyone again. I added the word universal to emphasize that we're talking about all possible loops.

Let me remind you of a problem that Dennis mentioned yesterday, to describe the relevant compactification of moduli space for (universal) string topology operations on chains.

We have known for some time that the homology of open moduli space acts on the homology of the free loop space of a manifold, and there are versions on the paths. I'll stay with non-open or closed, and I want to put off until later whether I'm talking about the equivariant or non-equivariant version, and so I'll be casual about basepoints.

What we want to see is an action of chains of a compactification of the moduli space on chains of the free loop space. I'll give a fat graph model in a minute, but let's look at a particular example, using the harmonic functions approach that Dennis described yesterday.

Definition 6.1. *Let $M(g, k, \ell)$ be the moduli space of Riemann surfaces, I'll draw a picture of what I mean, surfaces with boundary, but I want to consider punctured surfaces, so boundary at infinity, and the punctures at infinity will be labelled as inputs at the top and outputs at the bottom (k and ℓ) and genus g. I'll decorate the outputs with weights w_i which are nonnegative numbers with sum 1. I might want to add decorations to the boundary depending on whether I'm living in the equivariant world, but I'll be casual.*

So we've taken the usual moduli space here and crossed with the simplex. I want to look at the compactification of $M(0, 2, 2)$ because that's the first place that interesting things happen. So this looks like $\mathcal{M}_{0,4} \times \Delta^1$. I'll draw this as the open disk with two disks removed, and then cross it with the interval.

So I can think of the compactification in two steps. Let me identify the pieces of the boundary of the picture. The top and bottom will be where the weights w_1 and w_2 go to zero. We want to think that the other components are happening when we get near a Deligne-Mumford stratum, where a curve gets small. The outside of my cylinder will be when I separate the inputs from the outputs. The two internal cylinders will pair an input with an output and there are two ways to do that. So first I could include all of the boundary in the picture I drew, and second, the harmonic function allows us to include the Deligne-Mumford strata (moduli space of nodal surfaces) but whether I include or not will depend on the weights, when the two weights are actually equal. At the slice where $w_1 = w_2$, I add a point. The difference now between the top and bottom cone has to do with what the output weights actually are. Above, w_1 is smaller and below w_2 is smaller.

Something that Dennis mentioned, in the metric I'm using drawing these pictures, the gradient flow preserves the lengths of the level sets.

So like I said, what I really want to do is use a fatgraph model. I wasn't sure if everyone would know what a fatgraph was so I'll define it.

Definition 6.2. *A* fatgraph *is a graph together with a cyclic order of half-edges adjacent to each vertex.*

Let me give you an example: [picture] This is the kind of fatgraph I'll use later on. I'll use the blackboard orientation at each vertex. The name is suggestive. A thing that you can do with a fatgraph is find an orientable surface with boundary that has this as a deformation retract. We fatten the vertices to disks and each edge to a strip, and the cyclic order tells you how to identify the ends of these bands along that disk. I do this so that what I get is orientable. From that fatgraph we've produced a pair of pants. Here's another picture. If I thicken this one, the surface that I get looks like this. So I've got a surface with four boundary components. If it's not obvious what the genus is, you can tell that it has genus zero. I'll come back to that one. That's a fat graph. I want to use a particular kind of fatgraph. I'll give a technical definition, sweeping some details under the rug. This is in progress.

Definition 6.3. *A* string diagram *of type (g, k, ℓ), this is not the kind in category theory, is a sequence of metric fatgraphs $\Gamma_0 \subset \cdots \Gamma_N$, constructed inductively.*

- *Γ_0 is k disjoint circles of length one*
- *Γ_{n+1} is obtained from Γ_n by adjoining a collection of metric trees to Γ_n along their leaves*

such that Γ_N has genus g and $k + \ell$ boundary components, and I'm going back and forth between the surface and the fatgraph. I want k of my boundary components to come from the k circles I started with.

This may seem a little mysterious, this should also have a continuous parameter, an n-tuple $s_1, \ldots s_N$ in $[0, 1]^N$, which I might call "spacing parameters."

The metric trees must satisfy some length condition.

I'll give you too examples of the length condition. Chords have length one, and for the next tree up might be a tripod like this. If my edges have length a, b, c, I need $a + b + c = 2$, one less than the number of leaves, and none of a, b, c can be greater than one. This is giving a family of graphs that form a two-simplex sitting inside a bigger one.

[People think in terms of homotopy theory, this sounds like a homotopy invariant idea, but the equations she'll talk about depend on a pseudomanifold with a top chain with a boundary. It must be correct homotopically and also have a top chain.]

Thanks. The two fat graphs that I drew up here are examples of string diagrams. I've attached a single chord, and I've drawn in the other one attaching a chord, then another chord. You get a partial order on the set of trees that you're attaching. I think this guy has type $(0, 3, 5)$.

It's a good question, can you attach vertices wherever you want, that's a good question, because in previous constructions that wasn't allowed. In a different construction, well,

Definition 6.4. *A string diagram is* simple *if the spacing parameters are* 0 *and if $\Gamma_N - \Gamma_0$ is a forest. This type of object is called a "Sullivan" chord diagram in the literature.*

This is going to be our model of moduli space giving us string topology operations. Let me pick on a particular diagram and give you an idea of how it might give you an operation at the chain level.

The string topology construction for a string diagram Γ.

[It should be "associated to."] Your comments get written smaller. [Well, I use big words.]

Let me give a rough idea. So we have a closed oriented Riemannian manifold with injectivity radius ϵ. I want to feed this a chain in $C_.(LM^k)$ and it gives us a chain in $C_{.-|\chi(\Gamma)|d}(LM^\ell)$.

Maybe I'll start with an example where Γ looks like a barbell. Let me take σ to be a chain on $C_.(LM^k)$, so $\sigma : \Delta^n \to LM^k$. A point t goes to a map $\sqcup_k S^1 \to M$.

In previous treatments, there would be an operation that takes in two loops and producing one loop given by intersections. When the appropriate intersections are transversal, you can define an operation like that but at the chain level it's only partially defined. We want a fully defined chain level version of this. The main idea is to look at a map that takes the domain of σ and evaluates it the endpoints of the chord. I identify the input circles of Γ with my standard circles. So I want to extend $\sigma(t) : S^1 \sqcup S^1 \to M$ to a map $\Gamma \to M$. Then we could restrict to the output boundary and that would give us a single loop. We'll look at where $\sigma(t)$ sends the endpoints of that chord in $M \times M$. It evaluates $\sigma(t)$ at the chord endpoints.

Inside $M \times M$ we have the diagonal. If t lands on the diagonal, the two chord endpoints coincide in the manifold. But we want instead to look at an ϵ-neighborhood of the diagonal. A point inside the neighborhood corresponds to the endpoints of the chord being within ϵ of one another in M. I've got those endpoints lying in the ball and I can connect those two points by a geodesic segment, which looks like the image of a chord under that map. So I want to send the chord to the geodesic segment. I'm doing some behind the scenes intersecting with a Thom class near the diagonal.

I define a sequence of chain maps

$$C_*(\Delta^n) \to C_*(\Delta, \Delta - ev_\Gamma^{-1}(N_\epsilon)) \to C_*(ev_\Gamma^{-1}(N_\epsilon), ev_\Gamma^{-1}(\partial N_\epsilon)) \to C_{*-|\chi|d}(ev^{-1}(N_\epsilon))$$
$$\to C_{*-|\chi|d}(Maps(\Gamma, M)) \to C_{*-|\chi|d}(LM^\ell)$$

Another example would be for Γ being the second fatgraph I drew that isn't there anymore. In terms of the first part of the map, it's almost exactly the same, well, in the manifold, I have an ϵ-ball and then I map in the first chord with the geodesic segment. We know how far along the chord the two chord endpoints land, and we can map the next chord in as the geodesic between those points. For the other types of trees that we attach, we also have ways to map those guys in.

This is completely insensitive to Γ being simple. Previous constructions required simplicity. Another remark is that there exist string diagrams that are not equal to one another but which give the same operation. This was my reason for wanting to show this example. At this stage in the construction I have different Γ [picture] but they give the same operation. This suggests that the space governing string topology operations S should be a quotient of the space of string diagrams.

I want to look at that quotient. This space S is a pseudomanifold with boundary. and this quotient gets rid of some of the boundary. In the appropriate quotient, the string diagrams on the boundary come in two flavors. Either Γ has spacing parameter equal to 1 or Γ_N has a small output, an output boundary cycle with no input circle pieces. Before I take the quotient I have more boundary than this.

We're showing that cellular chains on this space act on singular chains of the loop space of the manifold.

So far we have operations coming from a particular subspace, if we look at the subcomplex with all spacing parameters zero and all trees chords, all chord endpoints on input circles, then the construction gives us a chain map.

Let me talk one more bit about the space govrning the two to two genus zero operations. [pictures and discussion].

