LOOP SPACES IN GEOMETRY AND TOPOLOGY

GABRIEL C. DRUMMOND-COLE

1. September 1: Nancy Hingston, geodesics and the structure of the free loop space

So I'm speaking on geodesics and the structure of the free loop space. I was asked to give an introductory talk. So experts, if you want to leave now, it's fine.

The main ideas in the talks are the following. First of all there are ideas coming from geometry, so we have closed geodesics, iteration and index growth, there'll be a couple more here in a minute.

From topology we have loop spaces, loop products, and something that I call Poincaré duality in the free loop space. That's a main theme here. These are the basic ideas from geometry and from topology. The applications will be in geometry to the existence of closed geodesics and a resonance theorem for spheres. There are also some analogues in Floer theory.

So the original motivation, how I originally got involved was from these questions in geometry. Given a compact Riemannian manifold M, I'm also going to assume, just for simplicity, that $\pi_1 M = 0$. More general statements are possible. Also just for simplicity any homology or cohomology will be with rational coefficients. Riemannian means we have a metric and so we have distance. A geodesic is a path which locally minimizes distance. If you start with a point and a direction there is a unique geodesic that starts at that point and goes in that direction. We ask the question whether this ever closes up and becomes periodic. This question goes back to Lusternik-Schnirellmann, Poincaré, Birkhoff, Morse, Bott (my advisor), many people have discussed this.

There are two basic approaches. The dynamical systems approach takes the point of view that a closed geodesic is a geodesic that happens to close up. Look among geodesics for closed curves.

The second point of view is the variational approach, this says look among closed curves for those that happen to be geodesics. That's the loop space, that's the subject of this conference, so now we're thinking of a closed geodesic as a critical point of, so, we're looking on the free loop space ΛM, the space of all maps $S^1 \to M$. I'm not going to worry about how smooth these are, what kind of maps they are, it doesn't seem to matter in practice.

The most obvious function is the length function. A better function is the energy function, $\int |\dot{\gamma}|^2 dt$, where $\gamma : S^1 \to M$. This isn't the best function, the best functinon turns out to be $F = \sqrt{\text{energy}}$. This function has the property that it's greater than or equal to the length, with equality if and only if the loop is parameterized proportional to arc length. I'll tell you right now, you may as well just think that F is the length. You won't go wrong thinking that.

There's actually a somewhat deep reason behind the fact that that's the best way to think about this.

I'm going to do a brief review now of Morse theory. If you understand the basic ideas you have some hope of understanding details, so I'm just going to do nondegenerate Morse theory on a compact manifold X, which the free loop space is not, of course. So suppose that $f : X \to \mathbb{R}$ is a nondegenerate Morse function, think of it as being a good function. Morse theory gives us a relationship between the topology of X and the critical points of f. It's a relationship between the topology and what you might think of as the geometry.

There's a beautiful picture we all learned from Milnor, from Milnor's book. Here X is a torus and the height function is the function f. The minimum has index 0, the saddle points have index 1, and the maximum has index 2. A critical point is one where the derivative vanishes. The index means we look at the dimension of the maximal subspace of the tangent space where the second derivative is negative definite. So if the second derivative looks like $x_1^2 + \cdots + x_k^2 - y_1^2 - \cdots - y_\lambda^2$, then this is a critical point of index λ. This tells us we can build this torus out of a 0 cell, two 1-cells, and a 2-cell. This is a way where we can use the geometry of the critical point to tell us something about the topology.

To go the other way around, we can use the topology to tell us something about geometry. For example, we can get the Morse inequalities. In this particular example, the Morse inequalities tell us that if we take $H_*(X, \mathbb{Q})$, this has rank 4, so there must be at least four critical points for f in the nondegenerate situation. One way to think about this is using relative homology. So if we look at the homology of the space X cut off at the level b, $H_k(X^{\leq b})$, where this is the subspace where $f(x) \leq b$. If we look at the relative homology $H_k(X^{\leq b}, X^{\leq a})$, then this is equal to 0 if there is no critical value in $[a, b]$, and if we assume there is exactly one critical point in the complement of $X^{\leq a}$ in $X^{\leq b}$, of index λ, then this homology $H_k(X^{\leq b}, X^{\leq a})$ is $\mathbb{Q}$ for $k = \lambda$ and 0 otherwise. You can state this in terms of relative homology. As the level increases, the homotopy type changes. The level homology is $H_*(X^{\leq a}, X^{<a})$, which is the limit as
epsilon $\to 0$ of $H_*(X^{\leq a+\epsilon}, X^{\leq a-\epsilon})$. This is the basic building block of Morse theory.

There's a more modern way of saying this, in terms of a spectral sequence. Morse didn't do this but it's a more modern way of saying what Morse was talking about. We cut off $X^{\leq a}$. This induces a filtration of the chains $C_*(X^{\leq a})$. The spectral sequence converges to the homology of X and the first or second page is the direct sum of the different homologies, $\bigoplus_{\text{crit values}} H_*(X^{\leq}\lambda, X < \lambda)$. This is bigraded by the level λ and by the degree.

That's the end of the brief review of Morse theory. The case we have in mind is the free loop space. I'm assuming the manifold has a metric and is simply connected. We also have the based loops ΩM, which fix the basepoint. We have the length function, the energy function. We want to use $F = \sqrt{E}$, which, well, the based loops are a subset of the real loops, and this is a function to the real numbers. Now critical points of F are precisely closed geodesics in the given metric. We need a metric to do this.

These are not compact manifolds, the loop spaces, but they have nice structure as infinite dimensional manifolds. With respect to any of the structures we're talking about, they have good finite-dimensional approximations. There's a beautiful finite dimensional approximation due to Morse that I'd be happy to talk about.

[Inside the free loops, we have a copy of M, these are constant loops and they are critical points. They have length zero and they are critical points.]

We can do Morse theory on the finite dimensional approximations. So Morse theory gives us a map, a correspondence between homology and closed geodesics. One way to set up this correspondence is using the critical level. From $h \in H_*(\Lambda)$ we can produce the critical level $cr(h)$, which is the infimum of lengths $\{a|h$ is represented in $\Lambda^{\leq a}$.

So you have this infinite dimensional manifold, take a homology class representative, and push it down inside the manifold and when it gets stuck you've hit the critical level. This is also the infimum over representatives $[x] \in h$ of the supremum over in $x \in X$ of $f(x)$, the minimax value. This is due to Birkhoff in this example. It's a theroem of Birkhoff that this is a critical value. of F.

Theorem 1.1. *(Birkhoff)*
The critical level of h is a critical value of F.

So there exists a closed geodesic γ with the length of γ equal to this critical level and the index of γ approximately equal to the degree of h. This theorem is true in much more generality.

In the non-degenerate case this is off by at most 1, in the degenerate case it's off by at most $2n$. Here n is the dimension of the manifold.

[You're talking about nondegeneracy in the loop space, are you perturbing this in some way? You always have a circle of symmetries.]

You always have a one-parameter family of reparameterizations, the group S^1 acts by isometries. In the best case there's still a dimension one family, so this is always degenerate. I was sweeping that under the rug. It's a valid point though.

You have this infinite dimensional manifold, you have lots of homology, does that give you lots of closed geodesics? There's a problem which is much bigger than the problem pointed out by Ralph, which is the problem of iterations. Unfortunately, lots of homology classes does not ensure lots of closed geodesics. Any time you have a closed geodesic γ, this is a map from S^1 to the manifold. If you have a closed geodesic you also have γ^2. You go around twice, you have $\gamma^n(t) = \gamma(nt)$, thinking of S^1 as $\mathbb{R}/\mathbb{Z}$. These are different points in the free loop space. In general these have different indices. If you have one closed geodesic, it gives an infinite family of critical points, and these are not points, these are circles, which are even worse. They have different lengths and indices. So one honest closed geodesic looks like an army. You can't just count homology classes.

However there's a wonderful theorem of Gromoll and Meyer that says if you have a LOT of homology you get a lot of geodesics:

Theorem 1.2. *(Bott, 1956) The index of the iterates grows approximately linearly. Fix a manifold and its iterates. The index grows approximately linearly in M.*

[Is there a formula in the typical case? Let me answer that later.]

Theorem 1.3. *(Gromoll-Meyer 1969) The rank of the level homology of the iterates of any fixed closed geodesic is bounded.*

It's a simple and beautiful idea. Fixing a closed geodesic, the index grows linearly. So the degree in homology grows approximately linearly. So all the homology, the rank can be infinite but it's somehow bounded. It's easy in the non-degenerate case, but this is true and important in the degenerate case.

My advisor was Bott. He told me to go read Gromoll and Meyer's paper and explain it to him. In the non-degenerate case, there's nothing to prove. He said "oh, is that all it is?" It's the degenerate case that's interesting. If you only have a finite number of closed geodesics, then the ranks of the homology of the free loop space are bounded.

Theorem 1.4. *If the rank of $H_k(\Lambda, \mathbb{Q})$ is unbounded as $k \to \infty$ then M has infinitely many closed geodesics for any metric.*

Then you might ask when this is true? This was settled by Sullivan and [unintelligible]-Poirrier, who showed (in 1976) that the Rank of $H_*(\Lambda, \mathbb{Q})$ is unbounded if and only if $H^*(M, \mathbb{Q})$ requires more than one generator. So it's almost always true. This is if and only if $H^*(M, \mathbb{Q}) \neq \mathbb{Q}[x]/x^n$.

Let me just finish up by saying, most manifolds you get infinitely many closed geodesic for any metric, manifolds you don't get it for include spheres and projective spaces. What about those? These are the spaces for which these are truncated polynomial rings in one generator. The rank is bounded, and Gromoll-Meyer does not apply. It's interesting because these are the spaces for which there is a metric for which all the geodesics are closed. If you perturb the metric it becomes a very difficult problem. Very little is known. Maybe I'll leave it at that, it's been about 45 minutes.

2. Kenji Fukaya: Cyclic homology in Lagrangian Floer theory and pseudo-holomorphic curves

I'm happy to have an occosaion to talk here. I will talk about some structures close to loop space homology. At least half af my talk will be sections 3.8 and 7.4 of my book. This does not mean that I assume all of that. It seems likely that 7.4 was not read by most people. It's a very similar structure that was discovered independently by many other people. These are open-closed or closed-open theory. Open strings are bordered Riemann surfaces and closed strings are Riemann surfaces. This story gives some relation between these.

We're always working with a symplectic manifold (X, ω). We'll let $\mathcal{L}M$ be based loops, ΛM the free loops and ΩM differential forms. So $\omega \in \Omega^2 X$ with $d\omega = 0$ and $\omega^n \neq 0$ In many parts of my talk, X will be compact. it's richer when X is not compact. It's studied in more recent work. Let $L \subset X$ be an n dimensional compact submanifold with $\omega|_L = 0$.

So q will be a closed open map, and will give a map from $H(X)$ to $HH(\Omega(L), m)$ with some twisted structure.

If you have $HH(\Omega L, d, \wedge)$, this is most likely to be equal to $H(\Lambda M)$. I used the twisted structure, though, not d and $\wedge$. So we use a deformation of this and get an A_∞ structure.

If X is non-compact, we replace $H(X)$ with the symplectic homology $SH(X)$, this kind of relation was discovered by many people. The first place I know about it is in Kontsevich, then Paul Seidel and others.

Let me say how this kind of thing works. The most typical non-compact symplectic manifold is T^*M, and in this case $SH(X) = H(\Lambda M)$ and the reason, I don't want to prove this, but Hamiltonian dynamics of T^*X is related to the homology of the free loop space. The Hamiltonian dynamics are related to the closed geodesics. So in this case this is your q and it's an isomorphism.

Let's take another case, the case of S^2. When X is non-compact you get symplectic homology, but you get the regular homology of X in the compact case. In the compact case, the Hamiltonian dynamics are controlled by HX. The number of periodic orbits of a Hamiltonian vector field is at least the rank of HX. So here this plays the role of the loop space homology. You can just do the de Rham complex, this is very likely to be the homology of the loop space. If you have the q-deformation, you will get something finite, that's different, it's not infinite rank.

Let $X = S^2$ and L the equator S^1. Of course, if you consider $\Omega(L)$, the homology of L is $\mathbb{Q}[x]/x^2 = 0$. The Hochschild cohomology is the homology of the free loop space of S^1. The structure is deformed, $HF(L) = \mathbb{Q}[x]/x^2 = 2T1$. You have two disks which are bounded by L. I don't want to explicitly take the Floer homology but I just want to say that $x^2 \neq 0$. So $\Lambda_0^R = \{\sum a_i T^{\lambda_i} | a_i \in R, \lambda_i \geq 0, \lim \lambda_i = \infty\}$

Then $HF(R)$ is a Clifford algebra. The Hochschild cohomology of this Clifford algebra is Λ, one dimensional. One of the two algebras in nilpotent and in the other case it's semisimple.

In this case there are two spin structures, s_1 and s_2, and Lagrangian Floer theory depends on the choice of spin structure. So $\hat{q} : HS^2 \to HH(\Omega S^1, s_1) \oplus HH(\Omega S^1, s_2)$ and this is an isomorphism. The situation is very different from the cotangent bundle but you still get an isomorphism.

There is another example that's harder to understand, let me take $X = T^2$ with a meridian as L. Then m is just $d, \wedge$. So then we have a map $H(T^2) \to HH(S^1, d, \wedge)$. This is not an isomorphism. The reason that this big difference happens, you have many L in the same homotopy class. In this case it should not be correct to consider the individual Hochschild cohomology, we should consider the whole family. I don't know how to make it precies yet. So something like $\cup_a HH(L_a, d, \wedge)$. If you move the equator up or down in the sphere, you get 0. The equator is isolated, so there you can use the Hochschild cohomology itself. So then the choice is not correct. We need the Hochschild cohomology of a family version of Floer homology.

That's the general picture. This is something about, I can also cook another map p also. So p is something in the opposite direction. It's better to call it p^+ maybe, so p_*^+ goes from cyclic homology $HC(\Omega(L), m) \to H(X)$. Let me take $\hat{q} : H(X) \to HH(\Omega(L), m)$, which is an L_∞*homomorphism*. The first is a trivial Lie algebra and the second one is the Gerstenhaber Lie algebra. This is an L_∞ morphism. This is the closed open map. The other thing, the other story, you have the same q, it's actually a ring homomorphism.

We found that the HQ is actually an A_∞ algebra. I'll explain that soon. Then a problem is, is q an A_∞ homomorphism? Then the things are kind of more confusing. You have an A_∞ homomorphism and an L_∞ homomorphism, and are they related? If anybody knows how to name and study this? One wants to prove that q is an A_∞ homomorphism. Then you make this consistent and this should give the relation between the two. That's the question I don't know how to answer. On the chain level cooking up this structure is difficult.

In May, I heard a talk in which it was explained that one can generalize to the non-compact case, Pascaleff said that one can do this in the non-compact case, with contact boundary, then symplectic homology of X to $HH(\Omega(L), m)$, and this is an L_∞ homomorphism. The $SH(X)$ is not yet defined but then this should be L_∞ I don't know if there's an A_∞ structure on $SH(X)$.

Let me look at $X \subset X \times X$. This gives the A_∞ algebra structure.

The next section is about the open to closed map. This is something which goes in the opposite direction. Let me write $p_+ : HC(\Omega(L), m) \to H(X)$, from the cyclic homology to $H(X)$. We need to understand the homological algebra behind this map. In general $HC(C)$ is an L_∞ module over $HH(C)$. So $H(X) \to HH(\Omega(L), m)$ acts on $HC(\Omega(L), m)$. Then p_+ is an L_∞ module homomorphism.

[What about the compositions?]

The cyclic homology $HC(\Omega(L), m) \to HQ(X)$ is an $HQ(X)$-module homomorphism. The linear part is again respectful of the associative structure. Everything is actually A_∞ Everything should come from the same A_∞ structure. This is the story of p_+. There is another story about p_0 which is in a sense more interesting, from $\Lambda_0 \to H(X)$. This is some extra part of p_+. This has various applications, like proving nontriviality of some Floer homology. To understand this map p_0 is some motivation. To define this we need p_+. To understand p_0 correctly is important.

Let me say more things that are expected to exist. So p^+, this is not written, from $HC(\Omega(L), m) \to H(X)$, probably everything generalizes to the non-compact case. Probably a similar strategy works. You can generalize to $p^+_{S^1} : HC(\Omega L, m) \to H^{S^1}(X)$, and in the compact case this is just $H(X) \oplus H(\mathbb{CP}^\infty)$, and there is a famous sequence of Connes

$HH \to HC \to HC \to HH$ and the equivariant Gysin sequence. t]Then p_* intertwines one and the other. You just replace symplectic homology with equivariant versions. The whole structure looks complicated and involved. I don't know how to give the homotopy algebra framework behind this. I don't have enough knowledge to discuss all of them. There should be p_0 which is different from usual. If you understand it correctly it gives some knowledge of the open closed theory.

3. Kathryn Hess: Simplicial and cosimplicial models for free loop spaces

We have homotopy theorists, symplectic topologists, I want to make sure everyone can understand what's going on. Let me say what I want the goals for me for what these lectures are.

(1) I want to describe various (co)simplicial models for the free loop space ΛX. These will have different levels of complexity and we'll be able to capture different parts of structure, not just the homological structure but the S^1 actions and what Nancy was calling the iterations, which come from the power maps. So these should capture some of the extra structure of ΛX. I call the power map $\lambda^{(r)}$ which takes $\gamma \mapsto \gamma^r$. You've realized already that these are important parts of the structure. We'd like to have as much of this structure as possible.

    Why would we want simplicial models? If I actually want to understand the structure, it helps to have a combinatorial model for the structures we're looking at. Perhaps, or sometimes, this makes them better adapted for computation.

    I imagine or expect that reaching this first goal will take the lecture today and about half of the lecture tomorrow.

(2) I also want to describe various chain complexes the homology of which is either isomorphic to the homology of the free loop space or the equivariant homology (its S^1 orbit), let me call this fls_* or ho_*, so that $H_*(fls_*) \cong H_*\Lambda X$ and $H_*(ho_*) \cong H_*^{S^1}(\Lambda X) = H_*(\Lambda X \times_{S^1} ES^1)$, the homotopy orbit

space. I'd like a small computable chain complex for these. We'd actually like to capture the algebraic structure we can here, so these should be isomorphisms taking into account as much algebraic structure as possible.

That's the kind of thing I'm going to talk about today and tomorrow.

Now let me talk more about simplicial and cosimplicial models.

[What is a cosimplicial model? What's a model?]

The sense of the word model, there is a pair of functors relating simplicial sets to topological spaces, $|\ |$ (geometric realization) and $S.$ (the singular functor), which is a Quillen equivalence. What I mean by model is that $L_\bullet$ is a model of LM if $|L_\bullet| \cong \Lambda X$. I mean something similar in the cosimplicial context, with totalization instead of geometric realization.

I'll give a two-minute introduction and then refer you to any of numerous classical sets. It's relatively simple to describe.

A simplicial set is a graded set $K_\bullet = \{K_n\}_{n\geq 0}$ together with face and degeneracy maps linking these together. There are $d_i : K_n \to K_{n-1}$ (face maps) and $s_i : K_n \to K_{n+1}$ (both for $0 \leq i \leq n$). I won't write the complete set of identities that these things satisfy but I'll give a couple of examples.

One example is that $d_i s_i = Id = d_{i+1} s_i$. Then $d_i d_j = d_{j-1} d_i$ if $i < j$. There's an old book by Peter May which has all of these identities on page one.

Let me write $sSet$ as the cotegary of simplicial sets. A morphism respects face and degeneracy maps. It's nothing but a category of functors from the (opposite of the) ordinal category to sets.

Remark 3.1. Δ *has objects* $\mathbb{N}$ *and morphisms order preserving maps* $\{0, \ldots, m\} \to \{0, \ldots, n\}$. *So then* $sSet = Set^{\Delta^{op}}$.

I won't explain how geometric realization works. The basic idea is to think of geometric simplices, zero simplices as points, one simplices as intervals, two-simplices as triangles, three-simplices as tetrahedra and then the face and degeneracy maps give you how you should glue these together.

Let me make a few more simplicial preliminaries. Let me start with twisting functions. If $K_\bullet$ is a simplicial set and G a simplicial group, then I can define a twisting function as a map of graded sets $\tau : K_\bullet \to G_{\bullet-1}$. This is a twisting function if the following condition is satisfied.

If for x in K_n, if I calculate $d_i\tau(x)$, for $i = 0$ this should be $\tau(d_0x)^{-1}\tau(d_1x)$ and for other i this is $\tau d_{i+1}x$. For degeneracy maps you just intertwine $s_i\tau = \tau s_{i+1}$.

If I have a twisting function τ and a simplicial action α from $G_\bullet \times L_\bullet \to L_\bullet$, then we can build the twisted Cartesian product (TCP) that looks as follows.

$$(K_\bullet \times_\tau L_\bullet) = K_n \times L_n.$$

The degeneracies are componentwise, $s_i(x, y) = (s_ix, s_iy)$. Faces are strange only for degree zero:

$$d_i(x, y) = \begin{cases} d_0x, \tau(x)d_0y & i = 0 \\ d_ix, d_iy & i > 0 \end{cases}$$

So the projection to $K_\bullet$ is a simplicial map and is a Kan fibration if and only if $L_\bullet$ is a Kan complex. These are nice (if you don't know what they are) because they have good lifting properties in simplicial sets.

The next thing I should remind you of is Kan classifying spaces and loop groups. We want, one of the points of twisting functions is that they give a way of expressing things related to loop groups.

Theorem 3.1. *There is an adjunction between reduced simplicial sets $sSet_0$ (where $K_0 = \{*\}$) and simplicial groups sGr, $\mathbb{G}$ and $\bar{W}$ which has the following properties. If I apply $\mathbb{G}(K_\bullet)$ it should look like a group, then $\mathbb{G}(K_\bullet) = Free(K_{n+1})/\langle s_0 K_n\rangle$. Let me just say the faces and degeneracies. If I have $x \in K_{n+1}$, that corresponds to $\bar{x} \in \mathbb{G}(K_\bullet)_n$. I want to tell you the faces and degeneracies look like. So*

$$d_i(\bar{x}) = \begin{cases} \overline{d_0 x}^{-1} \cdot \overline{d_1 x} & i = 0 \\ \overline{d_{i+1} x} & i > 0 \end{cases}$$

and the degeneracies are just $s_i \bar{x} = \overline{s_{i+1} x}$. I should say what the underlying sets of $\bar{W} G_\bullet$ look like. In degree n it should be $G_0 \times \cdots G_{n-1}$ for $n > 0$ and a singleton for $n = 0$.

Proposition 3.1. *We have τ_K from $K_\bullet \to (\mathbb{G}K_\bullet)_{\bullet-1}$ which takes x to $\bar{x}$, this is the universal twisting function. There's a "couniversal" twisting function $\bar{W}G_\bullet \to G_{\bullet-1}$ which takes $(a_0, \ldots, a_{n-1}) \mapsto a_{n-1}$.*

So $sSet_0(K_\bullet, \bar{W}G_\bullet) \cong Tw(K_\bullet, G_\bullet) \cong sGr(\mathbb{G}K_\bullet, G_\bullet)$.

Remark 3.2. *Homotpy classes of simplicial maps into $\bar{W}G_\bullet$ classify simplicial fiber bundles with fiber $G_\bullet$. There is a universal bundle*

$$G_\bullet \to \bar{W}G_\bullet \times_{\nu_{G_\bullet}} G_\bullet \to \bar{W}G_\bullet.$$

Remark 3.3. *Part of the story is that $\eta_{K_\bullet} : K_\bullet \to \bar{W}K_\bullet$ is a weak equivalence, this means that geometric realization takes it to a homotopy equivalence of topological spaces.*

So now I wnat to talk about the bar and cyclic bar constructions. Let A be either a topological (or simplicial) monoid, so a topological space (or simplicial set) endowed with an associative multiplication $\mu : A \times A \to A$ and unit $\eta : \{*\} \to A$. The bar construction on A is a simplicial space (bisimplicial set).

$$B_n A = A^{\times n}.$$

So I should tell you what the faces and degeneracies do. d_i takes $(a_1, \ldots a_n)$ to

$$\begin{cases} (a_2, \ldots, a_n) & i = 0 \\ (a_1, \ldots a_i a_{i+1} \ldots a_n) & 0 < i < n \\ (a_1, \ldots, a_{n-1}) & i = n \end{cases}$$

The degeneracy maps insert an e in the ith place. It's easy to check that this is a simplicial object.

The cyclic bar construction takes the same input and the same type of output. $Z_\bullet A = \{Z_n A\}$, $Z_n A = A^{\times n+1}$,

$$d_i(a_0, \ldots, a_n) = \begin{cases} (a_2, \ldots, a_n a_0) & i = 0 \\ (a_1, \ldots a_{i-1} a_i \ldots a_n) & 0 < i \leq n \end{cases}$$

At every level there's an action of the cyclic group of order $n+1$ on the elements. This is called a cyclic set. I'll come back to this notion maybe this afternoon. What I did want to say, there's a simplicial map, $\pi_A : Z_\bullet A \to B_\bullet A$, which drops off a_0. This will give us the free loop fibration down to our classifying space.

Let me take two more minutes for preliminaries so that tomorrow I can really start talking about free loop spaces.

The last thing I want to mention is Artin-Mazur totalization which goes from bisimplicial sets to simplicial sets. It has $Tot(K_{\bullet\bullet})_n$ as $(x_0, \ldots, x_n) \in \prod^n K_{i,n-i}$ such that $d_0^v(x_i) = d_{i+1}^h x_{i+1}$ for $0 \leq i < n$.

Theorem 3.2. *(Cegana-Remedios, 2005)*

$$|Tot(K_{\bullet\bullet})| \cong |diagK_{\bullet\bullet}| \cong ||K_\bullet|_\bullet|.$$

Here $(diagK_{\bullet\bullet})_n = K_{nn}$.

Thanks for your patience, next time I'll talk about loop spaces.

4. Latschev: Non-exact Liouville embeddings and symplectic homology

[We said we would announce the question session. This afternoon we'll have Nancy Hingston for half an hour, then Kathryn Hess. Now we start with Latschev.]

Thank you very much for organizing a conference that bridges two fields that I like. I will talk about work in progress with K. Cieliebak. I reserve rights to change pieces of the argument in the final version. Okay, so there are a few words in the title that I don't expect most of the topologists to know. So let me spend the first parts of the talk organizing the question. The basic object of study in what I'm going to say is a Liouville domain. What are these?

Start with a symplectic manifold (W, ω) and assume it has boundary, compact with boundary. Then for it to be a Liouville domain, I want, first, for $\omega = d\lambda$. Think of the λ as also part of the structure. Then I can find a unique vector field Y which satisfies the equation that $\omega(Y,)$ is λ. This is called the Liouville vector field. I require that it is transverse pointing outward at the boundary of W.

What are the examples? The first example is the ball. I take as my primitive, the standard symplectic form, take $\lambda = \frac{1}{2}\sum x_i dy_i - y_i dx_i$. The picture becomes that the Liouville vector field is the radial vector field. Another very similar example is if $W = T^*Q$, and here λ is pdq, the picture is roughly the same, I draw this picture in every fiber. This is radial. I should say the unit disk bundle in the cotangent bundle.

Just to give another one that's not the same, let's take a surface with boundary, a disk with holes, say, then this admits a Liouville structure. It's actually a nice little exercise to write one down.

That's domains. Now I want to talk about embeddings. We're interested in codimension 0 symplectic embeddings $W_0 \subset W$. We call φ exact if $\varphi^*\lambda - \lambda_0 = df$ for some f. In this case $\varphi_*\lambda_0$ exctends to a global primitive on W.

Say we are given (W, ω) and inside it there is a closed Lagrangian Q. Then this gives us $[\lambda|_Q] \in H^n(Q, \mathbb{R})$ and Q is called exact if $[\lambda|_Q] = 0$. Gien $Q \subset W$ there is a embedding of D^*Q in W. Then ϕ is exact if and only if Q is exact.

Theorem 4.1. *(Gromov, 85) There exists no exact Lagrangian embedding of a closed Q into $\mathbb{R}^{2n}$ with the standard structure.*

The cartoon version of the proof, the hard part of the argument, assume $Q \subset \mathbb{R}^{2n}$ is Lagrangian, the hard part of the argument, pick a compatible almost-complex structure (in this case the standard one) and there exists a holomorphic disk U with boundary on Q that is non-constant. The disk has positive area, so $\int_{\partial D} \lambda > 0$ which implies that $[\lambda] \neq 0$.

The next big development was a theorem of Kenji Fukaya, maybe it's a heorem, the details are a mess but a strategy for proving it is clear, around 2004. Let Q^n be closed, spin (w_1 and w_2 vanish so it's oriented), and a $K(\pi, 1)$. Assume there exists a Lagrangian embedding of Q into $\mathbb{R}^{2n}$ with the standard spin structure. Then there exists a symplectic disk with boundary on Q with Moslov index 2.

The idea of the proof, now we look at all disks, fix some class $a \in \pi_2(\mathbb{R}^{2n}, Q)$, and let $\mathcal{M}(a)$ be the moduli space of holomorphic disks in that class. This $\mathcal{M}(a)$ comes with an evaluation map to the loop space. I want my disks to have a marked point on the boundary and then I get a map to the loop space ΛQ. We think of this moduli space as a chain on the loop space. Now here comes the trick. If we take the formal sum $\mathcal{M} = \sum_a \mathcal{M}(a)$ and think of this in $C_*(\Lambda Q) \widehat{\otimes \mathbb{Z}[\pi_2}(\mathbb{R}^{2n}, Q)]$ then this thing satsifies the equation $\partial\mathcal{M} + \frac{1}{2}\{\mathcal{M}, \mathcal{M}\} = 0$. Then there is some $\mathcal{N}$ so that $\partial\mathcal{N} + \{\mathcal{N}, \mathcal{M}\} = [Q]$. The second equation tells you, the first says it's a Maurer-Cartan element. The second one is a twisting of the boundary, and in the twisted version of the boundary map, the cycle $[Q]$ has become exact. This cycle was manifestly non-exact before. My loop bracket had to be non-trivial and the $\mathcal{M}$ had to be non-trivial. Deform the right hand side in a one parameter family until you get no solutions and that family is $\mathcal{N}$.

The statement comes out now by playing with indices. Once you're at this stage and on one has complained, you're home for free. I really like this point of view, whenever you have a Lagrangian embedding, looking at things with boundary on that Lagrangian gives a deformation of string topology.

I wanted to mention a corollary

Corollary 4.1. *If Q^3 is closed oriented and irreducible, then there exists a Lagrangian embedding into $\mathbb{R}^6$ if and only if Q is $S^1 \times \Sigma_g$.*

That these do embed goes back to Givental or Gromov, but the fact that no one else does is quite surprising. There was an expectation on the minimal number of intersection points for immersion. A recent result of Eckholm-Eliashberg-Murphy-Smith showed that for all 3-dimensional oriented closed manifolds, if I take one connect sum with $S^1 \times S^2$, there is a Lagrangian embedding into $\mathbb{R}^6$. So there's an embedding with one double point.

What we don't know is the following open question. What about $Q_1 \# Q_2$ with both of these $K(\pi, 1)$. We don't know.

One of my students is working on proving nonexistence of Lagrangian embeddings of products where essentially you prove that string brackets vanish.

As a manifold it embeds by Whitney, but you want there to be a Lagrangian embedding. Immersion is then a similar easy homotopy theoretic condition. But embedding is harder. Asking these questions is probing symplectic topology in interesting ways.

This is all stuff that other people did. Where do I come in? OUr plan is to make this heorem a theorem, and the idea is to bypass technical difficulties coming from string topology by rephrasing the argument purely in symplectic terms. This brings me to the second half of my title, namely symplectic homology.

The experts have forced me to mention monotonicity. The Maslov index in this context (Q in $\mathbb{R}^{2n}$) is a map from $\pi_2(\mathbb{R}^{2n}, Q) \cong \pi_1(\mathbb{Q}) \to \mathbb{Z}$, which basically measures twisting of the Lagrangians in a loop.

The other thing of course you have is the symplectic area $\omega : \pi_2(\mathbb{R}^{2n}, \mathbb{Q}) \to \mathbb{R}$. Monotone is you want these two to be proportional with a positive constant.

So okay, symplectic homology. Clearly I won't try to define it. So for those who don't know, just wait until Thursday and then you'll get a little bit. So $SH(W, d\lambda)$ is Floer homology for Hamiltonians which grow at infinity. Technically you should complete your W and do all kinds of funny stuff. The generators are periodic orbits for your Hamiltonian functions. You get a chain complex with two types of generators, critical points in the interior of W and Reeb orbits on the boundary. Something I should have said in the beginning, the vector field being transverse to the boundary, I get a contact form on the boundary and on that thing I get closed orbits (sometimes). Once you have one, you get all its iterates. This is a highly infinite dimensional chain complex.

Let's go back to my examples and see what you get. So the symplectic homology of the ball is zero. You have one critical point in the interior. That cancels with [missed].

In the cotangent bundle case, working with $\mathbb{Z}_2$ coefficients, this is $H_*(\Lambda Q, \mathbb{Z}_2)$. Mohammed will explain what to do in the general case. For the surface with holes the answer is not quite as nice.

Now here comes a useful theorem due to Viterbo, who pioneered this kind of homology. Suppose thet $\varphi : (W_0, d\lambda_0) \to (W, d\lambda)$ is an exact Liouville embedding. Then there is an induced map $\varphi_* : SH(W, d\lambda) \to SH(W_0, d\lambda_0)$. Nowadays we know much more about this map.

Remark 4.1. *φ_* is a ring homomorphism which preserves even more structure*

Then we get the corollary:

Corollary 4.2. *There are no exact Liouville embeddings of $(W_0, d\lambda_0)$ with $SH(W_0, d\lambda_0)$ into a (W, λ) with zero symplectic homology.*

You don't have a ring homomorphism from a zero ring to a non-zero ring.

Here's the strategy, what does it mean to rephrase Kenji's argument in these terms? The strategy is, what do we need? We need the loop bracket in symplectic homology. I should have stressed this more before, it was important for the equations to be on the chain level. So the first step is to produce an L_∞ structure on the chain complex, I should say *a* chain complex because there are choices here, for symplectic homology. For the topologists it will not become a surprise that it is an L_∞ structure and not just a bracket. The underlying complex is very small compared to chains. If you want them on a small complex is that they become infinity versions. This is obvious to topologists and less so to symplectic people.

A second step is from an embedding of $(W_0, d\lambda_0)$ into $(W, d\lambda)$, we want to produce a Maurer-Cartan element in this L_∞ algebra $(W_0, d\lambda_0)$. Then we get a twisted differential.

The picture is that the Maurer Cartan element should count the caps in W of things on the boundary of W_0. These will not exist when the cobordism is exact, then this element will vanish. So we get a twisted version $\widetilde{SH}$ where the complex has the same generators as the complex for W_0 but the differential is twisted in some way coming from the embedding.

We also get a ring structure on this, also twisted by the Maurer-Cartan element. Now I want to assert the existence of an analogue to Viterbo's map which goes from $SH(W, d\lambda)$ to $\widetilde{SH}(W, W_0)$ which is a ring homomorphism.

5. Craig Westerland: homology of stabilized moduli of Lefschetz fibrations

I want to apologize for what will be a very jet-lagged talk. What I want to do, I'm in the homotopy theory crowd, I want to talk about using homotopy theory to talk about symplectic topology. I'm far from an expert, so pleas bear with me.

A general goal is to study the homology of moduli spaces of "structures" on Riemann surfaces which degenerate at some marked points. What "structure" means, I don't want to give a talk that is so general as to have no content. I want to start with a framework, though. My favorite example are those of branched coverings, where you're studying the moduli space coverings which degenerate to branched coverings. The other case is the case of Lefschetz fibrations. There's much more of substance you can say in the first setting. I'm hoping the symplectic people in the crowd can tell me what you can get from the second situation, I'll give some information and you can tell me what it means.

So let M be a connected n-manifold, possibly with boundary, let $(X, *)$ be a connected pointed space, and choose $c \subset \pi_{n-1}(X, *)$ which is invariant under the action of π_1. Basically I want a subspace of the free classes $[S^{n-1}, X]$.

With these assumptions, let the *configuration mapping space* $CMap_k^c(M, X)$ be the space of pairs $\{\underline{z}, f\}$ where $\underline{z} \in Conf_k(\circ M)$ and $f : M\backslash\underline{z} \to X$. The function $f|_{\partial B_\epsilon(z_i)}$ should lie in c, it's a homotopy class of S^{n-1} to M. So here $Conf_k(\circ M)$ is tuples in M^k such that $z_i \neq z_j$ modulo S_k.

Let me tell you how to topologize this space. One thing to note is that $Homeo(M, \partial M)$ acts transitively on $Conf_k(\circ M)$ as long as the dimension is at least 2. So $Conf_k(\circ M) \leftarrow Homeo(M, \partial M)/Homeo(M, \partial M, \underline{y})$ for $\underline{y} \in Conf_k(M)$. This map is a homeomorphism defined by taking the value on $\underline{y}$.

There's a bijection $Homeo(M, \partial M) \times Maps^c(M\backslash\underline{y}, X) \to CMap_k^c(M, X)$ by taking (g, f) to $g(\underline{y}), f \circ g^{-1})$. The same argument tells me it factors through the homeomorphisms that fix $\underline{y}$. So we topologize the right hand side as this product.

There are variants, we can choose f to be based for a basepoint in the boundary of M.

If I do it without the k, I mean the disjoint union over all k of $CMap_k^c$.

What we're going to focus on is the case that M is a surface and X is BG for your favorite group. I want to at least give you some reason to care about these spaces. If G is a finite group, then $CMap_k(M, BG) = \{(\underline{z}, f) | f : M\backslash\underline{z} \to BG\}$. So these are G-fibrations, so covering spaces over the complement with Galois group G. There is an essentially unique way to fill in the covers to give a branched covering space.

Another natural thing is to mod out by diffeomorphisms of M.

If I take $G = Mod(\Sigma_g)$, then BG is the orbifold moduli space of surfaces of genus g, because there's isotropy on Teichmüller space. Now I want to restrict to a specific set of conjugacy classes, positive Dehn twists. That's c. I have the same situation, notw $CMap_k^c(M, \mathcal{M}_g)$ is pairs where $f : M\backslash\underline{z} \to \mathcal{M}_g$ and this is a surface bundle. Now I have X^4 mapping via q ot $M\backslash\underline{z}$. This is a surface bundle over the base. I can't fill this in to produce a 4-manifold that maps to all of M, but I can do it since c is only Dehn twists. I produce nodal fibers by collapsing the Dehn twist. Then I get a Lefschetz fibration from $\bar{X}$ to M with nodal fibers at $\underline{z}$.

Let me follow Kathryn's model, who wants to see the definition of a Lefschetz fibration? Probably the complement of those who wanted the definition of a simplicial set. So $q : X^4 \to M$ is a Lefschetz fibration if q, X, and M are smooth, oriented, and there exist a finite number of critical points. On a neighborhood of this q appears to be the map $\mathbb{C}^2 \to \mathbb{C}$ sending (x, y) to xy. This is at best a continuous Lefschetz fibration. It's probably the same as a smooth Lefschetz fibration. If you have any wisdom, this isn't algebraic, the map from algebraic into smooth Lefschetz fibrations, if you know when that's a homotopy equivalence or that it is in a range of degrees, that would be really nice to know.

We'll assume a couple of adjectives about these spaces. First, we'll assume there is at most one critical point per fiber, that it's relatively minimal (no nodal S^2 in the fiber), that it's irreducible (the pinched curve is non-separating). I want to study the homology of these spaces $Cmap_k^c$. What does the homology give me? Characteristic classes for families of Lefschetz fibrations.

Let's bring this down to earth and talk about the homotopy type, for $M = D^2$. The homomorphisms preserving the boundary are contractible. Fixing the boundary and fixing my favorite configuration elements pointwise is the mapping class group, and this is the kth braid group.

That was one of the ingredients of the topology on the configuration space. I also needed maps (based now) from $D^2 \backslash \underline{y}$ to BG. This is the homotopy type of a wedge of circles, so this is G^k. If I want to restrict to c, this is c^k. So the configuration mapping space

$$CMap_k^{c,*}(D^2, BG) \cong c^k \times_{\beta_k} E\beta_k,$$

the Borel construction. This is a computable-looking right hand side. The homology is the homology of a braid group with funny coefficients, if you pick a c and a k I don't know what to do. But I can say something stably.

So I need to talk about multiplicative structures.

Proposition 5.1. *If I look at $CMap^{c,*}(D^n, X)$, this is an E_{n-1}-algebra.*

Why is this? Let me give you a picture proof. Well, E_{n-1} embeds into E_n equatorially.

It's worth noting that $CMap^{c,*}(D^n, X)$ is homotopy equivalent to the one where the entire southern hemisphere maps to $*$. Let me take some of these, and I can plug them into my swiss cheese, and say the function at any point is its value when I project vertically to the boundary of the little disk.

There's a variant of this that allows me to glue for other M. If $\partial M \neq 0$, then $CMap^{c,k}(M, X)$ is a $CMap^{c,*}(D^n, X)$-module. Again, a picture should suffice.

This takes some work to set this up. I need two more definitions.

If $V \in CMap^{c,*}(D^n, X)$, this has its set of components a monoid $M = \pi_0 CMap^{c,*}(D^n, X)$. Look at $[V]$ in M. This is called a *central stabilizer* if $[V]$ is in the center of M and for every element m of M, there are m' and $m'' \in M$ and $k, \ell \in \mathbb{Z}_{>0}$ so that $mm' = V^k$ and $m''m = V^\ell$.

For any element, I should eventually be able to multiply to get to this stabilizer.

Maybe I won't say straight away, these do exist in basically every example I've looked at. Let me define one more thing. $A(G, c)$ is the pushout of (L is my free loops), well, LBG has a lot of components, it's conjugacy classes, let's pick the ones from c. So it's the pushout $D^2 \times L^c BG \leftarrow S^1 \times L^c BG \to BG$ (via evaluation).

Theorem 5.1. *(Ellenberg-Venkatesh-W.)*
If $CMap^{c,*}(D^2, BG)$ admits a central stabilizer V, then

$$H_*(CMap^{c,*}(M^2, BG)[V^{-1}]) \cong H_*Maps((M, \partial), (A(G, c), BG).$$

The left hand side is a limit over V.

This crowd is probably fairly happy with function spaces, even if they're function spaces built out of function spaces. The homology is quite tractable for $A(G, c)$.

I do ultimately want to say something, maybe I'll shortcut through something really quick, when we take $G = Mod(\Sigma_{g,1})$, then c are positive non-separating Dehn twists, what do I get when I build this thing? The pushout is $D^2 \times BC(T_\gamma) \leftarrow S^1 \times BC(T_\gamma) \to \mathcal{M}_{g,1}$.

Here $C(T_\gamma)$ is the centralizer of γ. So $BC(T_\gamma)$ is almost $\mathcal{M}_{g-1,3}$ but things line up so it's an S^1 bundle over this. This looks almost like the very first layer of the Deligne-Mumford compactification of $\mathcal{M}_g$.

Maybe I'll take two or three minutes. You can compute with Meyer-Vietoris and Seifert-van Kampen that $\pi_1 A(g, c)$ is zero and then that π_2 is $\mathbb{Z} \oplus \mathbb{Z}$. I want genus at least four. Then these are the dual of κ_1 and degree. This says that $\pi_0(Map(D^2, S^1), (A(G, c), BG)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus Mod(\Sigma_{g,1})$.

What this ends up proving is that, maybe I'll finish by stating a theorem of Denis Auroux and you can compare.

Theorem 5.2. *(Auroux)*
For $g \geq 3$ there exist Lefschetz fibrations f_g^A, f_g^B, f_g^C, and f_g^D with $V = f_g^A \# \cdots \# f_g^D$ so that if f and f' are genus g Lefschetz fibrations without reduced fibres and the same boundary monodromy, then there exist positive constants a, b, c, d, k, ℓ so that $f \# a f_g^A \# \cdots \# d f_g^D \cong f' \# (a + \ell) f_g^A \# (b - \ell) f_g^B \# (c + k) f_g^C \# (d - k) f_g^D$.

That's exactly what we're getting, the boundary monodromy and the k and ℓ.

My question is that, in principle we can compute things about the homology of this mapping space. What do these invariants tell us about Lefschetz fibrations. We can't expect them to give us more than this theorem but perhaps they can talk about families thereof.

6. September 2: Kathryn Hess, Part II

So, thank you all for your patience yesterday. Last time I told you about basic simplicial things. I'm going to give you the payoff today. We'll start with section 1b, the Burghelea-Fiedorowicz-Goodwillie model. This article appeared in Topology in 1986 back when Topology was still a good journal in topology. Then there was Goodwillie, 1985, also in Topology, who more or less simultaneously came up with the same model. Research articles are often not as easy to read as syntheses that come later. So I suggest you look at chapters 6 and 7 of Loday's book. I'll omit some details and for the details you can look to Loday.

Recall that we have $Z_\bullet$ from topological groups or simplicial groups to simplicial topological spaces. This has an extra structure, a cyclic structure, an action of an appropriate cyclic group. We have that $Z_\bullet G$ is actually a cyclic space. I'm not going to go into great detail about what this means, but in particular, the cyclic group $\mathbb{Z}/(n + 1)\mathbb{Z}$ acts on $Z_n G$ for all n in a way that is appropriately compatible with faces and degeneracies.

If you want to see the exact formulas, you can look in Loday's book, for example. As a remark, a cyclic space can be seen as a functor from a category with multiple different names, I'll call it $\Lambda^{op} \to Top$, where Λ has the same objects as Δ, it's the natural numbers, but the morphisms from m to n is a morphism in Δ followed by an element of the cyclic group. One thing to observe is that there's a natural inclusion of Δ into Λ. We have this simplicial functor ι from Δ into Λ.

Theorem 6.1. *Geometric realization induces a functor* $|\ |$ *from cyclic topological spaces to* $S^1 - Top$*, spaces endowed with an* S^1*-action. I'll give you a little sketch of the proof. This inclusion of categories induces an adjunction between cyclic topological spaces and simplicial topological spaces that looks as follows.* ι^* *has a left adjoint which I'll call* L*. It's a left Kan extension. It has this back and forth between cyclic and simplicial topological spaces. It turns out that* $(LK_\bullet)_n = \mathbb{Z}/n+1\mathbb{Z} \times K_n$ *but the* d_i *and* s_i *are twisted by the group action.*

It turns out that the geometric realization of $LK_\bullet$ turns out to be homeomorphic, well, there exists a homeomorphism from $|\iota_* L(K_\bullet)|$ to $S^1 \times |K_\bullet|$, where $S^1 = |\mathbb{Z}_{\bullet+1}\mathbb{Z}|$.

All right, so now what? Now let $K_\bullet$ be a cyclic space. It has the simplicial and cyclic structure. The point is that you get an S^1-action on the realization $|\iota^* K_\bullet|$ by looking at $S^1 \times |\iota^* K_\bullet|$ and running the homeomorphism backwards, this is $|\iota_* L\iota_* K_\bullet|$, and there's a natural transformation ϵ from $L\iota_* \to Id$, the counit of the adjunction, and you can use this to get to $|\iota_* K_\bullet|$. This composite gives you an S^1-action.