7. Kevin II

Yesterday we had the definition of a classical field theory as a sheaf of differential graded Lie algebras with a quasiisomorphism $\mathcal{L} \cong \mathcal{L}^![-3]$. We defined supersymmetry on $\mathbb{R}^4 = V_\mathbb{R}$, and the key ingredient was the $N = 1$ supertranslation Lie algebra, $T^{N=1} = V_\mathbb{C} \oplus \Pi(S^+ \oplus S^-)$, with $\Gamma : S_+ \otimes S_- \to V_\mathbb{C}$ a $Spin(4)$-representation isomorphism The bracket is $[Q, Q'] = \Gamma(Q \otimes Q')$ for $Q, Q' \in S_+, S_-$.

Let me say how to get more supersymmetry. Suppose we take a complex vector space and then we can form from our vector space W a supertranslation Lie algebra $T^W = V_{\mathbb{C}} \oplus \Pi(S_+ \otimes W \oplus S_- \otimes W^*)$. The bracket is the obvious one:

$$[Q \otimes w, Q' \otimes w'] = \Gamma(Q \otimes Q'), \langle w, w' \rangle$$

The dimension of W is the number of supersymmetries.

Note that the group $GL(W)$ acts on this Lie algebra T^W, as does $Spin(4)$. This group, $GL(W)$, is called the R-symmetry group. If $G_R \subset GL(W)$, then a supersymmetric theory has R-symmetry group G_R if it is acted on by G_R in a way compatible with the action of T^W.

What I'd like to talk about, the next topic, is twisting. You've heard people talk about topological twists, let me tell you how that goes.

The idea is to choose a Q in $S^+ \otimes W \oplus S^- \otimes W^* \subset T^W$ with $[Q, Q] = 0$.

Suppose we have a supersymmetric field theory given by a sheaf of differential graded Lie algebras $\mathcal{L}$. Then Q maps $\mathcal{L}$ to itself. This commutes with the differential and is a Lie algebra derivation.

We know what to do with an odd square zero element in a Lie algebra, we can do deformation theory and form a deformation of $\mathcal{L}$ into a new differential graded Lie algebra with differential $d_{\mathcal{L}} + Q$. The properties tell us that this still squares to zero. However, it's not really a kosher object in this world, because the differential is not of cohomological degree one. Q has cohomological degree 0 and is degree one in the *other* (fermionic) grading.

What we have to do instead, there's a solution, it'll take a few lines.

Definition 7.1. Twisting data *is a pair Q (as above) and $\rho : \mathbb{C}^\times \to G_R \subset GL(W)$ and a homomorphism to the R-symmetry group with $\rho(t)(Q) = tQ$.*

Definition 7.2. *Given twisting data, the* twisted field theory *is defined as follows. Let s be a parameter of cohomological degree one which is fermionic (so s is even). Then take the differential graded Lie algebra $\mathcal{L}((s)), d + sQ)$, and take the $C_\rho^\times$-invariants, where s has weight -1 under $\mathbb{C}^\times$.*

If you work out what's going on, this has the effect of modifying the gradings so that it's $\mathbb{Z}$-graded.

Finally we can say, what are these twisted field theories.

Theorem 7.1. *The twisted $N = 1$ supersymmetric gauge theory on $\mathbb{C}^2$, well, any time you can write down a moduli space you can construct a field theory the cotangent theory, from last time, and this is the cotangent theory for the moduli of holomorphic G-bundles on $\mathbb{C}^2$.*

For $N = 2$, this is the cotangent theory to the moduli of G-bundles with a holomorphic section of the adjoint bundle of Lie algebras, $\phi \in H^0(\mathbb{C}^2, \mathfrak{g}_P)$.

For $N = 4$, this is the cotangent theory to the (derived) moduli of Higgs bundles on $\mathbb{C}^2$.

Let me explain what a Higgs bundle is. It's a holomorphic G-bundle with $\phi \in H^0(X, T^*X \otimes \mathfrak{g}_P)$ such that $[\phi, \phi] = 0 \in H^0(X, \wedge^2 T^*X \otimes \mathfrak{g}_P)$.

What are we twisting by? All of what I wrote is about a minimal twist. I choose $Q \in S^+ \otimes W$ and assume that it's a decomposable tensor, that is, $Q = \alpha \otimes w$.

[some discussion]

You might ask, why did we find holomorphic things when we twisted? The answer is, well, let's look at the $N = 1$ case for simplicity. We have some $Q \in S_+$

and we've deformed our theory, roughly speaking, by adding Q to our differential. You can ask what happens to this supersymmetry after I do this (we also changed the $\mathbb{Z}$-grading).

This implies that the differential graded Lie algebra $S^- \to V_\mathbb{C} \to S_+$ acts on the twisted theory (the differentials are all $[Q, \quad]$.

Let's see what this tells us about the twisted theory.

In the twisted theory, translation by $Q \otimes S^- \subset V_\mathbb{C}$ is homotopically trivial. Now we're giving a two dimensional complex subspace which turns out to be isotropic, and then we get a complex structure on $\mathbb{R}^4$ where the $0,1$ part is $Q \otimes S^-$ inside $\mathbb{R}^4 \otimes \mathbb{C}$. In this complex structure, $\frac{\partial}{\partial \bar{z}_i}$ acts trivially, so it's holomorphic.

From this we can also see what it means for a twist to be topological, which is saying that any translation is in Q.

Similarly, a toplogical twist is one where the image of Q is all of $V_\mathbb{C}$. So the $N = 2$ and $N = 4$ series admit topological twists.

What I'd like to do is just describe the Kapustin-Witten topological twists of $N = 4$ supersymmetric gauge theory.

For general culture: they were talking about the Langlands program in terms of this gauge theory. They said there was a $\mathbb{P}^1$ of such gauge theories.

We see for $N = 4$ gauge theory where we've minimally twisted on a complex surface is the cotangent theory to the space of Higgs bundles for the group G, so this means if I look at the whole space of solutions to the equations of motion, that space is $T^*[-1]Higgs_G(S)$. If I chose some holomorphic G-bundle, then the differential graded Lie algebra describing the field theory is, well,

$$\begin{tikzcd}
 & \Omega^{0,*}(S,\mathfrak{g}_P) \arrow[r, "t\partial"] & \Omega^{1,*}(S,\mathfrak{g}_P) \arrow[r, "t\partial"] & \Omega^{2,*}(S,\mathfrak{g}_P) \\
\Omega^{-1,*}(S,\mathfrak{g}_P) \arrow[r, "t\partial"] \arrow[ur, "s\cdot Id"] & \Omega^{1,*}(S,\mathfrak{g}_P) \arrow[r, "t\partial"] \arrow[ur, "s\cdot Id"] & \Omega^{1,*}(S,\mathfrak{g}_P) \arrow[ur, "s\cdot Id"] &
\end{tikzcd}$$

I want to write down a two-parameter deformation here, and the parameters are s and t.

When $s = 0$ we're adding the de Rham differential, that's well-known, taking $T^*[-1]Higgs_G(S)$ to $T^*[-1]Loc_G(S)$.

When $t = 0$, we're taking $T^*[-1]Higgs_G(S)$, this is isomorphic since the moduli space is symplectic to the tangent bundle, and this deforms to the de Rham stack $Higgs_G(S)_{dR}$.

From this you see, I'll say this quickly and stop, you have $\mathbb{P}^1$ possible twists, and at one point you have $T^*[-1]Loc_G(S)$, at another point $Higgs(S)_{dR}$, but generically you find $Loc_G(S)_{dR}$ with a certain symplectic form. Loc means local systems. I don't know if this description is helpful. I found Witten's version hard. Next time I'll say some things you can see at the quantum level with these theories.

8. Scott Wilson: Chern Simons form, loop spaces, and K-theory

Thanks for inviting me, it's very nice to be here. Before I forget, this is joint work with Thomas Tradler and Mahmoud Zeinalian. This is related to differential cohomology theories and presumably topological field theories. I wanted to start with something elementary, the Chern character and a nice geometric presentation for it. The even cohomology of a manifold accepts a map ch from the K-theory $K^0(M)$, and I'd like to give an actual formula.

Given a complex bundle $E \to M$ with a connection ∇, you can define $ch(\nabla) = Tr(e^R)$ where R is the curvature of ∇.

A natural question is, how does this depend on the connection? This is a nice affine contractible space. Given two connections you can get a path between them, and get an odd form from that path that interpolates between the chern forms for the connections. So you can define the Chern Simons form for a path ∇_s of connections,

$$CS(\nabla_s) = Tr(\sum \frac{1}{k!} \int_0^1 R_s \cdots R_s \nabla_s'(t) R_s \cdots R_s)$$

I want to be careful, I'm going to do something very non-Abelian soon.

Here is the property that I was discussing, $dCS(\nabla_s) = ch(\nabla_1) - ch(\nabla_0) \in \Omega(M)$.