This explains why it's important to understand ι.

Henceforth I'll drop ι_*. I'll write geometric realizations without it. So in particular this means the geometric realization of the cyclic bar construction on a topological group is, well,

Theorem 6.2. *For any topological group* G*, there is an* S^1*-equivariant map* $|Z_\bullet G| \to \Lambda BG$ *that is a homotopy equivalence.*

So it's not an equivariant homotopy equivalence. Let me again give you a sketch of the proof. So what do we do? We know that $|Z_\bullet G|$ is an S^1-space by the previous theorem. That means we can consider the following composite:

$$S^1 \times |Z_\bullet G| \to |Z_\bullet G| \to |\mathcal{B}_\bullet G| = BG$$

and then take the transpose of the adjoint of this map, and we get a map $|Z_\bullet G| \to Map(S^1, BG) = \Lambda BG$. As I mentioned yesterday, there's a really important fiber sequence for the free loop space. We have the geometric realization of [unintelligble] $|Z_\bullet G| \to |\mathcal{B}G|$ and the fiber is $|C_\bullet G|$, the constant at G. So this is the fiber sequence $\Omega BG \to \Lambda BG \to BG$. Since you have these two fiber sequences with the same base, then the other two maps, the total space is an equivalence if and only if the induced map of the fiber is.

Now it turns out, I wanted to capture the nice structure on the free loop space. It turns out that one can also see the power maps in here, not just the S^1-action. If there's a question session this afternoon then maybe we can see the details. Putting together work by Bökstedt-Hsiang-Madsen in Inventiones in 1993 and some work that I've done with John Rognes, it turns out that there is a simplicial map $Z_\bullet G \to Z_\bullet G$, maybe I'll call it $\tilde{\lambda}^{(r)}$, that is homotopy equivalent in its realization to the usual topological power map.

That's the staritng point that lets us get to simpler simplicial models for the free loop space. Now let's start simplifying a little bit, move to the "Hochschild" model.

This time I'll start with a simplicial group, not a topological group. This is also in the paper with John Rognes on the arxiv (it doesn't have his name on it for reasons that only he knows). We want a model for the free loops on the realization of the classifying space of this group.

One remark before this definition,

Remark 6.1. *Any time you have a fibration, the based loops on the base B have a natural action on the fiber, the holonomy action, this twisting is what you do to combine the base and fiber together. With the path space the holonomy is left or right translation. Here the holonomy is conjugation.*

This adds motivation for the definition here:

Definition 6.1. *The* simplicial Hochschild construction on G *denoted* $\mathbb{H}G_\bullet$ *is the simplicial set that we get by taking the twisted cartesian product* $\bar{W}G_\bullet \times_{\nu_G} Ad(G_\bullet)$ *where* $Ad(G_\bullet)$ *is* $G_\bullet$ *seen as a* $G_\bullet$*-simplicial set via conjugation.*

This looks like the right thing, it fits into a fiber sequence between $G_\bullet$ and $\bar{W}G_\bullet$ and we've glued together using the conjugation action. You prove that it is with the following observation.

Proposition 6.1. *Remember you have this sequence*

$$c_\bullet G_\bullet \to Z_\bullet G_\bullet \to \mathcal{B}_\bullet G_\bullet$$

and apply Artin-Mazur totalization and get this sequence

$$G_\bullet \to \mathbb{H}G_\bullet \to \bar{W}G\bullet$$

and since we have that, it turns out that the geometric realization of $\mathbb{H}G_\bullet$ *is* $\Lambda B|G_\bullet|$.

We'll use the fact from yesterday that $|TotK_{\bullet\bullet} \cong |diagK_{\bullet\bullet}| \cong ||K_\bullet|_\bullet|$.

Remark 6.2. *You can see the power maps here with no problem.*

Now we simplify even further starting with a simplicial set rather than a simplicial group, we'll use the coHochschild model. I want a model for $\Lambda|K_\bullet|$. Keeping in mind the holonomy action, we'll use the other universal cochain, and the *co-Hochschild construction* $\hat{\mathbb{H}}K_\bullet = K_\bullet \times_{\tau_{K_\bullet}} Ad(\mathbb{G}K_\bullet)$. We again have a fiber sequence

$$\mathbb{G}K_\bullet \to \hat{\mathbb{H}}K_\bullet \to K_\bullet$$

and we get

Proposition 6.2. *There is a commuting diagram*

$$\begin{tikzcd}
\mathbb{G}K_\bullet \arrow[r] \arrow[d, equal] & \hat{\mathbb{H}}K_\bullet \arrow[r] \arrow[d] & K_\bullet \arrow[d, "\cong"] \\
\mathbb{G}K_\bullet \arrow[r] & \mathbb{H}\mathcal{G}K_\bullet \arrow[r] & \bar{W}\mathcal{G}_\bullet K_\bullet
\end{tikzcd}$$

I gave you that the right map was a weak equivalence so the middle map is a weak equivalence, so its geometric realization $|\hat{\mathbb{H}}K_\bullet| \cong \Lambda B|\mathbb{G}K_\bullet| \cong \Lambda|K_\bullet|$.

Again, we can very easily model the power maps.

Remark 6.3. *The power map* $\lambda^{(r)} : \hat{\mathbb{H}}K_\bullet \to \hat{\mathbb{H}}K_\bullet$ *which takes* (x, a) *to* (x, a^r) *is a model of* $\lambda^{(r)}$.

I was very briefly going to talk about one more model, a cosimplicial model this time, the Jones model.

Jones in Inventiones, 1987, defined for X a topological space a cosimplicial object whose totalization is the free loop space.

Definition 6.2. The Jones model *for* ΛK *is the cosimplicial space* $\mathcal{J}^\bullet(X) : \Delta \to Top$ *given by* $J^n(X) = X^{n+1}$. *The coface maps are built using the diagonal (and a cyclic action). The codegeneracy maps are given by dropping an index, by projection. It's very easy to check this is a cosimplicial space. It's got a cyclic structure as well. If we apply totalization, we'll get a cyclic space again. We get an* S^1*-space and can apply a similar construction to show that you get the free loop space.*

Remark 6.4. *There's another functor, unfortunately also called totalization, from cosimplicial spaces to spaces such that if you have a cosimplicial object which has a cyclic structure, then* $TotK^\bullet$ *is an* S^1*-space. Using that and the fact that* $\mathcal{J}^\bullet(X)$ *has a cyclic structure, we can use that cyclic structure, look at this totalization and it turns out to be equivalent to* ΛX. *What one uses here is the fact that* $\mathcal{J}^\bullet(X) = Map((\mathbb{Z}/\bullet+1\mathbb{Z}), X)$.

I'll stop there, thank you.

7. Richard Hepworth, string topology of classifying spaces

These are three lectures on string topology on classifying spaces. There are notes on my homepage. If you want to not take notes but just read along on your device, that might work. This subject began with a paper of Chataur and Menichi, these talks are based on joint work with Lahtinen. Any mistakes I make or lies I tell are solely my own.

I want to address the following question. Let G be a finite group. What is the structure, the algebraic structure, of the homology $H_*(BG)$ and the homology of $H_*(\Lambda BG)$? I'm using Λ for the free loop space. These are the strings in string topology. Homology is always taken with coefficients in a field $\mathbb{F}$ which I'll suppress most of the time.

The answer given by Chataur-Menichi is that they are part of a homological conformal field theory. I don't expect you to know what that means, but it's an algebraic structure governed by surfaces and their diffeomorphisms. This is the standard answer in string topology (of manifolds, say).

The answer that Lahtinen and I gave was, we looked at the existing constructions and realized that whenever you see a surface, you only need something that is pretty much a surface, and a diffeomorphism is a homotopy automorphism. Our answer is that they are part of a homological h-graph field theory. For the moment let me say in contrast to HCFTs, these field theories are governed by things that are not quite surfaces and their homotopy automorphisms.

To give you a bit of a preview, this homological h-graph field theory will allow me to draw a picture of the standard pair of pants and that will give a product on H_* of the free loop space of BG. That's the kind of thing you find in homological conformal field theories. The homological graph theory lets me draw a trivalent graph and get a product on $H_*(BG)$. We can draw an interval with a loop at the base and that will also give us a structure.

If you look up the paper on the arxiv, you get a version where G is a compact Lie group. For simplicity in this lecture I'll use a finite group for simplicity.

The first aim is to tell you what a homological h-graph field theory is. Our answer says those vector spaces are part of such a field theory. Finally, those of you who are familiar with what happens in string topology. I'll hopefully end by explaining how we get some nonzero higher operations.

Now let's start properly with H-graphs and h-graph cobordism. If you're used to things governed by surfaces, bear with me. I'll tell you about a homotopy theoretic replacement.

Definition 7.1. *An h-*graph *is a space homotopy equivalent to a finite graph.*

Example 7.1. *A finite set is an example. A compact 1-manifold is an example. A surface, compact, with boundary in each component, is an example.*

In analogy with surfaces and 1-manifolds, h-graphs will be both our surfaces and our 1-manifolds.

Definition 7.2. *An H-graph cobordism $S : X \nrightarrow Y$ consists of h-graphs and maps $X \xrightarrow{i} S \xleftarrow{j} Y$ such that*

(1) *$i \sqcup j : X \sqcup Y \to S$ is a closed cofibration.*
(2) *$i(X)$ meets every path component of S*
(3) *there is a homotopy cofiber square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d, "j"] \\ B \arrow[r] & S \end{tikzcd}$$

where A has the homotopy type of a finite set and B is an h-graph.

Why do I impose these conditions? In order for spaces of homotopy equivalences to have the right type you need to do something like this.

Example 7.2. *Suppose we're given S a finite graph, X and Y finite sets, and maps i and j to S which are injections. Let's also assume that this satisfies the second condition. Then S determines an h-graph cobordism $X \nrightarrow Y$.*

Suppose I take S to be an arc and X and Y to be points on the arc, at the endpoints. This gives $i : pt \nrightarrow pt$. If I take S to be a Y-shaped graph, I get $m : pt \sqcup pt \nrightarrow pt$. I also get w from one point to two and c from a point to the empty set.

Suppose we're given a compact surface, a disjoint union of its incoming and outgoing boundary such that the second condition holds, then I get an h-graph cobordism from X to Y. I can take S to be the cylinder and I get $I : S^1 \to S^1$. I can take S to be the pair of pants and I get M from two circles to one. I get W from one circle to two. Finally, I get C from a circle to the empty set.

I'm going to write down a word and don't hate me for it.

Exercise 7.1. *Find more.*

I can do with these what I can do with ordinary cobordisms. I can compose them. How do I compose? I just glue together. I can also take disjoint union. More interestingly,

Definition 7.3. *Let $S \not\rightarrow Y$ and $S' : X' \not\rightarrow Y'$. A* two-cell *$\varphi : S \Rightarrow S'$ consists of compatible homotopy equivalences $\varphi_X : X \to X'$, $\varphi_Y : Y \to Y'$, and $\varphi_S : S \to S'$.*

For example $(c \sqcup i) \circ w \to i$.

Exercise 7.2. *Take ℓ to be the lasso, a line with a circle coming out of it. Take d the dunce's cap, the cone. These go from the poitn to the circle and back. Find two-cells $d \circ \ell \Rightarrow i$, $(d \sqcup d) \circ W \Rightarrow w \circ d$, and $W \circ \ell \Rightarrow \ell \sqcup \ell \circ w$*

Let S be an h-graph cobordism $X \not\rightarrow Y$. Then $hAut(S)$ is the topological monoid of homotopy equivalences α on S such that α respects i and j. This is our analogue of the topological group of all diffeomorphisms of a surface.

Proposition 7.1. *The map $hAut(S) \to \pi_0 hAut(S)$ is homotopy equivalence. In other words, the components are contractible and $\pi_0 hAut(S)$ is a group.*

Example 7.3. *Let's draw the second most simple h-graph cobordism, call it Q : $pt \not\rightarrow pt$ The proposition tells us that we might as well study the π_0. It's the semidirect product of the integers with $\{\pm 1\}$. We let $\{\pm 1\}$ act on $\mathbb{Z}$ by multiplication.*

Why do I get this? The group has two generators, the generator of the integers and the generator of ± 1. What does the generator of the integers do? It sends Q to, well, it sends the circle to itself by the identity map, it sends the arc to the circle plus itself. What does the ± 1 do? It sends the arc to itself and it flips the circle. It's not terribly difficult to see that this map is an isomorphism.

Exercise 7.3. *$\pi_0(hAut(cylinder))$ is the integers.*

Example 7.4. *For S a surface, the map from $Diff(S, \partial S) \to hAut(S)$ is a homotopy equivalence.*

Example 7.5. *If i modify Q so that it features a wedge of n-circles at its left hand end, then $\pi_0 hAut(Q)$ is $F_n \ltimes Aut(F_n)$, the holomorph of the free group on n letters.*

Suppose given a two-cell $S \Rightarrow S'$, I can cook up a zigzag

$$hAut(S) \leftarrow \mathcal{H} \to hAut(S')$$

of monoid homotopy equivalences.

That's the end of my first lecture, next time I'll tell you something about building them into a field theory.

8. Kathryn Hess, Part III

Now I'll talk about chain complex models for free loop spaces, hopefully with extra structure. I won't talk about string topology operations, but you should be able to see in the cohomology the structure not just as a vector space but the multiplication.

Some of the chain complex models will be built in analogy with the simplicial models, so I put some of the sequences, fiber sequences, up there.

Let me start again with some preliminaries. I won't say what a chain complex is. Let's fix $\mathbf{k}$ a commutative ring, and tensor products will be over $\mathbf{k}$. I'll work with the following three categories. Ch is the category of non-negatively graded chain complexes of $\mathbf{k}$-modules. I'll work with Alg, which is the category of differential graded $\mathbf{k}$-algebras (η the unit and μ the multiplication), monoids in Ch, and I'll assume that they are connected, so that in degree zero it's a copy of $\mathbf{k}$ and augmented so with a map $A \to \mathbf{k}$. I'll also work with $Coalg$, which is the category of dg k-coalgebras, counital. The counit is ϵ and the coproduct Δ. These will be 1-connected so $C_1 = 0$ and $C_0 = k$, and also coaugmented.

Given $A \in Alg$ and $C \in Coalg$, a twisting cochain is a $\mathbf{k}$-linear map of degree -1 $t : C_* \to A_{*-1}$ such that there is a Maurer-Cartan condition, so that $d_A t + t d_C = \mu(t \otimes t)\Delta$.

If M is a left A-module, that is, there is a chain map $\lambda : A \otimes M \to M$, associative and unital, and N is a right C-comodule, that is, there is a chain map $\rho : N \to N \otimes C$, satisfying dual conditions, so that $n \mapsto n_i \otimes c^i$, then there exists a chain complex, the twisted tensor product $N \otimes_t M$ such that in degree n, it's the usual tensor product, $(N \otimes_t M)_n = \bigoplus_k^n N_k \otimes M_{n-k}$ where the differential is defined, you have a standard way of computing the tensor product, but we'll perturb this using t, so $D_t(n \otimes m) = d_N n \otimes m \pm n \otimes d_m m + n_i \otimes t(c_i) \cdot m$.

We have this explicit formula for the twisted tensor product of these guys.

An important example is when A is a module over itself and C is a comodule over itself. Then you can construct $C \otimes_t A$ which fits into something that kind of looks like a principal bundle, $A \subset A \otimes_t C \to C$. It has the feel of a principle A-bundle.

The next thing that I want to talk about is the bar-cobar adjunction. This is analagous to the Kan loop group classifying space adjunction. This is old old classical stuff.

Theorem 8.1. *There is an adjunction $\Omega : Coalg \rightleftarrows Alg : \mathcal{B}$, these are called cobar and bar, and I'll say something about what these things look like. So ΩC will be the free algebra on the desuspended chain complex $Ts^{-1}C_{\geq 0}, d_\Omega$ where the differential is built from d_C and Δ. Dually $\mathcal{B}A$ is a cofree coalgebra, splitting words in all the ways you can, $\mathcal{B}A = TsA_{\geq 0}, d_{\mathcal{B}}$ where the differential is built from d_A and μ. This is the cofree coassociative coalgebra.*

If V is a graded $\mathbf{k}$-module, then $TV = \bigoplus_{k \geq 0} V^{\otimes k}$.

Twisting cochains mediate as before in the simplicial setting.

Proposition 8.1. *The $\mathbf{k}$-linear maps $t_\Omega : C_{>0} \to \Omega C, c \mapsto s^{-1}c$ and $t_{\mathcal{B}} : (\mathcal{B}A)_{\geq 0} \to A$ which takes $a_1 \otimes \cdots a_n$ to a_1 if $n = 1$ and 0 otherwise, are twisting cochains and mediate the adjunction:*

$$Coalg(C, \mathcal{B}(A)) \cong Tw(C, A) \cong Alg(\Omega C, A)$$

If I have a map of coalgebras g, I'll take it to $t_{\mathcal{B}} \circ g$ or an algebra map f goes to $f \circ t_\Omega$.

This adjunction plays nicely with homotopy in the sense that the adjoint give us unit and counit, and $C \mapsto \mathcal{B}\Omega C$ and $\Omega \mathcal{B} A \to A$ are quasi isomorphisms.

This bar construction is some sort of geometric realization, you can do everything in the previous lectures in the algebraic context, the bar is a realization and the cobar is a totalization.

What is the topological significance? In various versions and with varying degrees of extra structure, going back to Adams, Szczarba, Baues, and my work with others, if you have a one-reduced simplicial set $K_\bullet$, exactly one zero and one simplex, then there is a twisting cochain $t_K : C_* K_\bullet \to C_* \mathbb{G} K_\bullet$ (the C_* is the normalized chain complex functor to Ch, I'll come back to it) such that the associated dg algebra map $\alpha_K : \Omega C_* K_\bullet \to C_* \mathbb{G} K$ is a quasiisomporphism of algebras. We worked out that it's possible to make this actually respect the Hopf comultiplication as well.

One remark about this functor, so $C_* : sSet \to Ch$ behaves very nicely, it's the normalized chain complex, and if you have, it's a free $\mathbf{k}$-module in each degree, free on the nondegenerate simplices of your simplicial set. This says that the cobar can provide a nice model of the loop space.

The idea is to push this further now, find a model for the free loop space.

So the first thing to talk about is the Hochschild complex.

One can define this for ordinary $fieldk$-algebras. Define the Hochschild complex as a functor $\mathcal{H} : Alg \to Ch$ that is an extension, a twisted tensor extension of the bar of A by A. So

$$\mathcal{H}A = TsA_{>0} \otimes A, d_{\mathcal{H}}$$

which is a perturbation of the bar differential and the differential on A. So $d_{\mathcal{H}}(sa_1|\cdots|sa_n\otimes b)$ is $d_{\mathcal{B}}(sa_1|\cdots|sa_n) \otimes b \pm sa_1|\cdots|sa_n \otimes d_A b$ but then perturb this by adding $\pm sa_1|\cdots sa_n \otimes a_n b \pm sa_2|\cdots|sa_n \otimes ba_1$. We have an extension to a similar thing to one of our fiber sequences, $A \to \mathcal{H}(A) \to BA$, and let me also say that $HH_*A = H_*\mathcal{H}(A)$ is the *Hochschild homology* of A.

Why should someone interested in free loop spaces care about Hochschild homology? The big theorem in [B-F, G, J] (all cited in previous lectures) is

Theorem 8.2. *If X is a pointed connected topological space, then Hochschild homology of the singular chains of the Moore loops on X (a model which are literally associative), so $HH_*(S_*\Omega^{Moore}X) \cong H_*(\Lambda X;\mathbf{k})$. If X is one-connected, $HH*(S^*X) \cong H^*(\Lambda X, \mathbf{k})$ as graded $\mathbf{k}$-modules.*

Remark 8.1. *The key is the Eilenberg-Zilber equivalences, the shuffle, that gives you $S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$. I'll be happy if someone asks to see this proof in the questions session.*

One more remark

Remark 8.2. *There's work by Ndombol and Thomas in 2001-2002 in which they show that if $\mathbf{k}$ is a field and X is simply connected, there's a strongly homotopy commutative structure on $S^*(X)$ which lets us get the product on $H^*(\Lambda X;\mathbf{k})$ if $\mathbf{k}$ is a field, so that we have an isomorphism of algebras. Further, Menichi in 2001 showed that if X is path connected, then you can turn the* cohomology *of the Hochschild complex into $H^*(\Lambda X)$ as algebras.*

This is a chain model which is analagous to line number three, now let's quickly see one analagous to line number four, a coHochschild complex, sticking a "co" in front of everything. Remarkably that makes things almost easier, this goes back to Doi, Idrissi, and a paper I wrote with two others. I twisted an algebra and the bar construction. Now we twist together a coalgebra and its cobar construction.

Definition 8.1. *The* coHochschild construction *$\hat{\mathcal{H}} : Coalg \to Ch$ is built by twisting together ΩC and C so that you have $\Omega C \to \hat{\mathcal{H}}C \to C$, This is analagous to the fiber sequence over a reduced simplicial set.*

Let me define also $\hat{HH}_*(C) = H_*(\hat{\mathcal{H}}C)$. We'll see if you do this appropriately you get the homology of the loop space.

The point is that if t is a twisting cochain it gives rise to a comparison system that looks like this:

$$\begin{tikzcd} \Omega C \arrow[r] \arrow[d, "\alpha_t"] & \hat{\mathcal{H}}C \arrow[r] \arrow[d, "\gamma_t"] & C \arrow[d, "\beta_t"] \\ A \arrow[r] & \mathcal{H}A \arrow[r] & \mathcal{B}A \end{tikzcd}$$

and α_t is a quasiisomorphism if and only if β_t is if and only if γ_t is.

So as an example, apply this to $t_{K_\bullet} : C_*K_\bullet \to C_*\mathbb{G}K_\bullet$. Then we know that $\gamma_K : \hat{\mathcal{H}}C_*K_\bullet \to \mathcal{H}C_*\mathbb{G}K_\bullet$ so $\hat{HH}_*(C_*K_\bullet)$ is a nice chain model for the free loop space.

Let's take a little look at this and then stop there.

Why would one care about this particular model? This thing is C_*K (which has finitely many generators) and you're twisting it together with ΩC_*K. If you can find a nice model for your space, this gives you a relatively small and computable model for your free loop space. You can pick up the multiplicative structure and power maps.

We have a natural chain equivalence $\hat{\mathcal{H}}C_*K_\bullet \to C_*\hat{H}K_\bullet$ extending $\Omega C_*K_\bullet \to C_*\mathbb{G}K_\bullet$.

9. Somnath Basu: the closed geodesic problem for four manifolds

[Editor's note: I was pretty tired for this talk, I apologize]

Thanks to all the organizers in what has been a very nice conference so far. I'll use LX for maps from the circle to X. This is based on joint work with S. Basu, another S. Basu. We're going to try to answer the question, let n_T be the number of distinct closed geodesics of length at most T in (M, g) (this is a Riemannian metric). One needs to restrict the metric g that you want to put.

The question is the following. Usually my manifolds are simply connected and closed. If (M, g) is a generic Riemannian manifold then n_T grows exponentially. I guess as I've stated it, it's a conjecture of Gromov. He said something like this is likely to be true. I'll say more about what I mean when I say it grows exponentially. The answer, the positive one, will be for 4-manifolds that are simply connected and closed.

Before I begin, let me explain what I mean by "distinct." Here we mean we consider iterates and inverses the same, so they are considered the same if their images coincide.

So that's what we're counting. If you look at the first simple example, trying to count n_T for (S^2, g_{Euc}), you see that you don't see any until you hit ∞. You want a number for each T and talk about how it grows. We'll see a reason this is not well suited is because g_{Euc} is not a generic metric.