An example you may know, if you have a trivial bundle on M^3 and write your connection as A, then the path from 0 to A is given by sA, so then the Chern Simons form is $Tr(A \wedge dA + \frac{2}{3}A^3)$ since $\nabla_S^1 = A$ and $R_s = dA_s + A_s^2$.

Theorem 8.1 (Bismut)**.** *The Chern form lifts to a form on the free loop space. Given a connection on $E \to M$, there is a form $BCh(\nabla)$ in $\Omega_{d+i}^{even}(LM)$ such that*

1. $(d + i)BCh(\nabla) = 0$
2. *restricted to the constant loops, this restricts to $Ch(\nabla)$.*

Here i is contraction by the vector field for the circle action.

What are the forms on the free loop space? You could use iterated integrals, cochains, this being a Frechet manifold, this form can be written down in an integral involving only the connection and curvature.

Bismut wanted to interpret the partition function for some field theory. I don't completely understand that part and it might not be totally rigorous, but this is a clear piece of mathematics.

Getzler-Jones and [unintelligible]looked at this in terms of iterated integrals and then [unintelligible]and Stolz-Teichner used dimension reduction in field theories.

I want to give a geometric interpretation using holonomy. We know that the Chern form in degree $0, 2, \ldots 2k$ is, well, on $\Omega(M)$ these are $Tr(\frac{R^k}{k!})$.

On LM, the zero part is the trace of the holonomy of the connection. This restricts correctly to the trace of the identity. If you want to solve holonomy, you need to solve $x'(t) = A(t)x(t)$, where $A(t)$ is a local expression for your connection. So locally, you get the following expression:

$$\sum_{n \geq 0} \int_{\Delta^n} iA(t_1) \cdots iA(t_n) dt_1 \cdots dt_n$$

So b_2, the Chern character for LM in degree two, is $Tr(\sum \int iA \cdots R(;) \cdots iA)$. This takes in two vector fields on the loop and outputs a number. Then the only terms that are nonzero have just the curvature R. More generally, b_{2k} is the integral where you put the curvature in k times in all possible positions and $n - 2k$ copies of A.

It matters what order this is in this case. The volume of the n-simplex is $\frac{1}{n!}$ which give me the coefficients of the exponential.

It's a lemma to show now that $(d + i)BCh(\nabla)$ is zero. You calculate that $db_{2k} = -ib_{2k+2}$ which shows we have the first condition.

This is not how Bismut did it, but it's a nice interpretation.

So how does this form depend on the connection? That's the next thing to ask. Here's the theorem, due to Tradler, myself, and Zeinalian.

Theorem 8.2. *Given a path of connections, there exists an odd form $BCS(\nabla_s) \in \Omega^{odd}(LM)$ such that*

(1) *$(d+i)BCS(\nabla_s) = BCh(\nabla_1) - BCh(\nabla_0)$, and*
(2) *$BCS(\nabla_s)$ restricts to $CS(\nabla_s)$ by restricting to constant loops.*

Corollary 8.1.

$$\begin{tikzcd} & H^{even}_{d+i}(LM) \arrow[d] \\ BCh : K^0(M) \arrow[ur] \arrow[r] & H^{even}(M) \end{tikzcd}$$

I wanted to say two things first. A natural question is whether this diagram of groups respects the ring structures. We proved that $BCh(\nabla \oplus \nabla) = BCh(\nabla) \oplus BCh(\nabla)$ and $BCh(\nabla \otimes \nabla) = BCh(\nabla) \wedge BCh(\nabla)$. The first part is easy, the second a little harder but not too surprising. You know this for degree zero with holonomy and you know it for the Chern character.

I wanted to give a short idea of where this comes from. I'm going to make a table of $BCS(\nabla_s)$, the formula is

$$\sum \iint (iA) \cdots (A'_s) \cdots (iA)$$

By our claim looking in degree one, we should get under i the difference of the two holonomies. For $2k-1$ I put in a bunch of copies of R_s. You need to do a little work to show that they don't depend on gauge and patch together. The proof of the closure is straightforward along the lines of the previous computation.

So what can you do with this? We heard some discussions earlier about differential cohemolyg. This is due to Hopkins-Singer and later Bunke-Schick. Work by Simons-Sullivan shows that you can make differential K-theory similarly to regular K-theory, adding a connection.

One sort of little fact about differential K-theory is that it fits into a diagram of ring homomorphisms:

$$\begin{tikzcd} & K(M)h \arrow[dr, "c"] & \\ \hat{K}(M) \arrow[ur] \arrow[dr] & & H^{even}(M) \\ & \Lambda^{even}_d(M) \arrow[ur] & \end{tikzcd}$$

Since there's a map from $\hat{K}(M)$ to $H^{even}(M)$, you might ask whether there's a lift to something for the loop space. The Bismut Chern form knows about the trace of the holonomy, that's more geometric. We'll make a new ring which contains this information.

We played a game earlier where we asked how something depends on the connection, and then made a path of connections. If you're really careful you can ask how it depends on the path. Given two paths of connections ∇_s and ∇_r, you can fill this in by a disk and associate an even form that mediates. There's an even form $H(\nabla_s, \nabla_r)$, such that $(d+i)H = BCS(\nabla_s) - BCS(\nabla_r)$. For any two connections, regardless of how you connect them, you get a well-defined class in the odd forms on LM modulo the image of $d+i$.

Now you can ask, this class, when is it zero? I'll use this to make an equivalence relation. We'll say ∇_0 and ∇_1 on $E \to M$ are equivalent if $BCS(\nabla_0, \nabla_1) = 0$ when is this? You have $BCS(\nabla_s \oplus \nabla_r) = BCS(\nabla_s) \oplus BCS(\nabla_s)$ and $BCS(\nabla_0 \otimes \nabla_s) = BCh(\nabla_0) \wedge BCS)\nabla_s)$ (similarly on the other side). You can tell then that tensor product and sum pass to the equivalence relation.

Let $\mathcal{M}$ be the commutative semiring under $\oplus$ and $\otimes$ of isomorphism classes of BCS-equivalence classes. One can form the Grothiendieck group $Groth(\mathcal{M})$ which I'll denote $L\hat{K}(M)$. It's built up out of vector bundles over M with connection up to an equivalence relation. This idea is taken from the Simons-Sullivan paper where they did something similar for differential K-theory.

Theorem 8.3. *There's a commutative diagram of ring homomorphisms*

$$\begin{tikzcd}
 & K(M) \arrow[dr] \arrow[dddr] & & \\
L\hat{K}(M) \arrow[ur] \arrow[dr, "BCh"'] \arrow[dddr] & & H^{even}_{d+i}(LM) \arrow[dddr] & \\
 & \Lambda^{even}_{d+i}(LM) \arrow[ur] \arrow[dddr] & & \\
 & & K(M) \arrow[dr] & \\
 & \hat{K}(M) \arrow[ur] \arrow[dr, "ch"'] & & H^{even}(M) \\
 & & \Lambda^{even}(M) \arrow[ur] &
\end{tikzcd}$$

These maps have kernels, and you get something like:

$$Groth(\text{bundles with connection})/\text{gauge equivalence}) \twoheadrightarrow L\hat{K}(M) \twoheadrightarrow \hat{K}M$$

For S^1 you get [missed] and that's the only space I know it for.

There's work going on to put this in a bigger commutative diagram and also work going on to do the odd version of this.

I should also mention that there's an analog for the Bismut Chern form for Abelian gerbes with connection. One takes the viewpoint that the Bismut Chern form is an equivariant extension of holonomy. You start with holonomy as a function on the loop space and try to extend it to be in the kernel of $d+i$. Instead of a bundle with connection you use a gerbe with connection, and you look at the trace of 2-holonomy as a function on maps from the torus into your manifold. We produce an element in the kernel of equivariant d that deals with rotation in two different directions.

A gerbe is an line bundle with connection over the loop space with a nice fusion product and a little bit more, or take differential cohomology, and take an element in degree one higher than the degree that determines line bundles with connection.

I wanted to end with a long list of questions. Ryan Grady's talk, they discuss an element in a derived loop space that recovers the Todd form when restricted to constant loops. So there are interesting things in that direction.

Is there a connection to string topology? There is a pairing between $\Omega^*_{d+i}(LM)$ with $(C_*(LM), \partial + \Delta)$. There's another version where you average over orbits of the circle action instead of contracting by the vector field. The Bismut construction still works there. Whenever you have a closed element on the one you can evaluate it on the other. Suppose you were interested in $H_{even}(LM)$, then you could compare with $H^{even}_{d+i}(LM)$.

So you could take $H_{even,\partial+\Delta}(LM) \to Func(Conn)$ where c goes to $\nabla \mapsto \int_c BCh(\nabla)$. I'm suggesting that the Bismut Chern character gives interesting functions on connections. Then you can ask how string topology is compatible with structure on functions on connections like the Poisson bracket.

9. Jacob Lurie

Thank you very much, it's a pleasure to be here, I guess this is the last one, but, c'est la vie. Loop spaces, p-divisible groups, and character theory is the title of my talk. I'll get back to toplogical field theories at the end. To start with, I want to talk about the representation theory of finite groups. I'll write $Rep(G)$ for the representation ring, using direct sum, and throwing in inverses. One of the first theorems is that these are determined by their characters. There is a map $Rep(G)$ to functions from G to $\mathbb{C}$. This takes V to χ_V, where $\chi_V(g)$ is $tr(g|_V)$. If we restrict in the range to conjugation invariant functions and tensor the domain by $\mathbb{C}$ we get an isomorphism.

I'll use K to mean complex K-theory. There's a relationship between $Rep(G)$ and $K(BG)$. A representation is a local system which gives a bundle which is in $K(BG)$. If you complete with respect to the augmentation ideal with virtual dimension zero you get all of $K(BG)$. Let's suppose G is a finite p-group. You can also p-adically complete. This is a more severe completion. Then this becomes an isomorphism, $Rep(G) \to K(BG)$. Now we have to p-adically complete both sides. To complete on the left, we tensor over $\mathbb{Z}$ with $\mathbb{Z}_p$. On the other side we use $\hat{K}$, which is not the same as the use of that symbol in the last talk.

Let's make use of that, and identify $\mathbb{C} \otimes Rep(G)$ with $\mathbb{C} \otimes_{\mathbb{Z}_p} \hat{K}(BG)$. Now the right hand side, the class functions can be identified with $H^0(Map(T^1, BG), \mathbb{C})$. What does the loop space look like? The connected components are conjugacy classes and the cohomology of this space is functions into the complex numbers. The main theory of representation theory of finite groups is telling us something about the relationship between K-theory and ordinary cohomology. The K-theory of a space, we kill all the torsion, and that's supposed to be the cohomology of the loop space.

There's a generalization of this theorem (Hopkins, [unintelligible], and Ravenel), of the flavor that taking a cohomology theory, applying to the space, and then tensor in this way, it looks like a different cohomology applied to a different space.

Definition 9.1. *A Morava E-theory is a cohomology theory E satisfying three conditions.*

(1) *E is represented (or can be) by a commutative ring spectrum. We're talking about a cohomology theory, and there may be representing cocycles. Regular cohomology can be represented by cocycles or differential forms. K-theory can be represented by vector bundles. We want E to be represented by a ring spectrum, which should be commutative and associative in a homotopy-coherent sense. I don't want to say E_∞ because there's an E on the board but that's what it is.*
(2) *$E^*(pt) \cong \mathbb{Z}_p[[v_1, \dots, v_n - 1]][U^{\pm 1}]$ where v_i is degree 0 and U in degree 2.*
(3) *$E^*(B\mathbb{Z}/p)$ is a free module over $E^*(pt)$ of rank p^n (same n)*

If you're not a stable homotopy theorist, this may feel unfamiliar and I sympathize. Let me say, for any p and $n > 0$ there exists a Morava E-theory denoted E_n, denoting n and not p. So p is fixed for all time. This is unique up to some kind of Galois twisting. If I put the algebraic closure of $\mathbb{Z}_p$ in, this would be uniquely determined. A canonical example is, if $n = 1$, then $\hat{K}$ is a Morava E-theory. You can check the conditions easily.

You might extend this to $n = 0$, which should be rational. In that case, you would get a free module of rank 0.

These other ones are not familiar because there is not a geometric interpretation, but as a stable homotopy theorist, these are your bread and butter.

Believe me that these are interesting because I'm going to talk about them from now on.

Let E be a Morava E-theory. What is E of your favorite space? Let's say your favorite space is BG where G is a finite p-group. Hopkins et al. computed what you get when you look at $E(BG) \otimes_{E(pt)} \mathbb{C}$, and it's conjugation invariant functions on the set $Hom(\mathbb{Z}^n, G)$. That's conjugacy classes of n-tuples of commuting elements of G. That's what the representation theory says when $n = 1$.

Let's call BG by the name X, we're saying that $E(X) \otimes_{E(pt)} \mathbb{C} \cong H(Maps(T^n, X), \mathbb{C})$, and this is true for any p-finite space.

Let me define p-finiteness. Say that X is p-finite if all homotopy groups $\pi_i X$ are finite p-groups and $\pi_i X$ vanishes for large enough i. So it's built out of Eilenberg-MacLane spaces for finite p-groups.

What I would like to do in this lecture is categorify this statement. I want this to be a consequence of the existence of a category with nice properties. Let me list those properties now.

Let E and X be as above. Here is a hypothesis. There exists a category $\mathcal{C}$ which has a unit object 1 such that the following things happen

(1) $Hom(1, 1) \cong E^0(X)$.
(2) We have $\otimes\mathbb{C}$ and $1_\mathbb{C} = 1 \otimes \mathbb{C}$, then $Hom_{\mathcal{C}}^{\mathbb{C}-lin}(1_\mathbb{C}, 1_\mathbb{C}) \cong H^0(Map(T^n, X), \mathbb{C})$
(3) $Hom_\mathbb{C}(1, 1) \otimes_{E^0(pt)} \mathbb{C} \cong Hom_\mathbb{C}^{\mathbb{C}-lin}(1_\mathbb{C}, 1_\mathbb{C})$.

Don't let me erase that board.

Let me give you a non-example. A caveat, when I say a cotegory, I mean ∞-category, but if you're happy, imagine I'm inserting them, otherwise ignore them.

The obvious attempt, I said E should be represented by a commutative ring spectrum, so there are modules. I'll write the category of those Mod_E. Try taking $\mathcal{C}$ to be the category of local systems on X with values in Mod_E. Then 1 is the constant local system with value E. What happens if you compute $Hom_{\mathcal{C}}(1, 1)$, well, what you recover is $E(X)$, tautologically. But the second condition isn't okay.

What if you calculate the $\mathbb{C}$-linear homs from $1_{\mathbb{C}}$ to itself? More or less tells us we'll get $H^0(X, \mathbb{C})$. The complexification of 1 will be the constant local system of $\mathbb{C}$ or a two-periodic version. This is the failure of the second one. Then the third condition also fails, and you'd be proving a much more naive statement than the Hopkins et al. statement. So the third condition also fails.

The idea is, Hopkins-Kuhn-Ravenel says there is a better version of the category of local systems. The existence of such a category would imply the theorem. I'd like to tell you about the theorem.

Now what's closer to working? I want to start with an observation. Let X be a space and $\mathcal{L}$ a local system of E-modules on X. I'm going to write $\Gamma(X, \mathcal{L})$ for global sections of $\mathcal{L}$. This is an object in Mod_E. You can say more than that, because it's a cohomology theory, and the cohomology groups of a point you can think of as the cohomology of X with coefficients in $\mathcal{L}$. But now you get modules over the E-cohomology of X.

This is a module over the spectrum $E(X)$, the spectrum of maps $X \to E$.

[discussion of whether this is flat enough.]

If you have a map of spaces $X \to Y$ and a local system on Y, $\mathcal{L}$, you can pull back to X and a global section of $\mathcal{L}$ gives a global section of the pullback, but these are modules over $E(X)$ and $E(Y)$, so you get a map $E(X) \otimes_{E(Y)} \Gamma(Y, \mathcal{L}) \to \Gamma(X, f^*\mathcal{L})$.

If we take Y to be a point and $\mathcal{L}$ the constant local system, we get something like $E(X) \otimes_{E(pt)} \mathbb{C} \to H^0(X, \mathbb{C})$. This is kind of what we want. So local systems aren't right and we want something different with that built in.

Definition 9.2. *A* twisted local system *on X is the following data:*

1. *For each map $\alpha : T \to X$, where T is the classifying space of a fiite Abelian p-group, a module $\mathcal{L}(\alpha)$ over $E(T)$.*
2. *If $T' \xrightarrow{\beta} T \xrightarrow{\alpha} X$, then $\mathcal{L}(\alpha \circ \beta) \cong E(T') \otimes_{E(T)} \mathcal{L}(\alpha)$.*

What I want to say is that this category of twisted local systems satisfies my three conditions. This is the right definition, now, after p-completion. There is a more complicated definition you can give without completing but let me neglect that.

Let me draw the picture of the scheme which is $Spec$ of $E^0(pt)$, so the prime ideals in $\mathbb{Z}_p[[v_1, \ldots, v_{n-1}]]$, I have this picture, and I want to add to the picture $Spec(E^0(B\mathbb{Z}/p))$. Over my special point there's only one inverse image. I'm making assertions that you can see directly in the examples I started with. For $n = 1$, we have $E^0(pt) = \mathbb{Z}_p$, and we have one close point and one open point. After p-adic completion, this is the representation ring. For $\mathbb{Z}/p$, there are p of these generated by the p irreducibles.

[Had to stop, too fatigued.]

10. Dan Berwick-Evans, spaces of field theories and the topology of manifolds

I want to start with the idea, due to Segal, to study manifolds by analyzing families of field theories over them. We'll use the construction of Stolz-Teichner, for a fixed $d|\delta$, which gets a sequence of sheaves $d|\delta - EFT^k : Man^{op} \to Grpd_*$.