What is a generic metric? Whenever you see a closed geodesic, you get an S^1 of these by rotating. You should not be able to get anything more by moving them. I should draw a picture in the free loop space, but what we need here is to restrict to metrics g which are generic, satisfying the property that any closed geodesic gives rise to $S^1 \subset LM$ and this is a nondegenerate critical submanifold.

It turns out by a result of [unintelligible] that these metrics are dense in the Frechet space of metrics. There for each T you get a number and the question is how fast this sequence can grow.

One can imagine that you can try to answer this from the geometric viewpoint or try to use algebraic topology.

Definition 9.1. *A sequence of non-negative numbers* $\{b_i\}$ *has* exponential growth *if there is a* $\lambda > 0$ *and* $c > 1$ *such that* $\max\{b_1, \ldots, b_n\} \geq \lambda c^n$ *for* $n > N$*. This is basically equivalent to saying that* $b_1 + \cdots + b_n \geq \mu D^n$ *for some* μ *and* D*.*

A key result relates the counting function n_T to the Betti numbers of the free loop space. In my notation, LM is the free loop space of M and so:

Theorem 9.1. *Gromov 78, Ballmann-Ziller 82*
For a generic metric on a simply connected manifold, we have the following inequality:

$$n_T \geq a \max_{j \leq bT} b_j(LM)$$

with a *and* b *positive constants for large enough* T *(*b_j *is the rank of* $H^j(LM, \mathbb{Q})$*).*

Why is this good? It reduces the problem to the question of how fast $b_j(LM)$ grows. How is this related to 4-manifolds or 5-manifolds in general.

There's another thing that I haven't quite mentioned. There's a twofold genericity, not only the manifold but the space should be generic. Given a simply connected space, it is either rationally elliptic or rationally hyperbolic. We are interested in the rationally hyperbolic ones.

Definition 9.2. *Consider a simply connected space* X *with bounded rational cohomology. In this setting, we say* X *is* rationally elliptic *if* $\sum dim \pi_j(X) \otimes \mathbb{Q} < \infty$ *and rationally hyperbolic otherwise.*

We want the space to be rationally hyperbolic. You choose your manifold and then put a metric on it where that thing makes sense.

So a fact is, if X is rationally hyperbolic then you have a similar phenomenon of exponential growth on the homotopy groups, $\sum^n dim \pi_j(X) \otimes \mathbb{Q} \geq \lambda A^m$ for large enough m.

So far, this sort of gives us a connection between $\pi_j(X)$ and the based loop space which you want to go compare to the free loop space, and that passage is usually not that obvious.

Theorem 9.2. *(—,Basu)*
The first one answers this question and the second calculates homology.

1. n_T *grows exponentially for* M_K^4 *where* M_k^4 *is a simply connected closed* 4*-manifold with* $H_2(M, \mathbb{Z}) \cong \mathbb{Z}^k$ *and* $k \geq 3$*.*
2. *There is an explicit formula for dimension of* $\pi_j(M_k^4) \otimes \mathbb{Q}$*.*

One might be tempted to say at this point, how do I obtain such a thing? One could write down the Sullivan minimal model but that is very hard to deal with.

Lemma 9.1. *(Well-known)*
Basically it says that if you have a four-manifold of this type, not necessarily smooth, you can assign to it a spin five-manifold, an S^1*-bundle* E_{k-1} *over* M_k*. This should satisfy*

1. E_{k-1} *is a* C^∞ *simply connected closed spin manifold.*
2. $H_2(E_{k-1}, \mathbb{Z}) = \mathbb{Z}^{k-1}$*.*

One way to try to construct circle bundles, corresponding to $H^2(\quad, \mathbb{Z})$ which are equivalences [missed some]

Let me assume that M is smooth, let me neglect the case where M is not proven. So these are the same as maps from M to $\mathbb{CP}^2$. How do you make this spin? If M is spin, then choose any α in $H^2(M, \mathbb{Z})$ which is part of an integral basis of $H^2(M, \mathbb{Z})$. This gives me a map. Look at the corresponding $f : M \to \mathbb{CP}^2$. Pull back from $\mathbb{CP}^2$ to M.

I'm essentially pulling back the classifying map to the larger space.

Then there exists a nonzero element in $\omega_2(M_k) \neq 0$. Make sure this is part of an integral basis so that the thing you get is rank one lower. Then choose it so $\alpha \mod 2$ is w_2. Then $E_{k-1} = f^*(S^5)$. The whole action, why is it spin? There are rather nice arguments, this is the second Stiefel-Whitney class. Then $TE_{k-1} = L \oplus \pi^*(TM_k)$. So this line bundle is a trivial bundle. We know that these are stable invariants. And by construction this is $\pi^*(\omega_2(M_k))$. It's 0 in both cases. Why is this useful? There are theorems of Smale that say what these things look like. The standard form is relevant for both these questions. So Smale has a classification for simply connected five-manifolds.

Theorem 9.3. *(Smale '62) Any simply connected closed spin* 5*-manifold is diffeomorphic to* $\#^\ell(S^2 \times S^3)$.

You read ℓ off of the rank of the homology group H_2. Take all spin manifolds, the E_{k-1} only depends on ℓ. So $H_2(M, Z)$ has dimension that determines $\pi_j(M^4)$.

Coming back to the two parts of the theorem, for $b_j(LM^4_*)$, this grows exponentially. if one has a handle on the Betti numbers of the manifold, you get a spectral sequence for the 5-manifold and it gives you expenential growth.

The idea is that $b_j(LE^5_{k-1})$ has exponential growth if and only if $b_j(LM^4_k)$ does. You can bypass the calculation or one can do this, but then you have to do something. You are pretty much done though by a result of Lambrechts, who says when such connect sums have exponential growth.

Theorem 9.4. *(Lambrechts 01) The quantity $b_j(L(M\#N)\}$ grows exponentially fast if neither of M and N is monogenic.*

You can apply this to the connect sum of spheres. You need k to be at least 2. [missed].

Okay, so that's a sketch of a proof. What about the last part? Some of you might hate formulas, some might like them, I at least like this one. The rank of $\pi^{\mathbb{Q}}_{n+1}(M^4_k)$ is

$$\sum_{d|n} (-1)^{n+\frac{n}{d}} \frac{\mu(d)}{d} \sum_{a+2b=n} (-1)^b \binom{a+b}{b} \frac{k^a}{a+b}.$$

The first few checks turned out to be right. It doesn't factor into irreducible factors. This is essentially the answer. As an easy corollary of this, it turns out that $\sum_{j=2}^{2n+1} \pi^{\mathbb{Q}}_j(M^4_k) \geq \sqrt{k-1}^{2n}$. So this shows the growth.

So how do you prove something like this? Well, the main tool would be to look at the based loop space and then try to calculate the rational homology of the based loop space. We know that $\pi_j(\Omega X)$ is $\pi_{j+1}X$. There is a famous theorem of Milnor and Moore that says that $H_*(\Omega X, \mathbb{Q})$ is the universal enveloping algebra of $\bigoplus \pi_j(X) \otimes \mathbb{Q}$. Here the Lie structure is the Whitehead product.

You compute using this the ranks of the homotopy groups which agree between the four and five manifolds, then use some Koszul duality to calculate this in another way and compare them to give the formula.

So define $p_V(t) = \sum dim(V_i)t^i$, then we can look at

$$p_{\wedge V}(t) = \frac{\prod_{odd}(1+t^i)^{dim\ V_i}}{\prod_{even}(1-t^i)^{dim\ V_i}}$$

and

$$p_{TV}(t) = \frac{1}{1-\sum dim(V_i)t^i}.$$

Why is that useful? It turns out that we know what the dimension ought to be using Milnor Moore. The generating function, the series for $H_*(\Omega E^5_{k-1}, \mathbb{Q})$ will be the generating series for $p_{\wedge\mathcal{L}}(t)$. This is the first part, the other part is that $H_*(\Omega E^5_{k-1}, \mathbb{Q})$ is isomorphic to $\frac{T(V)}{I}$ where V is an algebra generated by $x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}$, where you have x in degree 1 and y in degree 2. These are the generators of S^2 and S^3, shifted down. Then I is generated by $\sum[x_i, y_i]$, this is the five-cell. You need a little bit of duality of algebras. Calculating the generating series here is

$$p_{TV/I}(t) = \frac{1}{1-(k-1)t-(k-1)t^2+t^3}.$$

You solve for your dimesnions in terms of k and that's what you get. This is probably a good time to stop.

10. Thomas Kragh: A simple construction of the Fukaya Seidel Smith spectral sequence

So, I'll start with a little background. Let me fix some notation. So let N be a d-dimensional closed smooth (oriented spin but we can play around, start with $\mathbb{Z}/2\mathbb{Z}$ coefficients). The cotangent bundle T^*N has a canonical symplectic form $\omega = -d\lambda$ and on the disk bundle this is a Liouville structure. A Lagrangian is $j : L^d \subset T^*N$ such that $j^*\lambda$ is closed (or ω pulled back is zero). So $\alpha \in \Omega^1(N)$ we can look at its graph and that's a Lagrangian in T^*N if α is closed, exact if α is exact.

The nearby Lagrangian conjecture says that these are essentially all of them, any exact closed $L \subset T^*N$ is isotopic through exact Lagrangians to the zero section. That's the same as Hamiltonian isotopic. I won't dwell on that right now.

There are a lot of results on this. Gromov had a result early on, then Lalond and Sikorar, and later Viterbo proved something with spheres, but thene sort of, one of the biggest results is:

Theorem 10.1. *Fukaya-Seidel-Smith 2007; Nadler*
If N is simply connected and L is spin, and the Maslov index is zero, then $H_*(L) \cong H_*(N)$.

Their map is induced by the inclusion and then projection to N. They constructed a spectral sequence to prove this, it looks something like, on one axis, well there are some Lefschetz fibrations, you make thimbles and filter things in the Fukaya category and this gives a spectral sequence converging to the Floer homology of something or another. You have a generator for each thimble on one axis, some sort of Morse complex and then Floer homology of your Lagrangian, well, its tensor square.

In 2009, Abouzaid proved, took the ideas, gassed it up with local coefficients and a lot of stuff, and proved

Theorem 10.2. *If the Maslov index is zero, then $L \to N$ is a homotopy equivalence.*

Let me just finish this overview. I used another spectral sequence, motivated by this one, this looked like a Serre spectral sequence but it's not defined like this, combined with Viterbo's 1998 paper, I proved:

Theorem 10.3. *(Kragh, 2009)*
Up to a finite covering space lift of N, the map $L \to N$ is a homology equivalence. Combining with Abouzaid's result, this proves that $L \cong N$.

Before I explain this spectral sequence, I want to say a word about the Serre spectral sequence and Morse homology. If you have a fiber bundle $E \to B$ with fiber F and you want to talk about the Serre spectral sequence, one way of doing this is to pick a Morse function on B and now this has a bunch of critical points. If you plot E like a fat version, then you have entire fibers as critical manifolds. This is Morse-Bott. If you take the function on the base to be self-indexing, then if you have several critical points, you get disjoint unions of fibers as critical manifolds. So if you filter the chain complex by cutting off at halves, you recover the Serre spectral sequence.

What I'll try to explain for Floer homology, this is done on an infinite dimensional manifold and the function is the action integral.

So let L_0 and L_1 be exact Lagrangians in T^*N. Then define $FH_(L_0, L_1)$, well, I'll give two definitions, so here's definition one. Perturb L_1 to L_1' which is transversal to L_0. Now define the Floer chain complex as $\mathbb{Z}[L_0 \pitchfork L_1]$ with δ where the differential counts pseudo-holomorphic disks with corners in the intersection points.

So now this defines the Floer homology for the perturbation, and for some generic compatible almost-complex structure J on T^*N. This is to make sure that these are isolated, the rigid holomorphic disks.

Let me now instead give you the definition I prefer. I'll use both.

Definition 10.1. *This is version two. Define the path space $P(L_0, L_1)$ as the space of paths $I \to T^*N$ such that $\gamma(0)$ is in L_0 and $\gamma(1)$ is L_1. Then define $A(\gamma)$ to be $\int_\gamma \lambda + g_1(\gamma(1)) - g_0(\gamma(0))$. where g_i is a primitive for $\lambda|_{L_i}$, a map $L_i \to \mathbb{R}$.*

So I define Floer homology as the Morse homology of the action, but we'd perturb the action to make it Morse.

This is precisely what one does when one makes L_0 and L_1 transversal. The generic J gives nondegeneracy even on the path space.

If you look at the equation that these strips satisfy, and think of it as $I \times \mathbb{R} \to T^*N$, then the Cauchy-Riemann equation becomes the gradient equation. I prefer this picture.

So Floer saw that taking a small perturbation of L with itself, you can ook at $FH_*(L, L)$., this is isomorphic to $H_*(L)$ which is nothing like the loop space (well, it's actually the same homotopy type but let's pretend we don't know that).

So then let me also mention that however much you isotope L_0 and L_1 it doesn't change things. So this only depends on isotopy classes of these guys. So the plan is to isotope L away from itself. I should have said that the FSS spectral sequence converges to $FH_*(L, L)$ which is isomorphic to $H_*(L)$. So define L_t to be tL and define L'_t to be $tL + df$. If you look inside the cotangent bundle of M, you'll have the zero section and you'll be close to the zero section and very close to the graph of df. When df is zero, like a critical point, you get all your intersections.

This was also where FSS began their train of thought. But then they got into the Lefschetz thimbles instead, it has more structure. They were beginning to look at $FH(L, N)$, and then, well, that's a longer story, so let me just define something here that will help us with this spectral sequence. Let's look at one of these critical points for f. Then we'll talk about the bunch of intersection points lying around this critical point.

Let me argue that we can define a local Floer complex for L_t, L'_t, f, and q. So this will be free on intersection points near the critical point, with a differential. you don't want to include global critical points. You have one strip going there and one going back. That'll be bigger than some ϵ. By making t small enough, this is well-defined.

So this defines the local Floer homology, and this looks a lot like what Nancy was defining. That also happens in the Serre spectral sequence. That's why you get a copy of the fiber for each point in the base. The key point is to prove that this local homology thing is in fact a local system. There are all these groups, in the Serre spectral sequence, they're all fibers of the same fibration. We need to shaw that this is independent of q and f.

Lemma 10.1. *This is independent of q and f except for a shift from the Morse index of q.*

How do you prove this? A sketch is that, when you argue that this is isolated, you make some monotonicity argument, what goes on outside doesn't matter. you are still doing perturbative things. You get some sort of moving of negative directions. I don't think there will be. So now this is like a Serre spectral sequence.If we assume $\pi_1(N) = 0$ and the Maslov index zero, then you have a spectral sequence ith $LFH(L_t, f, q)$, and then nothing in the next vertical line and then a big lump. The Morse complex is along the bottom. You pick a function with no index one critical points. This works if the dimension is large enough (otherwise cross with S^6). You get two vertical lines of zeros near the beginning. You use some argument to show that certain parts have to be zero below a certain range and certain parts are zero above a certain range because this has to converge to the Floer homology which has nothing below a certain level. By Poincaré duality, this has consequences for homology above a certain level.

I should have said how this thing depends on the Morse index. If you do a birth of a critical point. Because this is local they have to cancel. So they're shifts of one another by one.

So if one wants to soo whath this means, you have $\mathbb{Z}$ then 0 then something then 0 then $\mathbb{Z}$. I have not proven that these differentials are the same as the ones they were for the Morse thing. This is already very close to a homology equivalence.

If you didn't know this, you'd actually know that, well, if the homology of N is free, then you have a Morse function with this number of critical points. For L the rank would be smaller but this can't be for a degree one map. You see putting it up against what you know about manifolds and degree one maps.

It looks like some parts of L'_t, you have a lot of intersection points. Look at their grading and the Euler characteristic. Since the parity of Maslov index follows these generators, this shows that χ is the square of the degree. I've not completely said that this is the same as FSS's fibers. If you look at one of these L, you have, well, it looks a lot like it. You have the same Euler characteristic, but it's just 1 and then you get ± 1.

What if $\pi_1(N)$ is not zero? How can one look then? In that case, one can go to the universal cover. If one has L inside T^*N, then going to the universal cover you get $\tilde{L} \subset T^*\tilde{N}$. Now we don't know that $\tilde{L}$ is connected. But still this local Floer homology up in the cover will be the same picture, you'll have the exact same picture. You have a projection. Everything works upstairs, things are periodic, but now there's one important thing. If you take a disk from one of these to another, when this is lifted to the universal cover, the same monotonicity argument says it cannot be lifted so it cannot close back at the same point. This says it's got to be the same as it was downstairs. Now the degree at least is finite. You have $\chi < \infty$ so that the degree is finite. This recovers the result about the projection being surjective, since the cokernel is finite.

Now can we get degree 1? If we look at T^*N and take $\bar{N} \to N$, the cover associated to the image of $\pi_1(L)$ inside $\pi_1(N)$, we look at this, because it's associated to the image, that means that L lifts to $T^*\bar{N}$. But this is $L \times \{1, \ldots p\}$, the associated cover there. If you assume this has degree k and this has degree p, then the lift has degree $\frac{k}{p}$. Taking the lift again you're back at degree k. Upstairs the Euler characteristic, well, you get k^2 is the Euler characteristic of the local Floer homology. So this is, well, if you got to the universal cover, this is still p copies of L. This means that the local Floer homology is in degree zero. We can use the same argument, so that the Euler characteristic is just going to be the number of components. It's just p. That means that $k^2 = p$. But the same formula downstairs tells you that $(k/p)^2 = 1$. Was that right? I confused myself a little bit but I think that's right. Now you're back to homology equivalence if you believe that the differential is the usual one, but very close if you're not.

You need to do something for general Maslov index. So there you need to look at product structures. The difficult part that I haven't been able to solve in this picture is the fact that this spectral sequence is going to be, is going to respect the Serre spectral sequence. You have an algebra structure and know something about grading. There you need a lot more structure and I don't have time to talk about it. So let me just stop.

11. September 3: Kenji Fukaya, Part II

Thank you very much. So in the first talk, after what I talked about in the first talk, it was supposed to be an introduction. Today I want to talk more about how we construct those maps. Today I want to do geometric [unintelligble] so I'll use the loop space and string topology. So $\hat{q}$ will be a map we can cook up from a chain model S_*X of X to the Hochschild complex $CH(S_*L, S_*L)$. Now S_*L is a Lie algebra with trivial bracket. The Hochschild complex is also a Lie algebra with the Gerstenhaber bracket. I actually want to replace this with $S_*(\Lambda L)$ where this is a Lie algebra with the loop bracket.

The statement that there is an L_∞ morphism is rigorously proved for the Hochschild case but not when it's replaced with $S_*(\Lambda L)$. I'd like to prove this one but there is still some difficulty. One difficulty is constructing a chain level thing on the loop space. There are even more difficulties. There is a map to de Rham homology. There is a map $S_*(\Lambda L) \to CH^*(\Omega(L), \Omega(L))$ and this is easy to see, that this intertwines the loop bracket with the other bracket. This composition is constructed, but not the map to $S_*\Lambda L$.)

Let A_1 and A_2 be chains of ΛL, then let's look at the bracket, find (a_1, a_2, t) in $A_1 \times A_2 \times S^1$ such that $\ell_{a_1}(0) = \ell_{a_2}(t)$ and define a map $A_1 * A_2$ to ΛL by going around a_2 to the basepoint of a_1, then going around a_1, then finishing with a_2. The bracket is the commutator of this.

So $\hat{q} : S_*(X) \to S_*(\Lambda L) \otimes \Lambda_0$, where $\Lambda_0 = \sum a_i T_i^\lambda$

So $\hat{q}_\ell : E_\ell S_*(X) \to S_*(\Lambda L) \otimes \Lambda_0$. So $E_\ell C = C^{\otimes \ell} / \sim$ where $a_1 \cdots a_\ell \sim \pm a_{\sigma(1)} \otimes \cdots a_{\sigma(\ell)}$, so this is symmetric. You have a family of such maps, and the condition that this is an L_∞ homomorphism is the following. First of all, take this total sum $\bigoplus E_\ell C = EC$. Then $\Delta : EC \to EC \otimes EC$ is the unshuffle:

$$x_1 \otimes \cdots \otimes x_n \mapsto \sum_{I \sqcup J = \{1 \ldots n\}} x_I \otimes X_J$$

If we have $\oplus q : EC \to C'$ then we can lift this to a coalgebra morphism $EC \to EC'$ uniquely so that $\pi_1 \hat{q} = q$. So if C is a Lie algebra, then you get a differential on EC, where $\hat{d}(x_1 \otimes \cdots \otimes x_n) = \sum \pm \{x_i, x_j\} x_1 \otimes \cdots \hat{x_i} \cdots \hat{x_j} \cdots k_n$ Now $\hat{q}$ is an L_∞ morphism if and only if $\hat{q}$ is a chain map.

So we want to cook up $q_\ell : E_\ell(S_*(X)) \to S_*(\Lambda L) \otimes \Lambda_0$ so that q_ℓ becomes an L_∞ homomorphism.

The construction is by using pseudoholomorphic curves. So $\int \mathcal{M}_{1,\ell}(\beta)$ is the set of $(u, z_1, \ldots z_\ell)$ where $u : D^2, \partial \to X, L$ is holomorphic and has homology class β, and z_i are disjoint points in the interior of D^2 modulo G where G is the space of $D^2 \to D^2$ biholomorphic and preserving 1. We have an evaluation map $\mathcal{M}_{1,\ell}(\beta) \to \Lambda L$ taking u to $u|\partial D^2$. Once you fix the S^1 part you get a contractible choice. We want to fix the parameterizations in a consistent way. There are another natural map $\mathcal{M}_{1,\ell}(\beta) \to X^\ell$. Here we take this cycle Q_i, actually to make everything rigorous we need to use a different model than the singular model, but let me stay in the singular model. So $(Q_1 \times \cdots \times Q_\ell) \times_{X^\ell} \mathcal{M}_{1,\ell}(\beta) = \mathcal{M}_{1,\ell}(\beta, Q)$. So then $\hat{q}(Q) = \sum_\beta ev^\partial \mathcal{M}_{1,\ell}(Q) T^{\omega \cap \beta}$.

The hard thing is transversality. After we are working on this for a couple of years, my feeling is that the easiest way is to use de Rham theory. You have to use infinite dimensional forms. But we can use iterated integrals to work out the de Rham theory.

Okay, so this is the map, I want to explain why this is an L_∞ homomorphism. The main formula we need is how to understand the boundary. What is the boundary of $M_{1,\ell}(\beta)$? You know something about Floer theory, you know that there were bubbling phenomena, sphere bubbling and disk bubbling. The former is codimension two so you only need to worry about disk bubbling for the boundary. [Picture].

So you see that this picture looks like a string bracket, splitting things into two pieces. Suppose you take the comultiplications of this vector, then what we know is that the boundary of $\hat{q}\beta, Q$ is

$$\sum_{\beta_1 + \beta_2 = \beta, c} \{\hat{q}(\beta_1, Q_{c,1}), \hat{q}(\beta_2, Q_{c,2})\}$$

where this is the string bracket. You can see this in the picture.

If you see this formula and remember the definition, this exactly means that $\hat{q}$ is an L_∞ morphism. To make it work transversally is rather cumbersome.

What's good about this, I need to mention one point, when I wrote EC, I consciously do not write, well, usually you use E_+C and start from $\ell = 1$ but maybe you want to start from $\ell = 0$. In a similar way to the A_∞ case you can define a curved L_∞ homomorphism. It wakes sense that this is a sense, but the curved corresponds to E_0. So you have $q_\ell : E_\ell C \to C$ so that $\hat{q}E_+C \to E_+C'$ is a coalgebra homomorphism and $\hat{q}\hat{d} = \hat{d}\hat{q}$. If you have the same thing for EC this is a curved L_∞ homomorphism.