Here $d|\delta{-}EFT^0(X)$ is $Fun^{\otimes}(d|\delta{-}EB(X), Vect)$. A twist is a functor τ between $d|\delta{-}EB(X) \to ALG$ and a τ-twisted field theory is a natural transformation $\tau \to 1$ of functors $d|\delta - EB(X) \to ALG$.

To extract topology, we consider $X \mapsto d|\delta - EFT^k[X]$ which is π_0 of the nerve of $d|\delta - EFT^k(X \times \Delta^*)$. A claim is that $d|\delta - EFT^k[]$ is an additive homotopy functor.

So, an example. If I look at $0|1 - EFT^*$, this reads off closed differential forms. This is the kind of thing to keep in mind. Then the concordance relation gives me the de Rham cohomology.

So maybe we're lucky enough that this always happens, maybe you always get a homology functor. You have additivity and homotopy, do you always get Mayer-Vietoris? You need extra data, but the answer is not always. When someone hands you a homotopy functor with no nice properties, how do you extract invariants? It's hard. The next question is how to extract computable homotopical data from field theories.

I need to describe some structures from field theories. Suspension is related to quantization, we'll use that and also dimension reduction. We're ready for some giant diagrams. I'll start with $d|1$-theories. I can integrate in families, if I have an orientation on $0|1 - EFT$ on $X \times Y$ of the appropriate kind, I can integrate to $0|1 - EFT(Y)$. In the $1|1$-case the map uses $Spin_X(Y)$ rather than $Or_X(Y)$ and for $2|1$ I need $String_X(Y)$.

Dimension reduction involves thinking of things as products, it's not that easy but it's been constructed. Dimension reduction factors through the kind of thing Scott was talking about.

$$\begin{tikzcd}
Or_X(Y) \times 0|1 - EFT(X \times Y) \arrow[r, "\int"] & 0|1 - EFT(Y) & \\
& & 0|1 - EFT^{S^1}(LY) \arrow[ul] \\
Spin_X(Y) \times 1|1 - EFT(X \times Y) \arrow[uu] \arrow[r, dotted] & 1|1 - EFT(Y) \arrow[uu] \arrow[ur] & \\
& & 1|1 - EFT^{S^1}(LY) \arrow[ul] \\
String_X(Y) \times 2|1 - EFT(X \times Y) \arrow[uu] \arrow[r, dotted] & 2|1 - EFT(Y) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

And let me draw a related diagram from passing to concordance, before doing it for other field theories.

$$\begin{tikzcd}
Or_X[Y] \times H_{dR}(X \times Y) \arrow[r, "P_!"] & H_{dR}(Y) & \\
& & H^{S^1}(LY) \arrow[ul] \\
Spin_X[Y] \times K(X \times Y) \arrow[uu] \arrow[r, "P_!"] & K(Y)h \arrow[uu, "c"] \arrow[ur] & \\
& & K(Y)[[t]] \arrow[ul, "t=0"] \\
String_X[Y] \times TMF(X \times Y) \arrow[uu] \arrow[r, "P_!"] & TMF(Y) \arrow[uu] \arrow[ur, "evTate"'] &
\end{tikzcd}$$

What do we get for $d = 2$?

$$\begin{tikzcd}
0|1 - EFT(X \times Y) \arrow[r, "\int"] & 0|2 - EFT(Y) & \\
& & 0|2 - EFT^{S^1}(LY) \arrow[ul] \\
Or_X(Y) \times 1|2 - EFT(X \times Y) \arrow[uu] \arrow[r, dotted] & 1|2 - EFT(Y) \arrow[uu] \arrow[ur] & \\
& & 1|2 - EFT^{S^1}(LY) \arrow[ul] \\
\{??\} \times 2|2 - EFT(X \times Y) \arrow[uu] \arrow[r, dotted] & 2|2 - EFT(Y) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

We only have a candidate for the middle arrows here and the left vertical arrows are even more difficult and fraught. This $2|2$ arrow amounts to doing Morse theory on the loop space.

We have the same dimensional reduction maps as before. What comes next? Well, $d|2 - EFT$s don't have suspension isomorphisms.

Similar pictures should exist for $d|4$, $d|6$, $d|8$, and $d|16$. These are all finite families but there are lots of them.

So let me fgive you something that this could be used for.

Definition 10.1. *The $d|\delta$-sigma model of X is the image of $\mathbf{1}$ in $d|\delta - EFT(X)$ under quantization. We denote this by σ_X.*

We can take the partition function Z_X, which is $\sigma_X(T^{d|\delta}) \in \mathbb{R}$, the evaluation on the torus. This varies smoothly as you move around flat torii.

Why is this interesting?

Theorem 10.1. *Z_X is a manifold invariant together with its orientation data.*

What I want to explain in the last few minutes is how to recover a homology theory from one of these sequences of sheaves.

Theorem 10.2. *There exists a functor $T_1(d|\delta - EFT^k)$ with Mayer-Vietoris satisfying, for any sequence of sheaves $\mathcal{C}$ with the Mayer-Vietoris sequence, there is a lift of $\mathcal{C}^* \to d|\delta - EFT$ to $T_1(d|\delta - EFT)$.*

Definition 10.2. *A field theory is linear if it has Mayer-Vietoris sequences.*

Theorem 10.3. *For $\delta > 1$, if $d|\delta - EFTs$ satisfy a certain condition (quantization is related to integration) then the theory is* not *linear.*

It's kind of a funny statement, this is a precise condition, this is saying that unless some part of the picture is wronge, these homotopy functors give rise to something more complicated than cohomology theories. Let me give a final example as to why this might be.

This is an example I found later. Flat vector bundles with $O(\delta)$ action sits in $1|\delta - EFT(X)$, so you shouldn't expect to see stable data alone. So the real challenge is seeing what data you can get and and how to pull homotopical data out.

11. Joey Hirsh, deformations with non-commuting parameters

Thank you for organizing a really great workshop. It's the first time I'm camera so I should say "hi mom and dad." I won't just talk about a non-commutative case but with parameters over any operad. I think we've been hearing about deformations and moduli space, I want to say a few words about what deformation theory is before I do that. I'll work in characteristic zero.

Definition 11.1. *A* commutative moduli problem *is a functor from commutative algebras over k to sets.*
A derived commutative moduli problem *is a functor from differential graded commutative algebras into spaces $\mathcal{T}$.*

You might ask why this is a problem, if you had a set of things and you made it into a space, you could find a space M and your moduli problem would be $Maps(\quad, M)$.

So you'd expect spaces, not rings, and maybe you'd expect this to be an op-functor.

The opposite category of Algebro-geometric spaces is equivalent to commutative algebras.

I love functors but that's not a good enough reason to do instead of moduli space. Writing down a space might be harder. So typically if you wanted to define a functor into sets, you'd define modulo isomorphism. In the derived moduli problem you put in the isomorphisms. The derived version might be representable even if the moduli problem is not.

We have a functor from spaces to sets called π_0, which gives a moduli problem from a derived moduli problem.

When Kevin gave his talk, he said that a quantum field theory should be like a space with a sheaf of moduli problems. Then he immediately said "that's too hard."

I was going to justify that but Kevin said it's too hard so it's too hard. So Kevin looked around a point and looked around that point, calling that a deformation theory. So we can think that this point is like an object. We'd be ignoring tons of difficult overlap, local to global questions, and we'd just look at a zoomed in neighborhood of X.

How do we do this if a moduli problem is a functor? Conveniently, there is a class of commutative algebras that only see this infinitessimal information, called local Artin rings, which I'll denote $ComArt$. Then this gives us our definition.

Definition 11.2. *A* commutative deformation problem *is a functor from commutative Artin rings to sets.*
A derived commutative deformation problem *is a functor from differential graded commutative Artin rings to spaces.*

These functors should satisfy other criteria, called Schlessinger's criteria, which I won't write.

Let me give you an example of a local Artin ring. It's $k[\epsilon]/\epsilon^2$. I'm saying that this line is so tiny. A polynomial ring would be a line. You just get a tangent vector. Mod epsilon cubed would give you a second order derivative. You can come up with a tangent space using that, and for an arbitrary functor you can make the following definition.

Definition 11.3. *The tangent space of either a commutative deformation problem or derived commutative deformation problem, TF, is defined to be $F(k[\epsilon]/\epsilon^2)$.*

Lemma 11.1. *TF is a vector space (respectively a chain complex)*

If I started with a moduli problem and wanted a deformation problem I'd have to pick a point, but in deformation problems, I only have one point.

I wanted to do an example that came up at least three times in the last week.

Fix a differential graded algebra A. This is a restriction of a larger example. I'm fixing a point in some larger space. Then we have a derived commutative deformation problem which should go from differential graded commutative Artin rings to spaces.

Let me give you the points of the space first. To R, it will give points which are multiplications $\mu : (A \otimes R)^{\otimes 2} \to A \otimes R$ swhich are associative and so that $A \otimes R \to A$ is a differential graded map.

These Artin rings have nilpotence, and look like they have some variables, and I can always set these to zero. The classical limit is the original structure.

If I wanted to get the paths, I'd take $R \otimes k[t, dt]$, this is the differential forms on the one-simplex, and replace R with that everywhere.

So here's maybe an illuminating fact. The tangent space of F_A is isomorphic to the Hochschild cohomology of A as vector spaces.

Here's another example of a deformation problem. Given a Lie algebra L, I have the following functor MC_L from differential graded commutative Artin rings to spaces. I take R to elements of degree 1 in $(L \otimes R)$ so that γ satisfies the Maurer Cartan equation $d\gamma + [\gamma, \gamma] = 0$. These are the points, and the paths are as before.

Let me make a remark, this construction $L \mapsto MC_L$ is a functor itself, from differential graded Lie algebras to derived commutative deformation problems. So $T(MC_L) = Z(L, d_L)$. Dan was talking a bit about, if you take $k[\epsilon]/\epsilon^2$, you'll get concordance classes, so the homology of your Lie algebra.

Theorem 11.1. *(Schlessinger 68, Deligne 86, Manetti 01)*
I like to say it like this: MC is an equivalence of $\infty - 1$ categories from differential graded Lie algebras to derived commutative deformation problems.

Corollary 11.1. *Given a derived commutative deformation problem F, there exists a differential graded Lie algebra structure on TF such that $MC_{TF} \cong F$.*

There are existence and construction problems, and this states existence and gives you a little bit of the construction, you can build the bracket, and the rest of the stuff is sort of fuzzy.

Maybe you're dissatisfied with Maurer-Cartan, because I haven't talked about mapping into something. Let's see what a Maurer-Cartan element looks like. It's like a map from k to the shifted version of L, which generates a cofree coalgebra BL, and you lift a coalgebra map, and call that e^γ, and we'll send γ to something like $\gamma + \gamma \wedge \gamma 2! + \cdots$. So we have $D(e^\gamma) = 0$ that's true if and only if e^γ is a differential graded map, which is true if and only if $d\gamma + \frac{1}{2}[\gamma, \gamma] = 0$. So $MC(R)$ is something like $Map(R^\vee, BL)$, so BL is the moduli space I'm talking about.

Okay, so here's an example, we have one other eaxmple of deformation problems, I said you got the Hochschild cohomology of A for an associative algebra A, so $F_A \cong MC_{Hoch(A)}$. So the Hochschild cohomology is the representing object for that functor. You might ask, how will I prove that, the nice thing is that these things behave like tangent spaces, if you have a map of deformation functors, you can just look at the tangent spaces.

Let me make a definition and then get to a theorem. Earlier this week, Dennis talked about a remarkable fact about field theories is that they deform themselves. To me that says that the moduli space is a not just a commutative moduli space but one over quantum field theories.

Definition 11.4. *Given an operad P, a derived P-deformation problem is a functor from differential graded Artin P-algebras to spaces.*

If you don't know what an operad is, think of associative, commutative, Lie, BV, E_n. We'd like to exploit the symmetry or structure of our moduli space. You might be able to define your deformations over more (or different things). If P maps to commutative as an operad, you'll have map from commutative Artin algebras to P-Artin algebras, and then you could ask about extending a commutative deformation problem, like when Kevin had a pairing on the tangent space.

So let me say a word, Kevin mentioned, there's a theorem by Lurie in DAG X, this is not how it's stated there.

Theorem 11.2. *(Lurie)*
For $P = E_n$, the functor MC from E_n algebras to derived E_n deformation problems is an equivalence.

So for E_n algebras you have a equivalence of categories then, well, now we're seeing Koszulity. Here's a theorem.

Theorem 11.3. *(Hirsh)*
For $\mathcal{O}$, MC induces an equivalence between $\mathcal{O}$-algebras and derived $\mathcal{O}^!$ deformation problems is an equivalence.

Corollary 11.2. *For F a derived $\mathcal{O}^!$-deformation problem, there exists an $\mathcal{O}$-algebra structure on TF such that $MC_{TF} \cong F$.*

Let's say our operad is augmented. Then any chain complex is an algebra over our operad, and the tangent complex is still $k[\epsilon]/\epsilon^2$.

If you try to take advantage of symmetries, you won't have just a Lie structure, you'll have more.

I don't mean the Koszul dual in the non-Koszul case, but something like the linear dual to the bar construction. I use operads rather than cooperads to make it more familiar.

Let's go back to the case where $P = E_2$ and consider the following situation. We have the functor F_A that we discussed before, and we can ask whether we can extend the functor to differential graded E_2-Artin algebras. You'd get an E_2-structure on TF, and this structure would extend the Lie structure on TF_A, that is, the Hochschild cohomology.

So if we could do this in the right way, this would give us another proof of Deligne's conjecture, just by writing down a functor, so just as a corollary.

This uses the fact that E_2 is self-dual, which is non-trivial, but I don't know that it uses much else.

In the abstract I promised a discussion of Goodwillie calculus. I meant to say I'm not going to get there. Another example that I see that I haven't gotten the details of, is, if I choose the algebra to be the differential forms of M, I get the chains of the loop space. So the functor extension would be writing down the picture that you can see with your eyes, and then that functor would give you that universal action on the chains of the free loop space.

[What about Goodwillie calculus?]

Schlessinger's criteria include linearity. Manetti says he doesn't know how to do anything without the condition, but maybe, well—

Derived deformation functors are Goodwillie's linear functors. This theory allows you to approximate all functors with linear ones. The theorem is a classification of linear functors, which is the first step of Goodwillie calculus.

[During the sixties and seventies, many people would write a paper whose real title was "now I know how to prove the Atiyah Singer index theorem." Deligne's conjecture is serving in that capacity now.]

[What are Artin algebras over an arbitrary operad?] They're free so that the quotient is finite dimensional and nilpotent, something like that.

12. Chris Schommer-Pries

So, it's a pleasure to be here, to have been a participant in several of these over the years. What I want to talk to you about today is the comparison problem in the theory of ∞, n-categories. This is joint work with Clark Barwick, and you can read about it on the arXiv if you're so inclined, it's 1112-0040. There are two questions I want to talk about. Let me give you some idea of the landscape. One question you can ask is, what is an $\infty, 1$-category, what does that mean? A related question is, what is an ∞, n-category. Another question which is important which is related is, what is a homotopy theory? There are different answers to both questions. Let me start here, and there are many different answers. I'll eventually talk about axioms for the homotopy theory of ∞, n-categories.

You'll get different answers here about what a homotopy theory is. You might hear that it's a Quillen model category, which is a category M along with classes of maps, cofibrations, fibrations, and weak equivalences. That's just one example.

Maybe you'll be more happy with even less structure, what is called a homotopical category. I'll write that over here. This is a category and a collection of weak equivalences W which satisfy a two out of six property, saying if you have composable maps f, g, h, that if gf and hg are in W then all six maps are in W.

One structure you get from a Quillen category is a way to deal with homotopy limits and colimits, a bunch of structure. Dwyer and Kan said that you don't need the cofibrations and fibrations, you can often get away with less structure, just some weak equivalences. You can even get away with less. A relatie category is a category and a subcategory of weak equivalences. The only requirement is that it contains all identities, so that every object is in W.

If you have a relative category, you can obtain mapping spaces between objects. This is enriched in simplicial sets. You can form the hammock localization and get a simplicial category. Instead of modding out by zig-zags, you have grids of zig-zags. This is hard to work with, but you could do it, and extract something you might call a homotopy theory. There are theorems about this, if you have a model category, it's often a simplicial model category. It doesn't give an identical category but it gives you one with equivalent mapping spaces.

One hint that the model category structure is overkill, is what is a map of model categories? It's a Quillen adjunction, which doesn't preserve the cofibrations or weak equivalences. So that's maybe extraneous parts of the structure. You can get everything you want from just the weak equivalences.

[Do you have cohomology, homotopy, and obstruction theory?]

There are unusual model categories. You can ask "can I build a Postnikov tower?" Maybe it doesn't have enough of them. There's a notion of Quillen cohomology in most model categories. You look at Abelian group objects in your model category, try to do stuff with that.

I want to talk about another proposal for a homotopy theory, due to Charles Rezk. Suppose I have a notion of homotopy theory, I should be able to get certain data out of that. Suppose I have some homotopy theory. I should be able to extract from that a moduli space of objects or a classifying space of objects X_0 In the regular homotopy category of spaces, this will be a space, of say, self-homotopy equivalences. If I have a map to this, I'll have a fibration whose objects are objects of the homotopy theory. You should also be able to create a moduli space of maps X_1, and pairs of maps X_2, and assemble these. The data you extract is some simplicial space. It satisfies a variety of axioms, and we'll call it a complete Segal space, and call the category of them CSS.