Now you have $q_0 : \Lambda_0 \to C'$, and $\hat{d}\hat{q} = \hat{q}\hat{d}$ implies that $\partial q_0(1) + \frac{1}{2}\{q_0(1), q_0(1)\} = 0$. So $q_0(1)$ satisfies the Maurer-Cartan equation. So then q_ℓ is curved means you can use this $q_0(1)$ to perturb C' to an L_∞ algebra, then $q_1, \ldots, q_\ell$ is a usual L_∞ homomorphism.

In our situation, our q_0 is nontrivial, it is $\Lambda_0 \to S_*\Lambda L \otimes \Lambda_0$. so $q_0(1)$ is in this loop space homology and satisfies Maurer Cartan.

This deformation theory relating this is the following thing. We have something like a sum of $Hom(\Omega L^{\otimes k}, \Omega L)$. This is a Lie algebra homomorphism. This is by definition the Hochschild cochain complex. The Maurer-Cartan element in $S_*\Lambda L$ gives a Maurer-Cartan element of $CH(\Omega L, \Omega L)$. Then m_k is $f_k : \Omega L^{\otimes k} \to \Omega L$ for $k > 2$, is $f_2 + \pm\wedge : \Omega L^{\otimes 2} \to \Omega L$ for $n = 2$ and f_1 is $\pm d$. You can see that m_k is an A_∞ structure. This m_k is Lagrangian Floer theory. So then since we have this q, we have slightly more. We start from $S_*(X)$ and then $S_*(\Lambda L) \otimes \Lambda_0$. So we get transformation of Maurer Cartan elements, so Maurer-Cartan elements on the singular complex of X give deformations for this A_∞ algebra. Since the bracket is trivial, the Maurer-Cartan equation is easy, the solutions are $\partial q = 0$. Any homology class deforms the de Rham complex of L to an A_∞ algbera (curved). This is the story of bulk deformations.

Let me just say a few words about what happens in the non-compact case. If you replace singular homology with symplectic homology, you can put Hamiltonian perturbations near the boundary, and this was dealt with by some people and you can get an L_∞ homomorphism from symplectic homology to loop space and Hocschild homology. What's different in this case, in the compact case the Lie algebra on the ambient homology is trivial. In symplectic homology you have non-trivial bracket so that the Maurer-Cartan equation is non-trivial. Then the elements you have go to the Maurer-Cartan elements and deformations on the Hochschild side.

12. Nancy Hingston, Part II

I refer you to a very nice paper by Alex Oancea, the title is Morse theory, closed geodesics, and something about the free loop space. Here are some nice results.

(1) There exists at least one closed geodesic on any sphere or projective space. I'll outline a proof.

Definition 12.1. *A* circle *on S^n is the intersection of S^n with a 2-plane, not necessarily through the origin.*

This also includes the point loops of the sphere.

We'll also be interested in the set of all circles beginning at $*$ with the tangent vector v, a fixed v. This is the cycle A, it has dimension $n-1$. If you take $(A,*)$, that represents a nontrivial element in $H_*(\Lambda, \Lambda_0)$ the free loops modulo the constant loops. If you change the metric on the sphere, you still get this thing. You can't shrink these all, so for any metric, the critical level of this homology class is positive, so there's a closed geodesic. There exists a closed geodesic γ with the length of γ the critical value of this homology class A.

(2) There exist at least three simple closed geodesics for any metric on S^2. There's a very nice story, but the basic idea is that you look at the three parameter family of circles, if you look at the space of circles, there have to be at least three homology classes and, well, there's a beautiful idea behind this and it's often true in this field, it takes a while to work out the details. It originally appeared in 1929 and there were people working on it over the years and in 1989 we had a complete proof, by Grayson, who finished this up. He went to Wall Street since.
(3) There exist infinitely many closed geodesics for any metric on S^2. This is a long story. It started off with Birkhoff. We need at least on simple one. There's a part by Bangert, by Franks, for finishing this up, and with my method you get a growth rate. $\liminf n_T \frac{\log T}{T} > 0$. That's the best so far.
(4) There are infinitely many closed geodesics for a generic metric on S^n (also projective space). Some names are [missed]-Takens, Hingston, Rademacher (who had a better proof than mine)
(5) There are Finsler metrics with finitely many closed geodesics. These are due to [unintelligible]
(6) There exist at least two closed geodesics on any Finsler S^2. This is due to Bangert and Lang using my results. This looks like a miserable number but it's sharp by the previous examples.
(7) It's also known that there are greater than or equal to 1 close to the standard metric. If you look at the topology of the Morse Bott, close to the standard metric you will not go far. There are various results like this.

So Dennis has said, there should be an algebraic topology proof that there are infinitely many closed geodesics on the sphere. This led to this question of mine. You've got lots of homology classes, but you can't distinguish between closed geodesics and their iterates. So a question is, is there an algebraic operation on $H_*(\Lambda M)$ which corresponds to iteration?

The answer is sometimes. In some critical cases, yes. This brings us to products on loop spaces. We'll start with the Pontryagin product. This is a product from $H_j(\Omega) \times H_k(\Omega) \to H_{j+k}\Omega$. Assume you have two "cycles" which I'll pretend as subsets of the loop space. These are sets of loops with base point at the basepoint of the manifold. Take the homology class of $[X] \times [Y]$ to be $[\{\alpha \cdot \beta\}]$ for α in X and β in Y. So let me do some examples on S^n. A lot of you have seen this. But let me do some examples.

I already did the class A. You do all the circles going through A with direction v. You can also take U, the constant loop. Let's compute some products. What is $[A] * [U]$. It's $[A]$ again. So $[U]$ is the identity element in the Pontryagin ring. So, what's $[A] * [A]$? It has degree $2(n-1)$. It's not zero, and in fact it's a generator of the Pontryagin ring. The Pontryagin ring is $\mathbb{Z}[A]$, so $[A]^n \neq 0$, so A is non-nilpotent.

What about the critical levels? For the standard metric the critical level for U is 0. The critical level for A is 2π. What's the critical level of $[A * A]$? It's 2π. Here's an exercise, $[A * A] \sim [B]$, the cycle of all circles beginning at $*$. In fact, it's true that the critical level of $[A]^{2m-1}$ is the critical level of $[A]^{2m}$ is 2π. So we say $[A]$ is level-nilpotent. But its level-nilpotent because the level drops. This is the statement that $Cr([A]^N) < NCr([A])$ for some N.

The next product I want to talk about is the Chas-Sullivan product. The Chas-Sullivan product is on the homology of the free loops. Again, assume we're given cycles, so $X = \{a\}$ and $Y = \{b\}$. Now we're in the free loops. A cycle will just be a set of points or will be a submanifold. We're going to assume that the base points of X are transverse to the base points of Y. Then $[X] *_{CS} [Y] = [\{\alpha * \beta | \alpha \in X, \beta \in Y, \alpha(0) = \beta(0)\}]$.

We take α then β, the same product, but we do this when they have the same base point. If you take a j-dimensional class and multiply it by a k-dimensional class, you lose n, the dimension of M.

Let's add some more cycles. There's this beautiful cycle C which is the set of all circles. This has dimension $(3n-2)$. We also have E, the set of all trivial loops on the sphere. What is $[C] *_{CS} [E]$, it will be another copy of $[C]$. So $[E]$ is the unit in this ring.

What about $[A] *_{CS} [A]$? This is 0. If you take anything from the based loops and take a Chas-Sullivan product, you get 0 because you pull them apart and you're parameterizing the empty set.

One more computationg. What is $[C] *_{CS} [U]$? It's $[B]$. It is a fact that $[C]^m$ is nonzero. What does $[C]^m$ look like? A composition of m circles at that basepoint. It's non-nilpotent and also level-non-nilpotent. In other words, the critical level of $[C]^m$ is $m2\pi$. It's true that multiplication with $[C]$ is injective on homology. It's true that the homology ring, $H_*(\Lambda, *_{CS})$ is finitely generated. The Chas Sullivan ring was computed for spheres and projective spaces by Cohen, Jones, and Yan, in 2002, and they used the spectral sequence of the fibration of ΛM over M with fiber ΩM.

The transversality condition in the definition is hard to deal with. One definition is due to Cohen and Jones. This is the idea. In the simple, beautiful examples, the guy C is automatically transverse to everything. So the computations are easy to do.

There's a Morse-Bott-Samuelson connection. If you look at A, B, C, E, U, with their products, Morse drew all these pictures. He would have recognized all these pictures. Especially powers of C, you can see them in Bott and Samelson on manifolds all of whose geodesics are closed. I first encountered the Chas-Sullivan product in 2005. I was sitting in the lunch room at the IAS and I heard someone give the definition and I said “I know that product, Morse knew that product” and then I said “Where's the other product?”

There's this recurrent symmetry, everything I've ever seen using closed geodesics looks like Poincaré duality. For a finite dimensional manifold this asserts an isomorphism between $H_K(X)$ and $H^{N-K}(X)$. You can't have something like this for an infinite dimensional manifold, but if you think of turning the free loop space upside down, the homology is generated by Morse relative chains. These are the things that go into the spectral sequence. The cohomology is generated by infinite dimensional upward facing chains. In a finite dimensional manifold, you can reverse a Morse function and everything is flipped around. The function f should go to $-f$ and the index to the coindex and you get this seemingly very non-trivial correspondence. I've never known it to fail. I'll give some examples of this next time and judge for yourself whether you believe in this principle or not.

13. Richard Hepworth, Part II

Remember I was asking about the algebraic structure of $H_*(BG)$ and $H_*(\Lambda BG)$. The answer is that they are part of a homological h-graph field theory. The first aim is to ask what these are. The second aim is to ask what *this* homological h-graph field theory is, and finally to give some nonzero higher operations.

So we had some examples, some coming from graphs, these all go from points to points. We have more traditional things coming from surfaces with boundaries. We have the rest, the strange and wild remainder. We have genus one from a point to a point. We have a lasso from a point to a circle, from a circle to a point, and there's one from two circles to one by identifying the outputs of two cylinders.

Okay, so if I take homotopy equivalences that fix the input and output pointwise, if it's a surface, it's $Diff$ of the surface. We did the interval with a loop, which is $\mathbb{Z} \ltimes \{\pm 1\}$. The cylinder has $\mathbb{Z}$. You can get $A^s_{n,k}$ studied by Hatcher and Wahl by looking at appropriate diagrams. That was last time. So moving on, homological h-graph field theories.

Definition 13.1. *An HHGFT ϕ consists of*

(1) *A symmetric monoidal functor ϕ_* : $h-$ graphs, homotopy equivalences $\to$ $grVect$. The symmetric monoidal structure on the left is disjoint union. On the right it's graded vector spaces. The second part of the data is*
(2) *for each h-graph cobordism $S : X \not\to Y$, a map $\phi(S)$ from $H_*(BhAut(S)) \otimes \phi_*(X) \to \phi_*(Y)$.*

This should be compatible with two-cells, composition, disjoint union, and identity. I won't write these things down, I heard someone else say this, I'm sorry and you're welcome. It'd take a board, you'd be bored, let's not do it.

In our case, $\phi_*(X)$ will be $H_*(BG^X)$. What does the second thing do for us? It's a rich algebraic structure on these vector spaces. Another point of view is that what this does is allow you to study $H_*(BhAut(S))$. These groups contain many interesting things. They contain diffeomorphisms of surfaces, holomorphs of free groups. Those things are largely mysterious, and this lets you study. So if you had a homology class and wanted to study it, you could ask whether it gives a non-zero operation.

Definition 13.2. *The degree 0 operation associated to $S : X \to Y$, called $\phi_S : \phi_*(X) \to \phi_*(Y)$ is defined as $\phi_S(a) = \phi(S)(1 \otimes a)$ where 1 is the standard generator of $H_0(BhAut(S))$.*

Let me add something that I forgot to say. I said that ϕ_* is a symmetric monoidal functor. Let me add that this gives vector spaces $\phi_*(X)$ for all X and isomorphisms between $\phi_*(X) \otimes \phi_*(Y)$ to $\phi_*(X \sqcup Y)$.

Example 13.1. *The value of ϕ_* on a point is a commutative non-unital Frobenius algebra. It has a product, $\phi_*(pt) \otimes \phi_*(pt) \cong \phi_*(pt \sqcup pt) \overset{\phi_m}{\to} \phi_*(pt)$. Similarly, the coproduct is*

$$\phi_*(pt) \overset{\phi_\psi}{\to} \phi_*(pt \sqcup pt) \cong \phi_*(pt) \otimes \phi_*(pt).$$

I also have

$$\phi_*(pt) \xrightarrow{\phi_\varsigma} \phi_*(\emptyset) \cong \mathbb{F}.$$

Definition 13.3. *Let $S : X \not\to Y$ and $\sigma \in H_i(BhAut(S))$ with $i > 0$. The associated higher operation is $\phi_*(X) \to \phi_{*+i}(Y)$ defined by $a \mapsto \phi(S)(\sigma \otimes a)$.*

For example, the BV operator Δ is the higher operation associated to the generator of $H_1(BhAut(\text{cylinder}))$. Now let's get to string topology of BG.

Theorem 13.1. *(H-Lahtinen)*
There is an HHGFT ϕ with $\phi_*(X) = H_*(BG^X)$. For example $\phi_*(pt) = H_*(BG)$ and $\phi_*(S^1) = H_*(\Lambda BG)$.

I'll give you an idea of how to construct the operations. Then I'll say how to compute what's going on here.

Let me give a sketch construction of $\phi(S) : H_*(BhAut(S)) \otimes H_*(BG^X) \to H_*(BG^Y)$. The zigzag $X \to S \leftarrow Y$ gives $BG^X \leftarrow BG^S \to BG^Y$. This has a parameterized version. The homotopy automorphisms of S act on all three spaces. In the middle they do something nontrivial. I can attempt to do a Borel construction and try to get a zigzag lying over $BhAut(S)$. Since it acts trivially on S, I get $BhAut(S) \times BG^X \xleftarrow{\alpha} BhAut(S) \times_{tw} BG^S \xrightarrow{\beta} BhAut(S) \times BG^Y$. The $\times_{tw}$ doesn't correspond to a definition, it's the name of a thing we're doing. The homotopy fibers of α are finite since the group is finite. Then there is a transfer map $\alpha^* : H_*(BhAut(S) \times BG^X) \to H_*(BhAut(S) \times_{tw} BG^S)$.

So $\phi(S)$ is $H_*(BhAut(S)) \times H_*(BG^X) \to H_*(BhAut(S) \times BG^X) \to H_*(BhAut(S) \times BG^Y) \to H_*(BG^Y)$. That after stepping back is the rough construction of the string topology operations.

Tomorrow I'll make an extremely explicit recipe for doing this, saying what these things are explicitly. To do that I'll use a little bit of language or notation about homotopy quotients.

There's a big theory of homotopy quotients and I'll tell you a tiny little bit of it.

Definition 13.4. *Let G be a discrete group and X a G-set. The* homotopy quotient *is called $X//G$ and is the Borel construction $EG \times X//G$ where G acts diagonally.*

This construction is functorial in the pair X, G, where a map of such pairs is a map of groups and an equivariant map of sets. All these spaces BG^X will be homotopy quotients.

Here are some examples.

Example 13.2. *For X a point, we have $pt//G = BG$. A different example is, basic G-sets are cosets of H, (G/H), the ideal orbit with stabilizer H. The quotient there is $(G/H)//G = pt//H = BH$. The homotopy equivalence comes from $(pt, H) \mapsto (G/H, G)$. If I start with an orbit, I get B of the stabilizer of the orbit.*

In general, if X is arbitrary, then I can decompose it into the disjoint union of its orbits $\sqcup Gx$ and the disjoint union runs over representatives of the orbits. The orbit stabilizer theorem tells me that Gx is isomorphic to G/G_x where G_x is the stabilizer. Then $X//G = \sqcup Gx//G \cong \sqcup(G/G_x)//G \cong \sqcup BG_x$. For a given choice of X I told you how to represent this as a disjoint union of classifying spaces.

I think I'm going to overrun. I'll just stop now.

14. Denis Auroux: Monotone Lagrangian Tori

Theorem 14.1. *There exist infinitely many different families of monotone Lagrangian tori in* $\mathbb{R}^6$.

The following is not due to me but to Vianna and also Galkin-Mikhalkin,

Theorem 14.2. *The same is true in* $\mathbb{CP}^2$.

Why is this new? What is a monotone Lagrangian?

A symplectic manifold is a manifold of dimension $2n$ with a closed nondegenerate 2-form. A Lagrangian is a middle dimensional submanifold on which the form vanishes. In the flexible world of symplectic geometry, the Lagrangian is one of the few irons to probe the symplectic with some kind of rigidity.

My focus for now will be on $\mathbb{R}^{2n}$ with the standard symplectic form. What can I say about compact Lagrangian submanifolds. It's a classical result of Gromov that necessarily L must bound a holomorphic disk of positive area. For example, this implies that $\pi_1(L)$ cannot be 0. You can ask for other conditions, what kind of topology is possible in $\mathbb{R}^{2n}$?

The simplest examples are tori. I can build a torus $S^1(r_1) \times \cdots S^1(r_n)$. Any simple closed curve in $\mathbb{R}^2$ is a Lagrangian submanifold. Isotopies by which the area is preserved can make this into a round circle of whatever appropriate radius. So the theorems should be understood to mean via Hamiltonian isotopies and rescaling.

How can we come up with more interesting examples? There are things like connect sum and interesting things about what manifolds can be embedded in $\mathbb{R}^{2n}$. I'll ask how many ways I can fit Lagrangian tori in a symplectic vector space. That's an interesting question, but it will be algebraic topology, very disappointing, if you allow all isotopies. In dimensions eight and above, there exist examples of topologically knotted Lagrangian tori. In lower dimension no examples are known. So the question is whether we can have exotic phenomena, things that are smoothly isotopic but cannot be moved to each other that preserves areas of disks.

I want to impose the condition of a monotone Lagrangian torus. So what is monotone. If I have a disk β in $\pi_2(M, L)$, then there are two quantities I can associate to that. One is the symplectic area $\int_\beta \omega$. It doesn't matter where I am by Stokes' theorem and the condition. The other quantity is the Maslov index $\mu(\beta)$ which is an even integer. If I have a map $u(D^2, \partial) \to (M, L)$, I have $u^*TM \to D^2$. Over the boundary of the disk I have a loop of Lagrangian subspaces in $\mathbb{R}^{2n}$, looking at u^*TL. Now $\pi_1 Gr_{Lag}(\mathbb{R}^{2n}) = \mathbb{Z}$. One way to think of this is that this is $U(n)/O(n)$. This is some sort of rotation number. You ask how many times does my line rotate and become nontransverse to a given direction. For orientation reasons this will always be even. The standard disk I draw has Maslov index 2. The Maslov index is defined by u^*TL.

What's the condition of being monotone? It means that these two linear maps are, well, the symplectic area is positively proportional to Maslov index.

This is a condition that comes up in various places. One place it comes up is Floer homology. It helps making things better behaved. One reason this is interesting is that classification of monotone Lagrangians turns out to be much more interesting than non-monotone Lagrangians. I can look at the real torus in $\mathbb{CP}^n$. These will be Lagrangian. Only one is monotone, which has equal sizes in every direction. If I return to a product of circles in $\mathbb{C}^n$, which ones are monotone? Disks in one factor, what is its area, it's πr^2, what is the Maslov index? 2. So I need all the radii to be the same, that's necessary and sufficient.

Besides this symmetric condition, another thing is that monotone Lagrangians are often not displaceable. In $\mathbb{CP}^n$ that seems to be the case.

We have one family of monotone tori, and that was it for a very long time. Then Chekanov found another example, different in the sense that it is not isotopic through monotone Lagrangians to a product torus, in $\mathbb{R}^4$.

It's actually an explicit construction, let me explain it in one of its many incarnations. This might be the only other monotone torus in $\mathbb{R}^4$, I don't have any evidence except that people have tried and failed.

I can map $\mathbb{C}^2$ to $\mathbb{C}$ by taking $(x, y) \mapsto xy$. So what does this map look like? It will look over most points like xy is a constant. When $xy = 0$, it looks like two complex planes joined at a point. I want to draw, in each of these fibers is a preferred circle where $|x| = |y|$. There's an S^1 action which rotates x in one direction and y in the other.

Take your favorite simple closed closed curve downstairs in $\mathbb{C}$, not enclosing the origin. Take all the equatorial circles in the fibers. This is Lagrangian. A small calculation shows that it is Chekanov torus. You can always rescale these. Why is this different from the usual ones? How do you see the usual ones in this picture? What are the products of the two circles in this picture? They are circles including the origin. The ones outside are isotopic via non-monotone Lagrangians to the ones outside, so there's more to monotonicity than meets the eye.

The invariant I want to use is a count of mimimal area holomorphic disks bounded by L. I mean the smallest area holomorphic disk or the smallest positive Maslov index. That's the same as Maslov index 2 disks. Why is this an invariant and what does it mean precisely?

I'll be in $\mathbb{C}^n$ for simplicity. This satisfies the holomorphic equation $\bar{\partial} u = 0$. If you're a symplectic geometer, you can give an almost complex structure but you want your differential to be complex linear with respect to your almost complex structure. You also want to have $[u_* D^2] = \beta$ for some fixed β in π_2. We'll quotient by something and I'll call this $\mathcal{M}_1(L, \beta)$. If these were disks, I'd allow holomorphic reparameterizations. I'll only allow those which preserve a marked point at the boundary of the disk, say $+1$.

If transversality holds this will be a manifold of dimension $n - 2 + \mu(\beta)$ so for $\mu(\beta) = 2$, this has dimension n. Then monotonicity ensures that $\mathcal{M}_{0,1}(L, \beta)$ is a closed manifold which has no boundary because there can be no bubbling. I'm looking at the smallest possible area so I can't split my area into two pieces. That's the technical reason that monotone is nice.

If L is oriented and spin, then this is also oriented. The other thing I should tell you is about a natural evaluation map. Then I can look at $deg(ev : \mathcal{M}_{0,1}(L, \beta) \to L)$ which is $n_\beta(L) \in \mathbb{Z}$. If I don't have transversality, perturb a bit. In practice the standard complex structure is always regular but in general you can make an argument that this is invariant of perturbation.

Okay, so n_β, $\beta \in \pi_2$ such that $\mu(\beta) = 0$ are invariant under monotone isotopies.

Okay in $\mathbb{R}^4$, I have standard product torus, there are two classes in $S^1 \times S^1$, either $D^2 \times pt$ or $pt \times D^2$.

If you take the Chekanov torus you only have one class with $n_\beta = 1$.

For $\mathbb{R}^4$ this is all we know at this point.

Let me jump ahead to $\mathbb{R}^6$. You can do $S^1 \times$ Chekanov, which gives you two families. You can do products and you have three families. You can look at something similar to Chekanov, (x, y, z) and look at the map to xyz and this gives a family.

There are also monotone twist tori from Chekanov-Schlenk. These tori seem to bound fewer families of disks. In fact, the claim is that in $\mathbb{R}^6$ you can build infinitely many.

How am I going to do it? I won't construct it for you in $\mathbb{R}^6$? I'll do it in a different manifold that's the same, $X = \{(x, y, z, w) \in \mathbb{C}^4$ such that $xy = h(z, w) = 10z^n + \frac{1}{10}w - 1\}$ where here 10 is your favorite number bigger than one (maybe bigger than three or so). This has the restriction of ω_0 from $\mathbb{C}^4$. These are symplectic demormation equivalent. In particular, arbitrarily large bounded subsets are symplectomorphic. Why are they the same? You can solve for w. You can think of this as $\mathbb{C}^3$, complex diffeomorphic anyways.

How do I build a torus in there? I use a projection map in there, project to $\mathbb{C}^2$ in the z and w coordinates. What's going on above a general point? I have xy equal a number, so I get a cylinder. I get some branch curve, my fiber looks like this singular surface. I have again a family of cylinders. What is the claim? If I take a monotone Lagrangian torus, I can lift it and get again a Lagrangian torus. If it were monotone in the reduced space that upstairs one will be monotone. Simple constructions like this did not exist, this is just an indication of how immature symplectic topology is. I'll draw T_{red} some torus. My numbers were chosen, my reduced torus is (isomorphic to) the product of unit circles in $\mathbb{R}^4$. It's a monotone Lagrangian torus. Then my favorite torus will be (x, y, z, w) so that $|x| = |y|$ and $(z, w) \in T_{red}$. I'm lifting the torus downstairs back to the space by taking the equatorial loop in the fibers. [Something very quick about why you want to use an isomorphism for T_{red}.]

Proposition 14.1. *T is a monotone Lagrangian torus which bounds holomorphic disks of $\mu = 2$ in $n + 2$ different classes and the total count is $2^n + 1$.*

In particular, if you do this for many n, then you get different classes.

Remark 14.1. *The case $n = 0$ is $S^1 \times$ Chekanov. The case $n = 1$ is likely the product torus.*

The projection of a holomorphic disk is again a holomorphic disk. We know that in $\mathbb{R}^4$ you bound two kinds of holomorphic disks. How many ways can you lift back to X? Every time you pass a singular fiber you get a choice, independently in each point. The other one doesn't interstect the curve at all.

So that's all I have to say about $\mathbb{R}^6$. Let's talk quickly about $\mathbb{CP}^2$. So in $\mathbb{CP}^2$ the classical monotone torus is the Clifford torus but you have to choose the size right. What do I want to say? The Clifford torus is the set of points $(x : y : z)$ so that $|x| = |y| = |z|$. It bounds three families of disks, disk times point, point times disk, and a third one from the other chart.

There is a Chekanov torus as well. In $\mathbb{R}^4$, this bounds *fewer* but here you bound *more*, here actually five. So my student Vianna a couple of years ago constructed one that bounds 10 families with the total number of disks 41. Where do these come from? There are toric degenerations to weighted projective spaces $\mathbb{CP}(a^2, b^2, c^2)$ for all a, b, and c such that $a^2 + b^2 + c^2 = 3abc$, called Markov triples. What is the structure of Markov triples? They're related by mutations. Replace one with, well $(a, b, c) \mapsto (a, b, 3ab - c)$. Inside this you have an orbit which is monotone, all you need to do is deform it back to get a monotone thing in $\mathbb{CP}^2$. This will very quickly get more complicated. The Clifford torus is in $(1, 1, 1)$. The next one is $(1, 1, 4)$. The next one is $(1, 4, 25)$. These come in infinite families.

It's not unreasonable to conjecture that in $\mathbb{CP}^2$ this is the complete list. I think I should conclude now.

15. Mohammed Abouzaid: Untitled

I see that there wasn't a background on Floer theory. You can interrupt if you want. The setup I care about is that I have smooth closed manifolds that I'll call Q and I want to look at its cotangent bundle. This has moment coordinates q and p and I'll write the Liouville form $\lambda = pdq = \sum p_i dq_i$ and this is a symplectic manifold with $\omega = d\lambda = \sum dp_i \wedge dq_i$.

What additional ingredient do we need to define Floer homology? It's $\mathcal{A}_H : \mathcal{L}T^*Q \to \mathbb{R}$ given by

$$\mathcal{A}_H(x) \int x^*\lambda - \int_{S^1} H \circ x dt$$

Floer noticed you can do Morse theory here. Critical points are time -1 Hamiltonian orbits. Associate to $H : T^*Q \to \mathbb{R}$ the form dH, and then you can dualize with ω to get a Hamiltonian vector field X_H, so x is a map $S^1 \to T^*Q$ with $\frac{dx}{dt} = X_H$.

So you can make H time dependent and then you add t to all your H. This is the setup for most definitions of Floer homology.

We should take gradient flow lines between critical points. I can just see what those loops trace out. So gradient flow lines in the loop space of the cotangent bundle correspond to cylinders in T^*Q with certain asymptotic conditions. The asymptotic conditions are given by these Hamiltonian orbits. In fact, the gradient flow equation corresponds to the holomorphic curve equation, meaning that writing our cylinders as $S^1 \times \mathbb{R}$, saying this is (t, s), then we pick an almost complex structure J on T^*Q and require that $u : S^1 \times \mathbb{R} \to T^*Q$ satisfy the equation $\frac{\partial u}{\partial t} = J\frac{\partial u}{\partial s} - X_H$. The point is that, if I had no X_H, this would intertwine little j on the cylinder with big J, but the Hamiltonian twists this. The main warning is that in general this is not well defined as a "Morse homology."

[Picture example].

Here these coordinates are zero by inspection, but you could do the following stupid thing, remove a small neighborhood of one of these and it's still a Morse function, it's an open manifold but d^2 is no longer zero. If you think about the geometry, the geometry, in the proof that $d^2 = 0$ you look at one dimensional trajectories, you are looking at things that, you prove the boundary are these specific situations that appear algebraically. Here there is some other thing that appears in the boundary of the moduli space, which is a curve that, well, you have to say something about compactness in order to be able to say something about manifolds with boundary.

So pick a Riemannian metric, require that $H = b|p|$ is not the length of any closed geodesic outside a compact set.

The symplectic form gives a diffeomorphism between the tangent bundle of Q and the cotangent bundle of Q. The Riemannian metric gives us a geodesic flow. On the other hand I have the flow of the, the Hamiltonian flow of norm $|p|^2$. This diffeomorphism intertwines these two flows. The presence of a geodesic of length b

gives you a time one flow line of norm p^2. By using linear growth with this slope, I get exactly the geodesics of length b, but I get infinitely many copies of them.

In the linear case I use rescaled geodesic flow.

[Picture].

In particular, if there were a geodesic of length b, you get infinitely many periodic orbits of the flow X_H. They're sitting on top of each other.

Okay. So any questions? Whenever I say geodesics, I mean closed geodesics. Is that okay? Fine. Whenever you encounter this, I think it's relatively natural to expect that the Floer homology of this Hamiltonian which is something we can write down, we can define to avoid the bad phenomena Viterbo expected, the Floer homology of such a Hamiltonian, $HF(T^*Q, H)$, where H has this "slope" b, is isomorphic to $H_*\mathcal{L}^bQ$, where this is the subset of the loop space consisting of loops whose length is less that b.

This is not a critical point of the length functional so there's no bifurcation crossing b. Maybe my lectures will be boring becaus what I'm focusing on is the fact that this is true but with sign. Kragh noticed that this was not quite right, and Seidel formulated something that was correct, that this works if Q is spin, and I'll talk about what happens in the non-spin case. My main goal will be to explain what is the answer in general. Today is "what is the answer?" Next time is how to prove it?

Okay. Questions. No? Well, before giving the answer, if you take the limit as $b \to \infty$ of the homology of the loop space, of course you get the homology of the loop space itself. The limit as the slope goes to ∞ is called the symplectic cohomology of T^*Q. There are many definitions in the literature of the group. This is convenient for constructing various algebraic structures and minimizing the amount of analysis that needs to be done. This was originally introduced by Floer and Hofer, but this version was introduced by Viterbo.

So there are statements whether or not Q is oriented.

Okay.

[Back and forth about homology versus cohomology.]

Before going further, what are we going to construct? We can start by saying in complete generality, it's only $\mathbb{Z}/2\mathbb{Z}$-graded. That's how it comes in nature. On the other hand, the homology of the loop space is $\mathbb{Z}$-graded. So why is there a difference? We need to use the special feature of cotangent bundles that $2c_1(T^*Q) = 0$. The next thing you have on the loop side is that S^1 acts on the loop space. So we obtain a degree $+1$ operator on this homologically graded thing. So we also have a nice circle action on SH^*. To define it we're supposed to count cylinders. By considering marked points, we should call them asymptotic markers on the cylinder which moves in a circle, you get something, you always have a ring structure on SH^* given by pairs of pants. If you use the natural way to lift the $\mathbb{Z}/2\mathbb{Z}$ grading to a $\mathbb{Z}$-grading, this is not compatible with the product if c_1 is not zero, if Q is non-orientable.

There is also something fishy about the product on the loop side. Laudenbach extended Chas and Sullivan's product structure to one on the homology of the loop space with coefficients in the orientation line of Q, and I've pulled back the orientation line by evaluation.

But that's not the group with S^1 action. This does not have the S^1 action.

You see that this won't work out in the non-orientable case. You can try to fix the problem on the string topology side by considering $w : \mathcal{L}Q \to \{0, 1\}$ be the orientability function. This is 0 on orientable loops and non-orientable loops. I can look at the homology of the loop space but $H_*(\mathcal{L}(Q), |Q|^{\otimes 1-w})$. The fantastic thing is that this has a product and a circle action. It looks like something you could compare to symplectic homology. It does not satisfy the usual axioms. This usually gives a BV algebra structure, it's a twisted one.

That's a little bit unfortunate because on the symplectic side with the usual $\mathbb{Z}/2\mathbb{Z}$-grading, it's got an honest BV structure with no twisting, but it can't be lifted to $\mathbb{Z}$.

Theorem 15.1. *If $w_2(Q) = 0$, then there is an isomorphism of twisted BV algebras between $H_*(\mathcal{L}Q, |Q|^{1-w})$ and $SH^*(T^*Q)$ where the $\mathbb{Z}$ grading on this does not lift the $\mathbb{Z}/2\mathbb{Z}$ grading. It fails to lift in a totally explicit way. Instead of lifting in the natural way, you subtract 1 from the nonorientable components.*

I should stop. Next time I'll discuss the nonorientable case.

16. September 4: Abouzaid, Part II

Let me correct some conventions I said about other peoples'. One way to fix conventions is to say that there are German conventions, where the symplectic homology of the cotangent bundle is isomorphic (up to the corrections we've been talking about) to the homology of the free loop space. These are used by Floer and Hofer and others.

Then there are French conventions, used by Viterbo, Seidel, etc., and in these conventions, the symplectic *cohomology* of the cotangent bundle is $H_{-*}\mathcal{L}Q$. I use $n - *$.

It may appear strange that we have different conventions. One reason it's easy is you start studying it on compact manifolds. There it satisfies Poincaré duality. So $H_*(Q) \cong H^{n-*}(Q)$. Over a field you would not know what you are doing if you're using a closed manifold. If you leave the compact world it makes a difference.

The second thing, I wanted to say something about $\mathbb{Z}_2$ gradings. I want to say something explicit. I didn't work this out when I was working out the whole theory. I wasn't planning to do computations. This specific example was explained to me by my student J. Zhao. Look at nonorientable geodesic in a Riemann surface of negative curvature. I want to identify two things flipping the surface. Look at the cotangent bundle of the surface and there's a Hamiltonian flow coming from the geodesic flow. There is a time one orbit, which gives you a generator of $4SC(T^*Q)$, the complex computing symplectic cohomology.

If were discussing loop homology, this would contribute in degree 0. In the symplectic theory we need to compute the Poincaré return map. The grading, take this return map and count the $\mathbb{Z}_2$ grading in Floer theory. Count the number of eigenvalues of the return map that lie in the interval $(0, -1)$. When you compute this you get, you split the tangent space as the direction along and the direction normal to the geodesic. You get something like

$$\left(\begin{array}{c|cc} e^{i\theta} & & \\ \hline & -e^{\lambda} & 0 \\ & 0 & -e^{-\lambda} \end{array}\right)$$

So how many eigenvalues are there in that range? There's only one. The contribution is in odd degree. But in ordinary homology it's in degree zero. So there's a degree mismatch.

Now I want to talk about the orientable case. First I want to use some terminology. I'll say that a local system of rank one on the free loop space is *transgressive* if the corresponding class lies in the image of $H^2(\mathbb{Q}, \mathbb{Z}_2) \to H^1(\mathcal{L}Q, \mathbb{Z}_2)$. There's a natural map in the other direction, $H_1(\mathcal{L}Q, \mathbb{Z}_2) \to H_2(Q, \mathbb{Z}_2)$. If you try to take the Serre spectral sequence for the loop space, it's this kind of guys.

Now my claim, which I'll try to elaborate on a little bit. Maybe I should call it a theorem but it doesn't deserve the name theorem. If ν is a tarnsgressive local system on $\mathcal{L}Q$, then you can make a group $SH^*(T^*Q, \nu)$, and you can also form $H_{n-*}(\mathcal{L}Q, \nu)$, and these are both BV algebras. As far as I know, this is a question, the definition that I use, relies on this, but the question is, does the BV structure depend on the choice of a lift to $H^2(Q, \mathbb{Z}_2)$. Let's think of it on the loop space. Of course you can compute loop space homology with twisted coefficients.

How do you construct these? There are two ways of constructing these theories. Let me focus on the symplectic side. The first one is a brute force construction. It has the advantage of being the most explicit. Fix a cycle V in T^*Q representing a class v in $H^2(Q, \mathbb{Z}_2) = H^2(T^*Q, \mathbb{Z}_2)$ which is our lift of ν. When you go through the definitions of Floer cohomology, in defining the Floer complex, I will assume or rather choose the Hamiltonian generically so that all time one orbits are disjoint from V. The complex is generated by orbits. The differential is obtained by counting cylinders. There's some recipe for counting the cylinders from x to y with some recipe for signs if you're doing usual Floer theory. Now you twist the sign contribution of every cylinder $U : \mathbb{R} \times S^1 \to T^*Q$ by the intersection number with V. This is one way of defining the group $SH^*(T^*Q, \nu)$. This is how you twist the differential. Then you can twist every operation in Floer theory in exactly the same way. We can define a product by counting pairs of pants. If you twist whatever signs you encounter by looking at what happens when you intersect with V, it works, so on and so forth.

This was the brute force method. There is also the more theoretical method. It's easier to describe the more theoretical method if we make a couple more assumptions. Assume that $v \in H^2(Q, \mathbb{Z}_2)$ is the second Stiefel-Whitney class of E where E is an orientable vector bundle on Q. I can do this in general. If you know a little bit about topology, you can prove that on the three skeleton of Q you have such a vector bundle. Let's just assume we have such a global vector bundle. Then there is a very nice way of describing the local system on $\mathcal{L}Q$ corresponding to the transgression $w_2(E)$. It has fiber at a loop $X : S^1 \to Q$, I can pull back E and now every vector bundle on S^1 is trivial. You form the local system generated by the two stable trivializations with the relation that the sum vanishes.

If you have a local system, you have these fibers, but you can also compute monodromy. The monodromy around a loop in $\mathcal{L}Q$ is $+1$ if you can take one trivialization and extend it across the whole family of loops, that is, if E is trivial on the corresponding torus. It's -1 otherwise. That's how you make the connection between this point of view and the previous point of view.

Now how will I use this? The different point of view is, instead of defining, let ν_E be this local system. By abuse of notation, I will write this as a local system on $\mathcal{L}T^*Q$. Remember that when we define Floer homology of T^*Q, it came from an action functional on the loop space of T^*Q. When you do the twisted theory, the generators, the complex that you use is the direct sum over all orbits (time one orbits of some Hamiltonian) of the fibers of the local system at that point. This is a rank one free Abelian group. How do we get a differential? There's a recipe for associating signs to holomorphic cylinders. You can (more importantly from our point of view) use parallel transport to move our local system from one end to the other. You can make this explicit. We have this vector bundle, if I fix a trivialization at one end, I can move it to the other. The same works for curves with many inputs and one output. If I just knew I had a local system on the loop space, I wouldn't know hat to do in this case. I have some x_1, x_2, and x_3 going to y, and I map this to T^*Q. The one skeleton could be these loops plus the arcs connecting them. If I fix the trivialization on the three loops, then you get a trivialization on the one loop. So then you get a map $\nu_E|_{x_1}\otimes\nu_E|_{x_2}\otimes\nu_E|_{x_3}\to\nu_E|_y$. The theorem is, for any local system we have an isomorphism between the symplectic cohomology $SH^*(T^*Q,\nu)$ and $H_{n-*}(\mathcal{L}Q,\nu\otimes\nu_{TQ})$. If we fix v in $H^2(Q,\mathbb{Z}_2)$ which lifts ν, then this is an isomorphism of BV algebras.

17. Richard Hepworth, Part III

My aim today is to tell you something about how to do computations and give examples of nontrivial higher operations. Here's how to understand spaces like BG^X. Here's a definition.

Definition 17.1. *A set of* basepoints *for an h-graph X is a finite subset $P\subset X$ such that P meets every component of X.*

Definition 17.2. *Let X be an h-graph with basepoints P. Then G^P is the group of all maps $g:P\to G$.*

$\Pi_1(X,P)$ is the fundamental groupoid of X with objects P.

Now $G^{\Pi_1(X,P)}$ are functions from $Mor(\Pi_1(X,P))\to G$ such that $f(\delta\gamma)=f(\delta)f(\gamma)$ when this makes sense.

Now G^P acts on $G^{\Pi_1(X,P)}$ via $(gf)=g(q)f(\gamma)g(p)^{-1}$.

Example 17.1. *Let X be the circle S^1 and $P=\{1\}$. In this case, $G^P\cong G$ under the map $g\mapsto g(1)$.*

If I choose my favorite generating loop γ, then $\Pi_1(S^1,p)$ has only the one object 1 and the morphisms are γ^n for n an integer.

So what's $G^{\Pi_1(S^1,P)}$? It's isomorphic to G via $f\mapsto f(\gamma)$.

In this case the action of G^P on $G^{\Pi_1(S^1,P)}$, let's have a look, $gf(\gamma)=g(1)f(\gamma)g(1)^{-1}$, which becomes conjugation.

We see from the theorem I'm about to state that $BG^{S^1}\cong G^{ad}//G$, which is well-known. This is the disjoint union over the orbits of the action of the classifying spaces of the stabilizers. Here, these are the centralizers so this is $\sqcup BC(h)$ over one h in each conjugacy class.

Theorem 17.1. *(H-Lahtinen)*
This is not a massive surprise, it took some work to get it the way we wanted it. There is a natural zig-zag of homotopy equivalences

$$BG^X\leftrightarrow G^{\Pi_1(X,p)}//G\cdot$$

Okay, how to compute operations.

Remember that the interesting bits of my field theory are , for $S : X \not\rightarrow Y$ a morphism, to compute $\phi(S) : H_*(BhAut(S)) \otimes H_*(BG^*) \to H_*(BG^Y)$.

So choose basepoints $P \subset X$ and $Q \subset Y$ and look at the terms in the domain and range of that map and write my combinatorial models of them.

So for example BG^X becaomes $H_*(G^{\Pi_1(X,P)}//G^P)$ and BG^Y becomes $H_*(G^{\Pi_1(Y,Q)}//G^Q)$. I'll replace $H_*(BhAut(S))$ with $H_*(BAs)$ where $As = \pi_0 hAut(S)$.

So I'll call this replacement Φ_S. So what is this? It's

$$H_*BAs \otimes H_*(G^{\Pi_1(X,P)}//G^P) \xrightarrow{\times} H_*(G^{\Pi_1(X,P)}//As \times G^P) \xrightarrow{\alpha^*} H_*(G^{\Pi_1(S,P)}//As \times G^P)$$

$$\xrightarrow{\beta_*^{-1}} H_*(G^{\Pi_1(S,P\sqcup Q)}//As \times G^{P\sqcup Q}) \xrightarrow{\gamma_*} H_*(G^{\Pi_1(Y,Q)//G^Q})$$

where I'm using both sets of basepoints because a priori I have basepoints on the left and on the right and they fight.

The maps come from the diagram

$$\begin{tikzcd}
G^{\Pi_1(X,P)}//As \times G^P & & G_1^{\Pi}(S,P)//As \times G^P \arrow[ll, "\alpha"'] & \\
 & & G^{\Pi_1(S,P\sqcup Q)}//As \times G^{P\sqcup Q} \arrow[u, "\beta"] \arrow[dr, "\gamma"] & \\
 & & & G^{\Pi_1(Y,Q)}//G^Q
\end{tikzcd}$$

You can see that β is an equivalence so I can invert it. This still looks very messy so how do I deal with this? I choose orbit-reps and so then β is an actual isomorphism and I can invert it, don't have to mess around. In this setting α^* is a group transfer map and γ is induced by homomorphisms.

Any questions? In the question session this afternoon I could try to carry out this procedure in some example.

Example 17.2.

$$\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet
\end{tikzcd}$$

Here I'm looking at $S : pt \not\rightarrow pt$ and $\phi(S) : H_*(BAs) \otimes H_*(BG) \to H_*(BG)$. It turns out that the group, which I said was $\mathbb{Z} \ltimes \{\pm 1\}$ is the free product of two copies of $\mathbb{Z}/2\mathbb{Z}$. One of these acts trivially and the other one doesn't. So let me replace it.

$$H_*B\{\pm 1\} \otimes H_*BG \to H_*BG$$

So here ± 1 acts on S by flipping strands. Call this map ψ_S. I'll tell you what $\Psi(S)$ is.

I need some notation. We have $\pm G$ acts on G by $(\epsilon, g) \cdot h$ by sending it to $gh^\epsilon g$. So you are conjugating either h or its inverse. The orbits of these things are called extended conjugacy classes*. The extended conjugacy class includes the conjugacy class of the inverse. The stabilizer is called the* extended centralizer $C^e(h)$*. If h is not conjugate to its inverse, it's just the centralizer. If it is conjugate to its inverse, it contains two copies which are the same size.*

There's a map $\xi_h : C^e(h) \to \{\pm 1\} \times G$ which is inclusion. There's also $\eta^h : C^e(h) \to G$ which sends $(1, g) \to g$ and $(-1, g) \to hg$. Why is this a homomorphism? I have no idea.

Then Ψ_s *is*

$$H_*(B\{\pm 1\}) \otimes H_*(BG) \xrightarrow{\times} H_*B(\{\pm 1\} \times G \xrightarrow{\bigoplus B\xi_h^*} \bigoplus H_*BC^e(h) \xrightarrow{\bigoplus B\eta_*^h} H_*BG.$$

I'd love to show you how this follows from what was written above but there's not time, I'm afraid.