I want to think about what an $\infty - 1$ or $\infty - n$ category is, and also what a category is. If we have an ordinary category, then we can extract a simplicial set called the nerve of the category. What does the nerve consist of? It's the set of objects, the first space is the set of morphisms, which has two maps to the objects, and so on, $N\mathcal{C}_2$ is functors from [2] into $\mathcal{C}$, think of it as a poset, you get a pair of composable arrows. You get three maps and so on, and you get a simplicial set in this way called the nerve. You can characterize those which are the nerves of categories in the following way. You have two maps $N\mathcal{C}_2$ to $N\mathcal{C}_1$, taking the first and second maps, and those agree when you go to $N\mathcal{C}_0$, and this is a pullback:

$$\begin{tikzcd}
N\mathcal{C}_2 \arrow[r] \arrow[d] & N\mathcal{C}_1 \arrow[d] \\
N\mathcal{C}_1 \arrow[r] & N\mathcal{C}_0
\end{tikzcd}$$

Here is a fun fact that I just thought of, capturing the idea that model categories don't touch what you want.

Theorem 12.1. *On the category of sets, there are exactly nine model category structures, with three different Quillen isomorphism classes.*

[Jacob: are those 0-categories, -1-categories, and -2-categories?] Yes, exactly. These capture the appropriate homotopy theories, but contain extraneous information.

[Dennis: they're wearing tuxedos while they do it.]

A category you can think of as a simplicial set satisfying this condition, this is fully faithful. An ∞, n-category should be a "higher category with morphisms at all levels" but they should be invertible above n. How do you capture that? Even for $\infty, 1$, how do you express that? You could think about a category as objects and then having sets of morphisms. You could think that for an $\infty, 1$-category you have a mapping $(\infty, 0)$-category (where everything is invertible).

There is a basic principle, called the homotopy hypothesis, which says that ∞-groupoids are the same as spaces. There should be a homotopy theory of ∞, n-categories, and the homotopy theory of ∞-groupoids should be the same as the homotopy theory of spaces. You could truncate to n-groupoids, which have only trivial morphisms above n, and these should be the same as n-types. In the special case $n = 1$, there's a well-known way to think about it, you take the fundamental groupoid.

This leads to two different notions. Let me first do Segal categories. Such a category has a set of objects X_0, and then for any two objects you have a mapping space $\mathcal{C}(x, y)$, a space of maps, and then you could take the union $\mathcal{C}_1$. For every x, y, z, you have the space $\mathcal{C}(x, y, z)$, and these things assemble into a simplicial space. They are required to satisfy the Segal condition, that if you take the space $\mathcal{C}_n$, you can project it to $\mathcal{C}_1$, and the condition is that the projection to $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \times \cdots \mathcal{C}_1$ is a weak equivalence.

There are two different model structures on this, a projective and injective one. So work by Bergner shows some of the following diagram of right Quillen equivalences.

So now you can see, a category enriched in spaces is close to an $\infty, 1$-category.

$$\begin{tikzcd}
 & & Cat_\Delta \arrow[dll] \arrow[d, bend right=40] \arrow[d, bend right=15] \arrow[d, bend left=15] \arrow[d, bend left=40] & & \\
Seg_{proj} & & QCat \arrow[dl, bend right] \arrow[dr, bend left] & & RelCat \arrow[ull, dotted, "L^H"'] \arrow[dl] \\
 & Seg_{inj} \arrow[ul] \arrow[ur, bend right] & & CSS \arrow[ll] \arrow[ul, bend right] &
\end{tikzcd}$$

There are models for ∞, n as well.

$$
\begin{tikzcd}
 & & Cat_{\mathcal{M}} \arrow[dll] & & & nRelCat \arrow[dd, dotted] & \\
Seg_{\mathcal{M}} & & & complicialsets & & & \Theta_n - spaces \leftarrow \pitchfork - Sh_n \\
nSeg & Seg_m & Seg_{\mathcal{M}} \arrow[ull] & & CSS_{\mathcal{M}} \arrow[ll] & CSS_n &
\end{tikzcd}
$$

Some of these work if you have Cartesian properties (red), some if you have simplicial structures (blue), some things are good if you have both (green).

Theorem 12.2. *There exist four axiams which characterize the homotopy theory (quasicategory) of ∞, n-categories up to equivalence*

Theorem 12.3. *The space of quasicategories satsifying these axioms is $(B\mathbb{Z}/2)^n$.*

So we have, well, it's hard to see what's going on with model categories, instead of viewing it as a diagram of model categories, let's look at these as relative categories, or as simplicial categories, we're looking at a diagram of objects which are all equivalent. This is governed by the derived automorphisms.

For $n = 1$ this is a theorem of Töen, that $Aut(L^H CSS) \cong \mathbb{Z}/2$. The space of ways to make this diagram, there is no choice in making it homotopy coherent. As a diagram of homotopy theories it does commute. If you had another theory with other wacky arrows, there's only a $\mathbb{Z}/2$ to check. You can take the opposite in 2^n different ways, taking opposites of each of the levels of morphism. Another theorem is

Theorem 12.4. *(Nearly) all these models of ∞, n-categories satisfy these axioms (not necessarily complicial sets, I don't know)*

Let me spend my last five minutes trying to explain the axioms and giving a couple corollaries.

These are for $\mathcal{C}$ a presentable category with objects C_i, the *cells*

(1) Every object of $\mathcal{C}$ is generated by the cells under homotopy colimits.
(2) The overcategories $\mathcal{C}/C_i$ have internal homs.
(3) a finite list of obvious colimits are what they should be.
(4) a universal property, being universal with respect to the first three, that if D satisfies the first three, there are functors $L : \mathcal{C} \to D$ and a fully faithful right adjoint $D \to \mathcal{C}$. An example of D is $(n+k, n)$-categories, and you should think of L as a localization

This is too hard, you really want to use canonical colimits, and you need to make a bigger category, defining the category υ_n of strict n-categories. It contains the cells and is closed under fiber products over the cells, and under retracts. The fiber product is natural because having internal homs in the overcategory means that crossing over a cell with X preserves homotopy colimits for all X.

Then the first axiom gets replaced by the axiom that we ask for the canonical homotopy colimit, that the homotopy colimit over $\gamma \to X$ (with $\gamma \in \upsilon$) is equivalent to X. Making this bigger makes it easier to see that these things are satisfied.

Let me list some consequences. We construct a theory satisfying these axioms and construct some machinery to test when localizations are equivalences. You can learn the following facts as well.

- cells generate everything unde homotopy colimits.
- if M and N are model categries satisfying the axioms and we have a Quillen adjunction, then if L preserves the cells, then the adjunction is a Quillen equivalence. This is a recognition criteria
- Rezk showed that the homotopy hypothesis is satisfied by Θ_n-spaces. The k-groupoids here are k-types, this implies that this is true in all these models.
- Jacob proved the cobordism hypothesis in CSS_n, and you learn that the cobordism hypothesis is true in all models.

There are lots of variants you could ask about. Symmetric monoidal theory works in the same way. You could ask about dualizability. Hopefully this will be good to test with transversality sheaves. This is a good time to stop.

Really there should be an $\infty, (n+1)$-category of ∞, n-categories. One model would be a category enriched in ∞, n-categories. One of the axioms says that you have internal homs, and one instance is the zero cell, so given any two objects, there is an ∞, n-category of homs between them. That gives you a canonical category weakly enriched in ∞, n-categories. As a consequence, you learn that the axiomization of ∞, n-category homs comes for free.

[Can you do an enriched version?] That's a great question. Some of these work over ∞-topoi of any kind, or n-topoi, or whatever. People mentioned some here. There's a similar axiomitization that you can write down but I don't know if it gives the right thing, that's an interesting question, and for some models, well, I have no idea.

[How is this related to n-spaces, internal n-categories in Top?] There's certainly a similarity, you have these compositions, your compositions are allowed to be infinitely homotopy coherent. I don't know if there's a model.

[Jacob: I have a question. There are exactly four arrows $Cat_\Delta \to QCat$?] No, there are a bunch. Joyal did one, your construction with homotopy coherent resolution, that's another. Dan Dugger and David Spivak gave a whole family. There are infinite number of arrows. [That could be true everywhere right?] Yeah, you want to know that those are the same, that it doesn't matter.

[Dennis: ordinary homotopy theory started in the 20s or 30s, and used the interval. How do fit "looking for the interval" in this picture?]

I'm not going to remember, it fits as part of the story. Suppose you're in a Quillen model category. One axiom says you can factor the map from two points to one point into a cofibration into a weak equivalence which is a fibration. So this you can treat as an interval. This level of abstraction says this is just one piece, this isn't the fundamental piece. Using this to access the higher information is difficult. Suspension and loop, you could try to get higher stuff.

13. Kevin Costello III

Thanks very much. What I'm talking about is pretty far from topology. I must have given about twonty lectures at these FRG conferences. It's been neat, I get to work out a lot about quantum field theory. The last time, we talked about how minimal twists of supersymmetric gauge theories have a nice interpretation in terms complex geometry. Today I'll focus on the $N = 1$ theory. We saw that the twisted $N = 1$ theory on a complex surface X is what I called a cotangent theory to the moduli space of holomorphic G-bundles on X. And so if you remember in the first lecture I said you can think of a classical field theory as a sheaf of Lie algebras with an invariant pairing. Explicitly, the differential graded Lie algebra describing this theory assigns to U the algebra $\Omega^{0,x}(U, \mathfrak{g}_P) \oplus \Omega^{2,x}(U, \mathfrak{g}_P)[-1]$.

Today I want to talk a bit about the structure of the quantum theory. This may not make sense yet:

Theorem 13.1. *This theory admits a unique quantization on any surface X with trivial canonical bundle $K_X \cong \mathcal{O}_X$, unique when considered as a quantization compatible with natural symmetries. The $N = 2$ and $N = 4$ can be quantized on any complex surface.*

How could we prove such a statement, and what does it mean? First I'll tell you how we can prove it. The construction is always by obstruction theory. The deformation obstruction complex controlling quantizations is cochains of jets of $\mathcal{L}$ tensored over D_X with ω_X. Here $J(\mathcal{L})$ is the D_X differential graded Lie algebra of jets of sections.

The computation is that the invariants of this complex under natural symmetries is just zero. There is no cohomology whatsoever. There is a completely unique quantization.

A more interesting question is, what does this mean? What do I mean by quantization?

[What are the symmetries?]

If I trivialize the canonical bundle, I can rewrite this as $\Omega^{0,x}(U, \mathfrak{g}_P[\epsilon])$ where ϵ is of degree 1, and then the symmetries are $\frac{\partial}{\partial\epsilon}$ and $\epsilon\frac{\partial}{\partial\epsilon}$.

What does it mean to quantize? Let's work on $\mathcal{C}^2$. Classically, a factorization algebra which assigns to $U \subset \mathbb{C}^2$ cochains of $\mathcal{L}(U)$, taken in the topological sense, I'll call this the observables $Obs^{cl}(U)$ of our classical field theory. This is a commutative factorization algebra with a Poisson bracket of degree one.

A quantization gives a factorization algebra $Obs^q(U)$ over $\mathbb{C}[[\hbar]]$ which deforms $Obs^{cl}(U)$. I've probably talked about this point several times, what does it mean to quantize a factorization algebra, such that modulo $\hbar^2$, we have an equation, when you deform from commutative to associative, the Poisson bracket tells you how to deform, and similarly, here, the bracket tells you how to deform, $d(a \cdot b) = d(a)b + adb + \hbar\{a, b\} \mod \hbar^2$.

Now should I remind people or have they seen it enough times? We saw it in Ryan's talk. This is possibly the first talk I've given in this subject without defining it.

So unfortunately I don't have much interesting to say about this particular factorization algebra. It's an algebra on $\mathbb{C}^2$. Because it was built from the Dolbeaut complex with an extra "holomorphic property," this is a two-dimensional analogue of a vertex algebra.

This is a complicated algebraic object but I don't have anything to say about it. I'll move on to something a bit more concrete.

A *deformation*, to lead us back to more familiar mathematics, of our theory which is partly topological and partly holomorphic, well, recall, the Lie algebra describing our theory on $\mathcal{C}^2$ is the Dolbeaut complex of $\mathbb{C}^2$, with coefficients in $\mathfrak{g}[\epsilon]$, with differential $\bar{\partial}$. We'll deform this by adding $\epsilon\frac{\partial}{\partial w}$ to the differential. Here z and w are coordinates.

This is the theory that I'm going to focus on for the rest of the talk, this theory also admits a unique natural quantization. Whenever someone gives you a quantum field theory like this, you should try to figure out the space of solutions to the Euler-Lagrange equations. For the Lie algebra we wrote down,

$$(\Omega^{0,*}(\mathbb{C}^2, \mathfrak{g}[\epsilon]), \bar{\partial} + \epsilon\frac{\partial}{\partial w})$$

describes holomorphic G-bundles on $\mathbb{C}^2$ with a flat connection in the w-direction.

If you think of the Maurer-Cartan equation you can think that the ϵ should be seen as [unintelligible].

Maybe another thing you should always try to do is look at the solutions in codimension one, and this will always give us some (possibly infinite dimensional) symplectic manifold called the phase space.

In this case, let's take some compact manifold of dimension three and try to figure out what the equations of motions are. If I take $S^1_z \times T^2_w$, we find is of course, holomorphic bundles with a flat connection in the torus direction. Another way to say this, equivalently, we find flat bundles on the torus where we use the loop group instead of G, flat LG-bundles.

At this point it might be useful to compare with Chern-Simons. In Chern-Simons, you find that the phase space on a surface Σ is the space of flat G-bundles on Σ. This gives us a way to think heuristically about what this theory is. We should think of our theory as behaving like Chern-Simons theory for the loop group.

Everybody knows that Chern-Simons theory is controlled by the quantum group $U_q(G)$. I'd like to explain that this four dimensional theory we're considering is controlled by the quantum loop group.

So to do this, we should look in a bit more detail at what the factorization algebra describing this theory is. This theory gives us a factorization algebra on $\mathbb{C}^2 \times \mathbb{C}_W$, this is flat in one direction, locally constant in the w direction and holomorphic in the z direction. In derived algebraic geometry VI, Jacob shows that locally constant factorization oalgebras on $\mathbb{R}^n$ are the same thing as E_n algebras. Holomorphic factorizaiton algebras on $\mathbb{C}$ are essentially vertex algebras.

The subtle part is that the two structures fit together, so what we find is something like an E_2-algebra in vertex algebras.

This is a little bit in quotes, they're not quite vertex algebras but they're very closely related.

More precisely, for all disks D_z in $\mathbb{C}_z$ we have an E_2-algebra of observables on $D_z \times D_w$. The underlying cochain complex is $C^*(\Omega^{0,*}(D_z, \mathfrak{g}[\epsilon]), \bar{\partial}+\epsilon\frac{\partial}{\partial w})$. Here I get holomorphic functions on the disk that are locally constant in w, $C^*(\mathcal{O}(D_z)\otimes\mathfrak{g})[[\hbar]]$. let's call this E_2-algebra $Obs^q(D_z)$

If I take two disks inside a bigger one, I get a map of E_2-algebras which varies holomorphically as I move these around.

If I fix D_1 at the center, moving around another disk, I get a holomorphic family of multiplications.

Unfortunately, the only way I know how to say the higher coherences is that we have a factorization algebra on $\mathbb{C} \times \mathbb{C}$.

I haven't gotten to tell you about quantum groups yet. To tell you that, I have to tell you that this E_2 algebra I assign to a disk. It's an incarnation of the Yangian, invented by Drinfel'd, which plays an important role in integrable systems.

We have to recall something which I first learned from Tamarkin, he says that E_2 algebras and Hopf algebras are closely related. Dima used this because he wanted to prove formality of E_2-algebras, and he used a theorem that constructs Hopf algebras. So an E_2 algebra is an E_1-algebra in E_1-algebras. If we have an E_1 algebra with augmentation, we can get a E_1-coalgebra (via the bar construction).

So let's apply this to one of these factors. Apply Koszul duality to one of these things. We see we can turn an E_2-algebra into an E_1 algebra in co-E_1 algebras. This is a bialgebra with some extra conditions [Jacob: none—wait, I withdraw my claim], Hopf algebras are basically the same thing as bialgebras, the extra conditions, this construction gives a Hopf algebra.

Let's apply this construction to our example and associated to a formal disk D, and this E_2-algebra looks like cochains on $\mathfrak{g}[z]]$ with $\hbar$ adjoined.

Apply Koszul duality, we know that cochains of a Lie algebra become a universal enveloping algebra. We get a Hopf algebra structure on the $U(\mathfrak{g}[[z]])[[\hbar]]$ deforming $U(\mathfrak{g}[[z]])$. I suspect that the punchline of this talk is something that means nothing to anybody.

Theorem 13.2. *This is the Yangian.*

There are a couple of things this tells us.

(1) Observables of $N = 1$ supersymmetric gauge theory after deformation contain the Yangian. I think this is new.
(2) The Yangian has this extra structure which was not known before. As well as being, well, its linear dual is a vertex algebra. What does this mean? Modules for the Yangian form a monoidal category. From the vertex structure they also form a "vertex category."

    There are other fun things along these lines. Generalizations, I'm out of time, this story associates more generally a Hopf algebra to any Riemann surface Σ with a nowhere-vanishing holomorphic 1-form. The case we've been discussing is the disk with dz.

    Conjecture 13.1. *With Σ the punctured disk and the one form $\frac{dz}{z}$, we get the quantum loop group.*

    This would give justification for the claim that reduction gives us Chern-Simons theory.