Let's take $G = \{\pm 1\}$*. Then let's work this out.*

I start with $H_*(B\{\pm 1\}) \otimes H_*B\{\pm 1\} \to H_*(B(\{\pm 1\} \times \{\pm 1\})$*. What are the extended conjugacy classes of* ± 1 *? Each extended conjugacy class contains one element and there's two of them. So* $H_*(BC^e(1))$ *and* $H_*(BC^e(-1))$*. I take their direct sum, and map both to the homology of* $B\{\pm 1\}$*. For the former it's* $B\eta_*^1 \circ B\xi_1^*$ *and for the latter it's* $B\eta_*^{-1} \circ B\xi_{-1}^*$*.*

So the centralizer is the whole group, it's $H_*B\{\pm 1\}$*. The transfer* $B\xi_1^*$ *is the identity map. The same is true for the centralizer of* -1*. What does* η_* *become? Recall that* $\eta_h(1, g) = g$ *and* $\eta_h(-1, g) = hg$*. So* g *is my second variable,* ± 1 *is my first variable. So* $\eta_1(1, g) = g$*, and* $\eta_{-1}(\epsilon, g)$*, well, let me just give the answer.* η^1 *is prjection on the second variable. The first factor doesn't do anything. So* η^{-1} *becomes the addition map. Everything looks like it's the same, trivial, but we get two different things. What happens? Let* $\mathbb{F}$ *be the field of order* 2*. then we have* $\mathbb{F}_2[a_0, \ldots,] \otimes \mathbb{F}_2[a_0, \ldots,] \to \mathbb{F}_2[a_0, \ldots,]$ *and this sends* $a_i \otimes a_j$*, the first one to* a_j *if* $i = 0$ *and* 0 *otherwise. The second sends me to* a_{i+j} *if* i *and* j *have disjoint* 2*-ary expansions and* 0 *otherwise.*

So for all $i > 0$ *there are infinitely many* j *such that* $a_i \otimes a_j \mapsto a_{i+j}$*. Here in string topology where we're accustomed to higher operations being trivial, the simplest cobordism, the simplest automorphism, the simplest group, give us infinitely many nonzero higher operations.*

One final remark, Lahtinen has shown that the situation is similar for products of cyclic groups of order two, for certain dihedral groups, for S^1*, and for* $SU(2)$*. I think this is a nice outcome. We find nontrivial structure everywhere.*

$$\begin{tikzcd}
 & H_*B(\{\pm 1\} \times \{\pm 1\}) \arrow[dr, "B\eta_*^1"] & \\
H_*B(\{\pm 1\} \times \{\pm 1\}) \arrow[ur, "B\xi_1^*"] \arrow[dr, "B\xi_{-1}^*"'] & & B(\{\pm 1\}) \\
 & B(\{\pm 1\} \times \{\pm 1\}) \arrow[ur, "B\eta_*^{-1}"'] &
\end{tikzcd}$$

becomes

$$\begin{tikzcd}
 & H_*B(\{\pm 1\} \times \{\pm 1\}) \arrow[dr, "B\pi_2"] & \\
H_*B(\{\pm 1\} \times \{\pm 1\}) \arrow[ur, "id"] \arrow[dr, "id"'] & & B(\{\pm 1\}) \\
 & B(\{\pm 1\} \times \{\pm 1\}) \arrow[ur, "B+"'] &
\end{tikzcd}.$$

18. Mohammed Abouzaid: Part III

I made basically two statements, one about what happens in the nonorientable case and one about an isomorphism for the orientable case.

How do you prove these things? At the level of homology, ignoring the BV structure, I know six different maps between these spaces. The first is due to Viterbo, the second due to Salamon-Weber. The third is [unintelligible]-Schwarz, the fourth is Cieliebak-Latschev, and then two more maps $\}$ and $\mathcal{V}$. I'm confident, even though I won't talk about all of these, that any of these will be able to do the same things. From my point of view the Cieliebak-Latschev is best to get the BV map. The second and third are most convenient for checking you have an isomorphism (there's a bijection between generators for the chain complexes and the flow lines for the differentials) while in those that I prefer use degenerations of moduli spaces.

Let me start by describing Cieliebak-Latschev.

Recall that the definition of the Floer homology of T^*Q with respect to the Hamiltonian H, this used maps $\mathbb{R} \times S^1 \to T^*Q$ with asymptotic conditions on the orbits of the H. The basic idea of the map is that instead of doing $\mathbb{R} \times S^1$ we no consider $[0, \infty) \times S^1$. We think of this as a positive half-cylinder and what you can do on the half-infinite end is put your Hamiltonian orbit. On the other side you impose Lagrangian boundary conditions and make your Lagrangian be the zero section.

There's a distinguished starting point of your Hamiltonian orbit. Follow along this and you get a distinguished point on Q. This gives me a moduli space. I'll call this $\mathcal{M}(X)$. It's the space of things like this that also satisfy an equation that I won't write down.

The main feature of this is that it has an evaluation map $\mathcal{M}(x) \to \mathcal{L}Q$ which sends u to $u|_{0 \times S^1}$.

Let me draw a schematic picture for what's happening. [Picture].

What you basically want to say is, in general, by doing the usual things we do in Floer theory (by choosing perturbations), we can ensure that $\mathcal{M}(x)$ is a manifold with boundary and that the boundary is basically obtained by the inclusion—well, I should be a little bit more careful, the boundary is stratified by things like this. Top dimensional strata are $\mathcal{M}(y) \times \mathcal{M}(y, x)$, where $\mathcal{M}(y, x)$ is the moduli space of cylinders.

[Picture].

If we don't worry about orientations, life is straightforward at this point. If we use a chain theory where "degenerate" chains don't contribute, then you observe that the evaluation map can be restricted to the boundary $ev : \mathcal{M}(y) \times \mathcal{M}(y, x) \to \mathcal{L}(Q)$ which cannot give you any contribution unless $\mathcal{M}(y, x)$ consists of points (has dimension zero). Why is that? In either case, this evaluation map factors through $\mathcal{M}(y)$. With $\mathbb{Z}_2$ coefficients, we get a chain $\mathcal{F}(x) := ev_*[\mathcal{M}(x)]$ which lives in $C_*(\mathcal{L}Q)$ and this satisfies, because the only contribution to the boundary, the boundary $\partial \mathcal{F}(x) = F(\partial x)$, since this is the Floer differential which counts rigid $\mathcal{M}(y, x)$. Only the zero dimensional ones contribute so you get a chain map.

If you want to work with $\mathbb{Z}$-coefficients you have to orient things, and you get that $\mathcal{M}(X)$ is naturally oriented relative to the pullback ev^* of some local system on the loop space. That local system is $\nu_{TQ} \oplus |Q|^{1-w}$. That is, we have a natural equivalence to $|T\mathcal{M}(x)|$. So you should evaluate into the loop space with twisted coefficients. These were discovered independently, where does this come from? You try to see orientations of holomorphic curves and the first time it appeared explicitly in the literature was in Vin de Silva's thesis (you have to go to Oxford and find it) and also was done by FOOO's book, which worked out the corresponding orientations for holomorphic disks, and this was not different.

Are there any questions before I continue? So. I will continue. Whatever construction you do, this ensures that all the moduli spaces I will be drawing are naturally oriented. There was really no reason to prefer Cieliebak-Latschev's point of view. We can see very easily that the map is compatible with operations. By the way I assumed everyone knows what a BV algebra is. I said there exists an operator Δ of degree one, in my theory of degree $+1$ on SH^*. I won't give it in detail but it basically corresponds to moving the starting point of $X : S^1 \to T^*Q$ along the circle.

If you do that, how do you define this evaluation, follow this starting point along the cylinder. If you move the point on the right, you move the point on the left. You draw this picture, I'm not even sure what to say, if I move the point once around, then the point downstairs will also move once around. This is the proof that $\Delta \circ \mathcal{F} = \mathcal{F} \circ \Delta$. This is the proof as long as these moduli spaces are oriented.

The proof for the product requires a nontrivial cobordism and is more interesting. We want to show that $\mathcal{F}(\underbrace{x * y}_{\text{pair of pants}}) = \underbrace{\mathcal{F}(x) * \mathcal{F}(y)}_{\text{Chas-Sullivan loop product}}$.

So let's start with the pair of pants and then look at the half-cylinder. We take x and y, we look at the moduli space, get z, and then look at the moduli space and evaluate at a base. We glue them and get a moduli space of pair of pants that only involve x and y. It loks the same but we have the Lagrangian boundary on Q. One has punctures and this has boundary.

Now the claim is that this is just one boundary of this moduli space. This is the part where these go infinitely far away. You can model this in different ways, think of x and y getting very close together. Instead let the saddle go down to where the Lagrangian lives. The other boundary is there. [Picture]. If you arrange it correctly, you see that this is what will happen, if you have chosen perturbations that were sufficiently generic, if $ev : \mathcal{M}(x) \to$, let me say this is ev_0 now, evaluation at the marked point. Say $ev_0 : \mathcal{M}(x) \to Q$ and $ev_0 : \mathcal{M}(y) \to Q$ are transverse, then the other boundary stratum is the fiber product $\mathcal{M}(x) \times_Q \mathcal{M}(y)$. The fiber product has its own evaluation map to the loop of Q, where you follow the loops. The assertion is that the moduli space I'm drawing in the middle, $\mathcal{P}(x, y)$ is a cobordism between $\mathcal{M}(z) \times \mathcal{P}(z, x, y)$ and $\mathcal{M}(x) \times_Q \mathcal{M}(y)$. This equality holds at the level of homology.

The main question is why this map is an isomorphism. Once you have a map of BV-algebras, you only have to check it on the linear part, we just have to build left and right inverses. I won't check that these are [unintelligible] maps. You get something up to sign. At some point you figure, why bother checking it directly.

Okay, so what's, here's a different kind of map that will use Lagrangian Floer theory, $\mathcal{V}$ (named for Viterbo). It will go from the homology of the loop space to symplectic cohomology. It's better to think of it as going from the homology of the part of the loop space in which the length is bounded, which you can model by piecewise geodesics. What is this $\mathcal{L}^r Q$. Assume that the injectivity radius of Q is bigger than 4. So $\mathcal{L}^r Q \subset Q^r$, which you should think of as $d(q_i, q_{i+1}) < 1$. This is a submanifold of Q^r. How do I want to go from a chain, what do I want to do with this? The main observation is the following. Pick a function $h : \mathbb{R} \to \mathbb{R}$ that has the following properties. I want h' to grow from zero to two. What will I do with h'? I'll do what I did last time, I'll take h of the norm, consider $h(p, q) = h(|p|)$, why do I consider this? If I look at a cotangent fiber at a point q and at q', I want to flow along the Hamiltonian flow of h, so I get something like this picture. [Picture]. What I see is a unique intersection point here. If the distance between q and q' is less than one, there exists a unique intersection point of $T^*_{q'}Q$ and the image of $T^*_q Q$ under flow. So there exists a unique Hamiltonian chord of X_h with these boundary conditions, that the cotangent fiber of q and the cotangent fiber of q'.

So what will I do with these chords? Before I had orbits, but now I have chords built in. The sequence of points, I can put Lagrangian boundary conditions on each on of these points. My asymptotic conditions are orbits, between these I want to put these chords. [Picture]. I've drawn a Riemann surface. I can put x at the bottom which is a Hamiltonian orbit. This means that for any point in the space of piecewise geodesics, there's something to study. We can then do this for families of points. Given a cycle $Y \subset \mathcal{L}^r Q$, we get a moduli space $\mathcal{R}(x, Y)$ for all orbits x.

This gives a chain map, the evaluation gives a chain map from Floer, from homology of the geodesic space to, my product will take five minutes of the question session and why the composition of this map with the one I described earlier is an isomorphism.

19. Ralph Cohen: Calabi-Yau categories, string topology and the Floer field theory of the cotangent bundle

I do not take notes during slide talks.

20. Alexander Berglund: Loop spaces and Koszul algebras

This talk will be very algebraic, I don't know if that's a good thing or a bad thing.

The goal of the talk is to try to see how to compute loop space homology $H_*(\Lambda M)$ using Koszul algebras.

We've certainly heard a lot about loop spaces, let me slowly review the Koszul algebras. Let me discuss the relation between formality of dgas and Koszul algebras, then present some calculations.

Okay, so these were introduced by Priddy in 1970 and since then a lot has happened. If you want some references, this is close to the presentation in Loday–Vallette's algebraic operads. It's late and I'll keep this elementary, I'll talk only about associative algebras, which will be enough for free loop spaces.

What is a Koszul algebra? It's a kind of quadratic algebra, $TV/(R)$, with $\mathbf{k}$ a field. Much of this works integrally if you're torsion free but let's work with a field for simplicity. Here V is a graded vector space, either cohomologically graded and concentrated in degrees 2 and higher or homologically graded and concentrated in degree 1 and higher. So the relations should be within $V \otimes V$.

Example 20.1.

$$H^*(BT^2) = \mathbf{k}[x, y] = T(x, y)/xy - yx$$

with x and y in degree 2.

If we look at the bar construction on such an algebra, we say $\mathcal{B}A$ is a tensor coalgebra on the suspension of A, the augmentation ideal, with a differential b, so $(Ts\bar{A}, b)$.

This is bigraded by weight and the bar length. For instance, in my example, the elements of the bar construction might look like $[x|xy]$, which has weight 3, the sum of the polynomial degrees, and the bar length is 2.

Then the way the differential works, I have something in degree zero and stuff on the diagonal and above the diagonal, and the differential goes to the right.

In general it's hard to compute the homology of the bar construction but in general, well, you can look at the homology of the diagonal, call that $A^{¡}$, which is the kernel of the differential intersected with the portion where the weight is equal to the bar length. This sits inside the bar construction $\mathcal{B}A$. This will not be all of the homology in general but it's computable because, if you take the linear dual of the coalgebra you get an algebra $A^!$ and that admits a quadratic presentantion of the form $T(s^{-1}V^*)/R^\perp$. What do I mean by $R^\perp$? A pairing between V^* and V extends to $(V^*)^{\otimes 2} \otimes V^{\otimes 2}$. So $\langle \alpha \otimes \beta, x \otimes y\rangle = \pm\langle \alpha, x\rangle\langle \beta, y\rangle$.

In the example, the Koszul dual, the quadratic dual of my algebra is $T(\alpha, \beta)/(\alpha^2, \beta^2, \alpha\beta + \beta\alpha)$, which is the exterior algebra on α and β. So these are degree 1 things.

To answer Richard's question, with wishful thinking, if the homology is concentrated on the diagonal, let's give that a name.

Definition 20.1. *A is* Koszul *if the inclusion $A^{¡} \to \mathcal{B}A$ is a quasi-isomorphism.*

Will this ever happen? Checking this, you'd have to calculate the bar construction. If I compose this with the universal twisting cochain from $\mathcal{B}A \to A$, I get a twisting cochain $A^{¡} \to A$. This composite pulls back, it's a twisting cochain κ. Then you can form the twisted tensor product.

Proposition 20.1. *A is Koszul if and only if $A^{¡} \otimes_\kappa A$ is equivalent to $\mathbf{k}$.*

The proof is, think of this as an A-bundle. You have

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & A^{¡} \otimes_\kappa A \arrow[r] \arrow[d] & A^{¡} \arrow[d, "\psi"] \\
A \arrow[r] & \mathcal{B}A \otimes_{t_{\mathcal{B}}} A \cong \mathbf{k} \arrow[r] & \mathcal{B}A
\end{tikzcd}$$

and you can prove a sort of two out of three thing so that the middle is an equivalence if and only if the right is.

Remark 20.1. *A is Koszul if and only if $A^!$ is Koszul, since $(A^!)^! \cong A$.*

That was my crash course on Koszul duality. Any questions?

Now I want to move to formality, which originates in rational homotopy theory.

Definition 20.2. *A differential graded algebra A is* formal *if it is quasi-isomorphic to its homology as a differential graded algebra.*

We view homology as a differential graded algebra with 0 differential. Quasi-isomorphic means there is a zig-zag of quasi-isomorphism connecting them. You can always find a guy in between if A is formal so that $A \xleftarrow{\sim} A' \xrightarrow{\sim} H_*A$. I'll say X is formal (over $\mathbf{k}$) if $C^*(X, \mathbf{k})$ is a formal differential graded algebra.

Remark 20.2. *If the characteristic of* $\mathbf{k}$ *is* 0 *then one usually works with the* commutative *differential graded algebra* $\Omega^* X$*, which is quasi-isomorphic to the cochains if the characteristic is zero. I don't know whether formality as differential graded algebras implies formality as commutative differential graded algebras. I doubt it.*

For example, Deligne-Griffiths-Morgan-Sullivan showed that every compact Kähler manifold is formal in this sense.

To give you an idea that this is something that happens, Miller showed that an $(n-1)$-connected closed monifold for $n \geq 2$ of dimension at most $4n-2$ is formal over $\mathbb{Q}$. In particular, all simply connected manifolds of dimension at least 6 are formal.

There's a dual notion of formality that hasn't been studied to the same extent. So

Definition 20.3. *We say that* X *is* coformal *(over* $\mathbf{k}$*) if the chains on the based loop space* $C_*(\Omega X, \mathbf{k})$ *is a formal dga.*

When $\mathbf{k} = \mathbb{Q}$ and X is simply connected, then one should really be working with Quillen's differential graded Lie algebra *lambdaX* instead. The universal enveloping algebra is equivalent to the chains on the based loop space.

I'm restricting the structure to just the E_1 structure to simplify a little bit.

Now I want to connect this to Koszul algebras.

Theorem 20.1. *Let* X *be simply connected of finite type and* $\mathbf{k}$ *a field. The first three are equivalent and they imply the fourth:*

(1) X *is formal and coformal.*
(2) X *is formal and* $H^*(X, \mathbf{k})$ *admits a Koszul presentation.*
(3) X *is coformal and* $H_*(\Omega X, \mathbf{k})$ *admits a Koszul presentation.*
(4) $H^*(X, \mathbf{k})$ *and* $H_*(\Omega X, \mathbf{k})$ *admit Koszul presentations that are Koszul dual to one another.*

I don't have a counterexample for the fourth implying the first three.

The direction I'll be focusing on today is a way to calculate the loop space homology when X is formal and coformal. Then you might ask for examples that are both formal and coformal.

Example 20.2.
- S^n *is formal, and we know its homology is* $T(x)/(x^2)$*, which is Koszul, and so* $H_* \Omega S^n = (H^* S^n)^! = T(\alpha)$ *where the degree of* α *is* $n-1$.
- *The products and wedges of formal spaces are formal. So for instance, in the wedge of two spheres, I get that* $H_*(\Omega(S^n \vee S^m)) \cong T^(\alpha, \beta)$
- *Here's an example that I like, not related to the rest of my talk. Take the configuration space of* k *points in* $\mathbb{R}^n$*, this is formal and coformal. The comonodromy, actually let me, the comonodromy is generated by* x_{ij} *for indices between* 1 *and* k *modulo* $x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = 0$, $x_{ij}^2 = 0$, *and graded commutativity. For formality let's take* $\mathbb{Q}$ *coefficients. The loop space homology, if you calculate, you get a free algebra on dual generators, modulo infinitesimal graded relations,* $[\alpha_{ij}, \alpha_{k\ell}] = 0$ *and* $[\alpha_{ij}, \alpha_{ik} + \alpha_{jk}] = 0$.
- *Every* $(n-1)$*-connected manifold closed manifold of dimension at most* $3n-2$ *is automatically formal and coformal if the dimension of* $H_*(M)$ *is at least four. Rationally this was observed by Neisendorfer-Miller. They prove this statement in that paper, but Koszul duality is what's new here.*

- $\mathbb{CP}^2$ *is formal, but its homology is a truncated polynomial algebra, there's no quadratic presentation of this algebra, so it' can't be Koszul. So it's not coformal.*

So sometimes this lets you calculate based loop space homology. Now I'd like to go on to free loop space homology.

The Hochschild cohomology of $C_*\Omega M$ is isomorphic to $H_{*+d}(\Lambda M)$. Here on the left we have a cup product and a Gerstenhaber bracket. On the right we have the loop product and the loop bracket.

So you can use $HH^*(C_*\Omega M) \to HH_{*+d}C_*\Omega M \to H_{*+d}\Lambda M$.

Theorem 20.2. *Let M be formal and coformal. Then $H_*(M)$ has some generators, $H_*(\Omega M)$ has some dual generators. We have $\kappa = \sum_i x_i \otimes \alpha_i$ in $H^*M \otimes H_*\Omega M$. This is a twisting cochain which means that it squares to zero. It's an exercise that this holds because of orthogonality. We can twist the algebra with the Maurer-Cartan element and get a differential, so $H^*M \otimes H_*\Omega M, [K, \quad]$ is a dga model for Hochschild cochains on $C_*\Omega M$.*

Proof. First of all, since, well, let U be $H_*\Omega M$. Since M is coformal, we have $U \sim C_*\Omega M$, so we can look at $Hoch^*(U)$, which is equal to the twisted Hom complex $Hom^{t_{\mathcal{B}}}(\mathcal{B}U, U)$. We can twist the structure here. This is always true for Hochschild cochains. Since U is Koszul, we know that the Koszul dual coalgebra injects quasiisomorphically into $\mathcal{B}$ and a spectral sequence says that this is the same as $Hom^{\kappa}(U^{¡}, U) \cong (U^{!} \otimes U, [K, \quad])$. □

There will always be an A_∞ model for these things that has this as an underlying chain complex. But this is saying we can make the algebra structure very explicit in this situation. Okay, I'm running out of time here.

Let me just finish by discussing this in the case of highly connected manifolds.

Let me restrict to the case where M is an $(n-1)$-connected $2n$-dimensional closed manifold ad $n \geq 2$. Choose a basis $x_1, \ldots, x_r$ of H^nM, and let me assume that the rank is greater than or equal to 3. Then M is formal and coformal and if you calculate the orthogonal relations, then $H_*\Omega M$ is isomorphic to $T(\alpha_1, \ldots, \alpha_r)/(w)$ where $w = \sum_{i,j} c_{ij}\alpha_i\alpha_j$ and $c_{ij} = \langle x_i \cup k_j, [M]\rangle$.

Theorem 20.3. *(B-Börjeson) Under these hypotheses, for $\mathbf{k}$ a field and $r \geq 3$, you get*
$$H_{*+2n}(\Lambda M) \cong \mathbf{k} \oplus s^1 \frac{DerU}{AdU} \oplus s^{-2n} \frac{U}{[U,U]}.$$

So this is a description of the loop space homology. We can also describe the loop product. Up to sign, the loop product of two such derivations,
$$\theta \cdot \eta = \sum_{i,j} c_{ij}\theta(\alpha_i)\eta(\alpha_j) \in U/[U,U]$$
with all other products zero. The Gerstenhaber bracket has
$$\{\theta, \eta\} = \theta \circ \eta \pm \eta \circ \theta, \qquad \{\theta, u\} = \theta(u)$$
We have a BV-operator if the characteristic is zero, Δ goes from $U/[U,U] \to DerU/adU$ where $\Delta(u)(\alpha_i^{\#}) = \frac{\partial^{cyc}(U)}{\partial\alpha_i}$ where $\alpha_i^{\#}$ is $\sum_j c_{ij}\alpha_j$ and ∂^{cyc} is [too fast].

Kai has worked out explicit formulas for the dimensions here and you can see that you have this exponential growth for the Betti numbers for the free loop space.

There has been previous work on this by Seliger-Beben, who use slightly different techniques. I should end, thanks for listening.

21. September 5: Nancy Hingston, Part III

It's amazing to hear all these different ponts of view. We talked about the ideas of llop spaces, loop products, and Poincaré duality, which I'm just about to get to. We began this discussion of loop products, we have the Pontryagin product on based loops and the Chas-Sullivan product on free loops. The free loop space is infinite dimensional, but many things work the same if you turn the energy upside down. I'll give some evidence for Poincaré duality and you can decide if you believe it or not. Report back to me if you see other examples of this.

The other product is joint work with Mark Goresky, who happened to be at the institute when I heard this definition.

Poincaré duality predicts another product $\circledast : H^j(\Lambda) \otimes H^k(\Lambda) \to H^{j+k+n-1}(\Lambda)$ and there's also a product on the based loops of the same degree, $H^j(\Omega) \otimes H^k(\Omega) \to H^{j+k+n-1}(\Omega)$. These should have the same relationship to one another as the Pontryagin and Chas-Sullivan products.

Mark and I wrote definitions of these, they're defined, they're non-trivial, they're finitely generated rings for spheres and projective spaces. Now Cohen, Jones and Yan described the relation of these two to one another and it follows the same relationship.

These are related to Sullivan's coproduct on the homology.

We used, there's a nice finite dimensional approximation to the free loop space by piecewise geodesics. We took two cohomology clases. We restricted them to the finite dimensional approximation, did Poincaré duality, took the Chas-Sullivan or Pontryagin product, undid Poincaré duality, and then forgot the finite dimensional approximation. It turns out that you want to use curves parameterized according to arc length.

This is, that's the first piece of evidence for Poincaré duality. But the finite dimensional approximation doesn't play nicely with arc-length parameterization. That's what makes writing this down difficult.

I talked a little about this in the question session, but next is Poincaré duality and index growth. Now γ is a closed geodesic. The formula for the index of γ^m, the most it can be is $m\ ind\ \gamma + (m-1)(n-1)$ and the least it can be is $m\ ind\ \gamma - (m-1)(n-1)$.

You might think this looks symmetric, that looks like Poincaré duality. Actually it's not really symmetric, let's use the sum of the index and the nullity.

$$m(index + null)(\gamma) - (m-1)n \leq (index + null)(\gamma^m)$$

Dual to index is not coindex but coindex plus nullity, that's where the 1 comes from. When you have equality on the upper bound you have maximal index growth, meaning that there is a nontrivial cohomology product. When there's minimal index growth, and then you get a nontrivial Chas-Sullivan product.

Okay I wanted to say something about Poincaré duality and critical levels. So from last time the critical level of a homology class is $\inf\{a | h \text{ is supported on } \Lambda^{\leq a}\}$. [Picture]. The critical level of a cohomology class $Cr(x) = \sup\{a \in \mathbb{R} : x \text{ is supported on } \Lambda^{\geq a}\}$. We have a formula

$$Cr(g *_{CS \text{ or } P} h) \leq Cr(g) + Cr(h)$$

on homology but

$$Cr(x \circledast y) \geq Cr(x) + Cr(y)$$

So the cup product does not have this property. This is supposed to be more evidence for Poincaré duality.

There are a number of theorems that were proved before the products were defined that are manifestations of what is happening with these products. So here are some old theorems rephrased.

Assume that M is compact orientable, and Riemannian. Here's a theorem of Bott, rephrased, from 1956. If all closed geodesics on M are nondegenerate then every homology class is level-nilpotent. There's a dual theorem that says every cohomology class is level nilpotent. That means that for some N, $Cr(h^N) < NCr(h)$. For some N, $Cr(x^N) > NCr(x)$.

The next pair of theorems is due to me, one in 1993 and one in 1997. They say that if there is a class in $H_*(\Lambda)$ or $H^*(\Lambda)$ that is not level nilpotent, then M has infinitely many closed geodesics. These fit together with the Bott result. The proofs are very different. The cohomology version is a lot harder. The reason I believed the second one was true was because of Poincaré duality.

Let me draw a picture of the picture of these. [Picture].

That's the end of Poincaré duality. A few applications to dynamics. So this theorem, I mean this pair of theorems is useful, is used in the following proofs, that there exist infinitely many for any metric on S^2, it's not the only way of seeing it, it's the only way of proving it with a bound on [unintelligible]. There's one argument for the nondegenerate case and in the degenerate case you have limiting index growth.

I also wanted to mention some relationships with things on the Floer side. There'll be another theorem rephrased in a minute. The Floer stuff, there are all kinds of different connections, So Abbodandolo and Schwarz, Viterbo, Salamon-Veber, Cohen-Hess-Verana, Cohen with his collaborators, says there's an isomorphism between $H_*(\Lambda M, *) \cong HF(T^*M, \text{pair of pants})$. I want to say that Abbondandolo-Schwarz have located the cohomology product on the Floer side. The second thing is another long list of names, Conley-Zehnder, Salamon-Zehnder, lots of other people dealt with the following question. You have a $\pi_2 = 0$ compact $2n$-dimensional symplectic manifold, and we look for periodic orbits of a periodic Hamiltonian. Conley-Zehnder said that in the nondegenerate case there are infinitely many periodic orbits. This left out the degenerate case. You use the statement that in the nondegenerate case, you cannot have limiting index growth.

Another restated theorem due to myself and Ginzburg-Gürel, there are infinitely many periodic orbits when the pair of pants product is level-nonnilpotent.

I have about ten minutes left? I want to mention some more recent work.

There's recent work, the *resonance theorem*, with [unintelligible]. Look at the set of critical levels as a function of degree, this is the critical profile of a manifold. There's a theorem about this, it's very interesting, but let me tell you just the consequences for dynamics.

There are other nice versions, but if n is odd and you have a metric where the curvature is between $\frac{1}{4}$ and 1, well, there's a number $\bar{\alpha}g$, a slope, which has unites of conjugate points per unit length. If you plot all these points, critical levels as function of L, they lie within a small distance from a line (if this is a sphere) then at least one of these is true:

(1) There exist at least two closed geodesics with $\bar{\alpha}_\gamma = \bar{\alpha}_g$, or
(2) There exists a sequence of closed geodesics with $\bar{\alpha}_j \to \bar{\alpha}_g$.

You don't want a theorem that just tells you the numbers, you want something that tells you how they're organized. It would be nice to say either one is true or the other is true.

I just wanted to mention briefly some other recent joint work, recent and ongoing. I've been working with Alex Oancea on the Chas-Sullivan product on path spaces, particularly paths in $\mathbb{CP}^n$ with boundaries on $\mathbb{RP}^n$. I briefly want to say, it's related to a lot, what I've been doing with Nathalie, something about higher order operations. We've heard that it's hard to define higher order operations, so for example you can draw a picture of the Chas-Sullivan product and you consider the ones where the basepoints are equal. So the Chas-Sullivan product has degree $-n$ because of the two loops and the one intersection that you're imposing. So this is a picture, and you can fatten this up and get a picture that's like a pair of pants. If you want to draw a picture of the associated coproduct, you have one loop and you do some intersection within that loop. You want $\gamma(0) = \gamma(t)$. If you don't let t vary you get a trivial coproduct. If you thicken this up, you get a pair of pants. You have a nontrivial cproduct from this one, degree $3 - 2n$, it's nontrivial which is a nice quality for a coproduct.

22. Fukaya, Part III

In this talk I wanted to talk about open closed maps. I talked twice and spent the last lecture about the closed open map. Today I'll say open closed. I haven't made much progress on the higher genus stuff. This open closed stuff is five years old but somehow not well known.

So let $L \subset X$, and $p_{\ell,k} : E_\ell \Omega X \otimes B_k^{cyc} \Omega L \to X$. So we have differential forms on the ambient manifold and cyclic differential forms on the Lagrangian. The cohomology of the cyclic bar complex is related to the equivariant loop space. There are some delicate differences. Basically this map is dual to q. I want to explain the main reasons we came to this map.

I'll explain soon, but here's the theorem we got.

Theorem 22.1. *Consider the map $H_*(L) \to H_*(X)$, and assume the even part is injective, then Floer homology of L is defined and is isomorphic to the ordinary homology of L.*

The main motivation to introduce the open closed map is to prove this kind of theorem. A typical example is X as the diagonal in $X \times X$. The homology is injective. So the Floer homology of the diagonal is defined and is equivalent to the ordinary homology of X.

This is a kind of important case. One typical idea is that you have this map $\tau(x, y) = (y, x)$, the antiholomorphic involution to try to prove this. That works over $\mathbb{Z}/2$ coefficients. Then if you try to prove this isomorphism over rational numbers, you need to calculate signs carefully. We failed to do so. In some cases, things cancel but in some cases the involution gives you two of something. You need some other argument, which is this inclusion.

Why is the open closed map useful to understand this kind of thing? Well-definedness of Floer homology is related to $m_0(1)$, which is the homology class of a bubble. This is $\mathcal{M}_{1,0}(\beta)$, where 1 means one marked point on the boundary, 0 means no marked points in the interior, and β is the class. There is an evaluation map at the marked point. So $m_0(1) = \sum[\mathcal{M}_{1,0}(\beta)]T^{\beta\cap\omega} \in S_*L$

This makes things not well defined. We have $m_1 m_1 x = m_2(x, m_0(x)) \pm m_2(m_0(1), x)$.

If you pick the smallest possible [unintelligible] then $[\mathcal{M}_{1,0}(\beta)]$ is a cycle in $S_*(L)$ and the claim is that $i_*[\mathcal{M}_{1,0}(\beta)]$ is 0 in $H_*(X)$. If you go to the ambient space the cycle is zero. The proof is the following. Consider $\mathcal{M}_{0,1}(\beta)$. I only have marked points on the boundary. The boundary is $\mathcal{M}_{1,0}(\beta)$. This is basically an argument. You want to use this moduli space which has an interior marked point. So $\mathcal{M}_{0,1}(\beta)$ is basically $p_{0,1,\beta}(1,1)$. Our assumption is that the homology map is injective, if your cycle is a boundary here then it's a boundary on L. That's an argument. It's a bit funny. Usually cyclic homology, you have $\bigoplus B_k^{cyc}\Omega L$. What do we use? Actually $B_0^{cyc}\Omega L$. In a sense, we can restrict the map B to the part with a trace 1 marked point, the story is somewhat similar, the most interesting applications came from the 0 part.

As I explained, the algebra works nicely for the >0 part, but the $=0$ part introduces new phenomena.

This is somehow some motivation. Now I write down the formulas. So $p_{k,\ell} : E_\ell\Omega X \otimes B_k^{cyc}\Omega L \to \Omega X$.

Suppose you have $Q_1 \otimes \cdots Q_\ell$, then you have $\Delta Q = \sum Q_{c,1} \otimes Q_{c,2}$. This is the shuffle. The formula for $p_{k,\ell}$ is

$$p_{k,\ell}(\partial\vec{Q}, \vec{P}) \pm \partial p_{k,\ell}(\vec{Q}, \vec{P}) \pm \sum_c p_{k_1,\ell_1}(\vec{Q}_{c,1}, \hat{q}_{\ell_2}(\vec{Q}_{c,2}, \vec{P})) = 0$$

Here q is a map $E\Omega X \otimes B\Omega L \to \Omega L$ and $\hat{q}$ is the extension to a coderivation on $B\Omega L$.

[Picture of $p_{k,\ell}$].

The simplest case is $\partial p_{0,0}(1,1) = p_{1,0}(1, q(1,1))$, this is like $\partial\mathcal{M}_{1,0}$ is $\mathcal{M}_{0,1}$.

This formula looks complicated, but you can interpret it in a simple way as follows. Look at the boundary of the moduli space. We can still use this formula to get results.

The best way to say what this is is the following thing. I mentioned this the first day but I'll repeat it. The cyclic bar complex of ΩL is an L_∞ module over the Hochschild cochain complex.

You have like $\varphi : B_k\Omega L \to \Omega L$, then φ acting on $x_1 \otimes \cdots x_n$ is

$$\sum \varphi(x_1, \ldots, x_k)x_{k+1}, \ldots x_n + \cdots + x_1 + \cdots \varphi(x_m, \ldots, x_{m+k-1}) \ldots, x_n + \sum \phi(x_j \ldots x_n x_1 \ldots x_a)x_{a+1} \ldots x_{j-1}$$

[missed some].

So now we have $B^{cyc}\Omega L \to \Omega(X)$ by p, and then $(E_\ell\Omega X) \otimes B^{cyc}\Omega L \to \Omega X$ are an L_∞-module homomorphism.

Let me remind you why this is an L_∞ module. So $q : E\Omega X \to CH(\Omega L)$ is an L_∞ homomorphism. Then $B^{cyc}(\Omega L)$ is a Lie module over $CH(\Omega L)$, then we can regard this as having a trivial structure. You have these two modules, and the claim is that this is an L_∞ module homomorphism.

However, actually, I have ten more minutes. Up to here, the story looks rather transparent. Then there are several things you have to be a bit more careful. The first thing is the following. The formula needs to be modified if we include $B_0^{cyc}\Omega L = \Lambda_0$. In the geometric way, this should correspond to this [picture]. So you have $p_{\ell,0} : E_\ell\Omega(X) \to \Omega(X)$. This is a kind of natural enhancement. This guy is actually one of the most interesting ones. For positive k, this is actually correct, for these guys you get corrections. I'll write the formula for this zero case. For the extra term, I want something like $\overline{GW}_\ell : E_\ell(\Omega X) \to \Omega(X)$ where $\langle \overline{GW}_\ell(Q_1, \ldots, Q_\ell), Q_0\rangle = GW(Q_1, \ldots, Q_\ell, Q_0)$. What is the Gromov-Witten invariant? It's a count of [picture]. Then you turn this around and get this map.

The new formula is $p_{\ell,0}\partial\vec{Q}, 1) + \partial P_{\ell,0}(\vec{Q}, 1) + \sum p_{\ell,1}(Q_{c,2}, q(Q_{c,1}, 1)) + \overline{GW}_{\ell+1}([L], \vec{Q}) = 0$.

We never found the meaning of this formula. The potential application, there are some cycles that are Lagrangian cycles that aren't algebraic, this gives some information about those. I don't know how to use this for example. But this gives some formulas. A corollary of this is the following thing.

Corollary 22.1. *When Q is empty, if $GW_1([L]) \neq 0$ then the Floer homology is obstructed and can't be defined.*

When $GW_1([L]) = 0$, some how $[\mathcal{M}_{0,1} \in Ker(H(L) \to H(X))$.

This GW term has an origin, consider a simple case [picture]. There are actually two pieces of boundary for one marked point in the interior. There is another case, where the loop shrinks to a point. This looks a bit like a sphere bubble, but this is codimension 1. This guy you can see from the Gromov-Witten invariant. This is also studied by [unintelligible] and Liu. This phenomenon comes when you have no marked point on the boundary. If you forget these cases, the story is usually less interesting. The most interesting cases are sometimes the ones where this occurs. You have to include these cases to get [unintelligible]

Then the homology of the cyclic bar complex when we hav $k > 0$, this is $H_{S^1}(\Lambda L)$. I don't know what the $k = 0$ component means. It's probably some central extension.

Maybe my time is up, I stop here, thank you very much.

23. Octav Cornea: some properties of the Grothendieck group of the derived Fukaya category

The talk is joint with Paul Biren. I want to talk about monotone Lagrangians. Look in your notes for the talk of Denis Auroux. He discussed that there are infinitely many and constructions to provide them in various spaces.

Let me start a little bit on the geometry that appears in my subject.

- You want to classify monotone Lagrangians, up to:
  (1) Hamiltonian isotopy, maybe the most important one
  (2) Lagrangian isotopy, one parameter deformations of Lagrangians to Lagrangians, and
  (3) my focus, Lagrangian cobordism (introduced in 1979 by Arnold)

Let me discuss this idea. To an algebraic topologist, cobordism in natural to consider. The picture I want to describe right now is a cobordism between two families of Lagrangians. This is a smooth manifold V with boundary divided in two classes of components $(L_1 \cup \cdots L_k) \cup (L' \cup \cdots L'_s)$. This sholud be a Lagrangian embedded in a Lagrangian way in $\mathbb{C} \times M$ and there is a projection onto $\mathbb{C}$ and under this projection I want my Lagrangians to look linear away from a compact set, and I'll say what I mean. I label the two ends L_1, L_2, and L_3, and then on the other side L'_1 through L'_3. So there it should look like $\mathbb{R} \times \{xi\} \times L'_3$, et cetera.

So any question about just this definition?

Now, so what can you do with this stuff? If you think in terms of algebraic topology.

The next stage is, why not define a Lagrangian cobordism category (of M) Here it's a little subtle but it's not too bad. The objects are Lagrangians. I fix a specific class of monotone Lagrangians, but this is a technical issue. What are the morphisms? They will be a little different from what you maybe expect. I want to go from one L to an L', so the most natural thing to pick is to have one L and one output L'. I'll actually allow some other ends. The top one should be L'. I want to do this because I want this cobordism to be not too simple. For this to be more complicated, that's well, [unintelligible]

You can continue these things if you want to worry about composing. All right, and once you get there you can define a cobordism group, a natural thing to do, let's call it $\Omega_{Lag}(M)$ which will be $\mathbb{Z}/2\langle L\rangle/R_{cab}$, where if you have a V with ends $L_1 + \cdots L_4$, you put the relation that the sum of these is 0.

Now let's move into algebra. I've been greatly helped in the last week with what people have said about Floer homology. If one looks at the algebra, then the thing is, how can one see what the Lagrangian is? If I have any subset, I can look at the intersection of that subset with L. If X is variable, if I move X a lot, I'll recover L completely. I'll assume X is a Lagrangian transversal to L. This will still recover L completely. I can take the vector space spanned by the intersection points. This is not yet algebra, just a vector space.

The natural structure is the Floer differential. You'll put d^F on this. Now I'll make the picture of the Floer differential. You count strips between the intersection points.

Now the coherence with respect ot X leads to the Fukaya category and let me say how. There are many people who have been involved in this [quick list of names]. So we'll define this A_∞ category. The space for Floer homology will be $CF(X, L)$. Then the objects are just Lagrangians and the only thing left to talk about are operations $\mu^K : CF(L_1, L_2) \otimes \cdots CF(L_K, L_{K+1}) \to CF(L_1, L_{K+1})$. The higher operations can be described by a picture which just generalizes the strips. We treat the points as inputs and the one point as an output. This defines, you'll have the relation $\mu \circ \mu = 0$ and this will have the structure of an A_∞ category.

What does this have to do with L and our dependence on X? The important thing here is that L, our complex, the Floer complex relative to L can be viewed as an A_∞-module over this A_∞ category.

So the upshot is to look at L algebraically, we could look at it as a module over this A_∞-category.

Now there is some bad news in this story, which is that this A_∞ category depends on many choices. You get a family depending on choices. They're all comparable but it's a huge construction.

There's something you can do. I want to talk about how to pass to the derived Fukaya category. If you have modules over $\mathcal{F}(M)$, $\mathcal{M} \to \mathcal{M}'$, with a morphism, then you can construct the cone over the morphism. Why can you do this and why? You can always attach cones, you can always get a new chain complex. You get the sum of the chain complexes with the map as a new component of the differential. This space is already a chain complex, and A_∞ modules are chain complexes. It's great that you can deal with them as with chain complexes. It's almost automatic.

Now we're going to define this new category in the following way. We'll take all geometric objects, $\tilde{L} = CF(\quad, L)$. We take $D\mathcal{F}(M)$ which, well you complete with respect to these triangles, $\langle \tilde{L} \rangle^{\Delta}$. So you have this geometric object and take all possible cones. You have new objects, new cones, and completed means you put in everything that can be generated with these.

$$D\mathcal{F}(M) = H_*(\langle \tilde{L} \rangle^{\Delta})$$

Taking homology identifies all the choices I made up to equivalence.

Now, what's a great property of this? It's triangulated. It's clear it is, sort of, morally, you just pass to homology. Then the K_0 group I mentioned in my title, that's the Grothiendieck group attached to this category $K_0(D\mathcal{F}(M)) = \langle A \rangle / (A \to B \to C) \rightsquigarrow A + C = B$.

All right, good, so now we want to go from geometry to algebra. That's the purpose of the talk. So the first thing is that there is a functor from this Lagrangian cobordism category with image in the derived Fukaya category of M.

The functor on objects sends geometric objects to themselves. Why is this of interest? In particular if you want to look to Floer homology relative to L, then you can view it as a sort of topological field theory except it's not quite that, you take $H_L(X)$ and view this functor $\hat{f}$, the Floer homology $HF(X, L)$. This writes Floer homology relative to L as a thing defined on a cobordism category, monoidal, lands in linear spaces. It's kind of a systematic way to look at this.

So what else? The next stage that I wanted to mention, a very important property for this category, is that it's triangulated. In fact, this functor behaves sort of nice with respect to the triangulation. One property that's important is a group morphism from the cobordism group into the K group,

$$\hat{\mathcal{F}} : \Omega_{Lag}(M) \to K_0(D\mathcal{F}(M)).$$

On objects all this stuff is trivial. This is induced by $\mathcal{F}$. However, of course the content of the statement is that the relations match. We had these relations that looked like multiple ends of the cobordism. This is non-trivial. Now it turns out that this sort of setup allows you to prove some additional facts concerning this group K_0.

So then I wanted to mention a few properties. One way to look at this morphism is to say, you have something geometric on one side and something algebraic on the other side, and this is π_1 of the classifying space of the triangulated category. I read that this thing exists. It's some sort of π_1 of something. So the reason it's maybe interesting is that you could look to the higher parts.

What else do I want to say? Sometimes these groups are, in symplectic topology mirror symmetry is very important. When mirror symmetry applies, you can put things on the B-side. Tang did this for T^2. In that case he showed that this morphism is an isomorphism.

Now essentially I have twelve more minutes. I can do two things. I can talk about what I was supposed to talk about, some properties of K_0, or I can go and talk a little about how this is related to the free loop space. Let's say the properties I wanted to talk about, I'll talk about them some other time. One property is that it's an algebraic cobordism group. You have relations given by cobordisms that you complete algebraically. This lets you write the whole thing as a sort of algebraic cobordism group. They also satisfy a stability condition. Then there's some more stuff if you look at Lefschetz fibrations. Now I'll show a simple example. The simple example is related to the $m_{1,0}$ space that is associated to some almost complex structure. In this case I fix my Lagrangian, I put that on the side, and my point right now is that the constructions, there is an interplay between symplectic topology and the free loop space. We're in a very special case. When I write a moduli space it'll be of a disk with boundary on the Lagrangian. If you remembered, we wanted also the Maslov class. I don't even separate them.

This space, Denis computed the dimensions, it's n. Kenji made the picture of the disk with one marked point on the boundary. I can evaluate $m_{1,0}(J) \to L$, which is a map of n-manifolds, so I can take its degree, call this n_L. Denis called this $\sum n$.

Remark 23.1. *(Chekanov) If you have a cobordism and want to compare the number at one end to the other ends. The n_L should be the same as all the other ends n_{L_i} (assuming connectivity of V.) Monotonicity ensures that the manifold is closed.*

The proof, you can look at a point in the fiber, and another point in the fiber over the other. Join the points by a path. By reasons related to monotonicity, this will be a manifold. There is one more step, a priori you don't know that all these holomorphic disks are in the fiber. They need to be in the fiber for n_L. First I prove that the number of disks is the same. Otherwise I don't have... Take an almost complex structure with holomorphic projections. A disk going through this point goes through the straight line, so apply the open mapping theorem. Then it's infinite area. They're vertical. That's Chekanov's beautiful argument.

We still haven't gotten to free loops. So now I want to take L to be a torus, maybe of dimension two, and here I want to take a triangle. I mean I have three points, and three simple curves. I'll look to the disks that pass through a vertex that cross the opposite edge. It's a strange number. So n_A is the number of disks through A crossing BC. This I can define. Then I write a number $\Delta = n_A^2 + n_B^2 + n_C^2 - 2n_An_B - 2n_Bn_C - 2n_Cn_A + n_{ABC}$. Here you need a spin structure and everything is over $\mathbb{Z}$. Now what's the statement about this? I should assume that the triangle is contractible. The claim is that this number is invariant with respect to all the choices. Then Δ is invariant with respect to ABC, if I split one end with my cobordism, then Δ on the single end is not a perfect square. I can take $m_{1,0}(J)$ and send the boundary to the free loop space. Then this I can rewrite in terms of Hochschild homology or cohomology. This is a ridiculous space, $HH^*(C^*(L), C^*L)$, so I get this to be a bilinear map on the cohomology and to this bilinear map I can associate a quadratic form and Δ is the discriminant. I'm done for my talk, but I want to say a work. So the basic idea is this is a very bizarre invariant. Kenji said these open closed maps can live in certain enumerative invariants. This is an example of a thing you can do. The things that made up Δ are not invariant but Δ is an invariant. The reason is that one one side you can look to the moduli space of disks. In the free loop space, you get Hochschild cohomology which parameterizes deformations classically. The product in the algebra that shows up here, the deformations that show up are the quantum products. Then this invariant is the same as the discriminant of the quadratic form. Okay, that's one little example how the loop space is related. The map is related to open closed and closed open. This is an example of an enumerative invariant. Thank you very much.

24. Dennis Sullivan: Algebraic models of manifolds

I had to leave early and was not present at this talk.