MAX PLANCK TOPOLOGY SEMINAR

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1. June 22: Benjamin Matschke: Arithmetic topology (Part I)

[Welcome to our MPI topology seminar. This semester we're running it in a more unusual way. We arranged for people to get together to learn about certain projects and together to give a talk about that topic. The talk today will be in two parts.]

I see many experts in topology and also number theory. I will give a lot of detail for the number theory since this is a topology seminar.

This is about arithmetic topology, particularly about primes and knots. There is an older story about who looked at the relationship between primes and knots. It started probably with Mumford, Manin, and Mazur, who found an interesting relationship. Well, arithmetic topology tries to relate topology to algebraic number theory. In particular, it turns out that certain 3-manifolds, usually oriented and closed, correspond to number fields. If you have not seen this before, this is not a bijection, more of an analogy, somewhat weak, but if you take the right definitions in three-manifolds, you will be able to translate them.

In some sense, knots, that is, tame embeddings of the circle into $\mathbb{R}^3$, correspond to $\operatorname{spec}\mathbb{F}_p$ in $\operatorname{spec}\mathbb{Z}$. In order to speak about the number theory side, let me define spec.

Throughout the talk, rings will always be commutative with 1.

Definition 1.1. Let R be a ring. As a set, $\operatorname{spec} R$ is the set of prime ideals p in R. This is topologized to become a topological space. Take the basis given by U_a, the prime ideals p that do not contain a.

There is a sheaf, which associates to U_a the set $\{\frac{f}{a^n} | f \in R\}$.

Here are examples. The easiest example is when F is a field. Then *Spec F* is just $\{0\}$, so topologically it is just a point, but it has extra structure which makes it much more rich than just a point. From the point of view of, well, of Galois theory, of the associated étale cohomology, it's reasonable to consider it not just as a point, but the quotient field of some finitely generated ring. So if $F = Quot(R)$ for R a domain, then you can regard F as the limit of $S^{-1}R$ over finite subsets of $R\backslash 0$.

If you look at the spec of all these localizations first, they will be much closer to the fundamental group that I will define later. Maybe I should come back to this later.

The next example, $\operatorname{spec}\mathbb{Z}$ contains the prime ideals $\{p\mathbb{Z}\}$ and $\{0\}$.

If you take $\mathbb{F}_p[t]$, It has spec equal to $\{(f)\}$ for f monic irreducible and 0.

I said that knots in $\mathbb{R}$ were like $\operatorname{spec}\mathbb{F}_p$ inside $\operatorname{spec}\mathbb{Z}$. In topology, what is S^1? It is $K(\mathbb{Z},1)$. If I look at finite connected covers of S^1, I get just S^1. So $\pi_1(S^1) = \mathbb{Z}$ and higher homotopy groups vanish. The covers are multiplication by n, and there is a universal cover $\mathbb{R}$. Is there a spec that has fundamental group $\mathbb{Z}$? I haven't defined that yet.

For each prime, and each prime power, there is a field with that many elements, I can look at field extensions $\mathbb{F}_{p^n}/\mathbb{F}_p$, and I will consider them as coverings. Maybe I should define π_1 to make this picture. I should also define what is an étale map. Suppose I have two rings.

Definition 1.2. An A-algebra B where $A \hookrightarrow B$ is a ring homomorphism, is called finite étale if

1. $A \to B$ makes B into a flat A-module, that is, $B \otimes_A$ is exact as a functor of A-modules, and
2. for each $p \in \operatorname{spec} A$, we have $B \otimes_A K(p)$ is a finite product of finite separable extensions of $K(p)$, where $K(p) = Quot(A/p)$.

In our correspondence, coverings correspond to étale morphisms.

When we talk about covering spaces and π_1, we'd like to talk about normal coverings, the image of $\pi_1(Y)$ inside $\pi_1(X)$ is a normal subgroup. How is the corresponding thing on the other side, a *Galois covering*, defined? First we need a basepoint. A base point for $\operatorname{spec} A$ at $p \in \operatorname{spec} A$, is a morphism $\operatorname{spec} \Omega \to \operatorname{spec} A$, where Ω is an algebraic closure of $K(p)$.

So a basepoint of $\operatorname{spec} \mathbb{F}_p$ is a choice of an algebraic closure of $\overline{\mathbb{F}}_p$ with a morphism of $\mathbb{F}_p$ into it.

The fiber $F_{x_0}(Y = \operatorname{spec} B) = Hom_{X=\operatorname{spec} A}(x_0, Y) = Hom_A(B, \Omega)$. We call $Y \to X$ a Galois covering if it is finite étale and $Gal(B/A)$ acts on the fibers simply transitively.

Definition 1.3. The *étale fundamental group* $\pi_1(X, x_0)$ is the inverse limit of $Gal(Y_i, X)$ over all Y_i which are finite connected Galois covers of X.

Before I continue this picture, let me give you an example.

$\pi_1(\mathbb{F})$ is the Galois group of $\overline{\mathbb{F}}$ over $\mathbb{F}$, where $\overline{\mathbb{F}}$ is the separable closure. This turns out to be generated for $\mathbb{F}_p$ by the Frobenius operator σ which sends x to x^p. This group is $\hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$, that is, the inverse limit of $\mathbb{Z}/n\mathbb{Z}$. For each n there is the field $\mathbb{F}_{p^n}$, which is an extension over $\mathbb{F}_p$ and has Galois group $\mathbb{Z}/n\mathbb{Z}$. So an irreducible polynomial, the roots are just permuted cyclically. Similarly, the points in the fiber in the finite cover of S^1 are permuted cyclically.

What is the inverse limit of these groups? It's just $\hat{\mathbb{Z}}$, the profinite completion, the group that comes closest to $\mathbb{Z}$.

What is an étale map between, suppose I have two number fields K over F, then the ring of integers $\mathcal{O}_L \to \mathcal{O}_K$ is finite étale if and only if L/K is unramified, so $p \in \mathcal{O}_K$, and I can write $p = \prod P_j^{ej}$ and this is a finite product, and unramified means that e_j is 1 for all j.

Then $\pi_1(\mathcal{O}_K)$, well, for $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$, this is the Galois group of the maximal unramified extension of K over K.

So does this make sense, to think of $\mathbb{F}_p$ as the circle?

[some discussion].

So what does one study in knot theory? Knots and their complements and the fundamental group of the complement. If you have a manifold M, and you take away the knot, call the complement X_K, and this corresponds to $\pi_1(\operatorname{spec} \mathcal{O}_K \backslash \{p\})$; in topology this is called the knot group, and some people call this the "prime group." On the side of topology, one also considers a tubular neighborhood. Why? For example, if you have a knot and take a tubular neighborhood,, one is interested in the meridian generator.

First of all, what is the tubular neighborhood of a prime ideal? Before answering this, there is an analogy that number theory people consider very very often, which is the analogy between number fields and function fields. So $\mathbb{F}_p[t]$, or let me go further to $\mathbb{C}[t]$, if I consider power series, $\mathbb{C}[[t]]$, what would spec of this be? It's basically an infinitessimally small disk. Power series are defined on a small disk and determined by what they look like on it. How is this constructed? It's the inverse limit of $\mathbb{C}[t]/\langle t^n \rangle$. What does this correspond to on the number theory side? I can take $\mathcal{O}_K$, the ring of integers if $\mathbb{Z} = \mathbb{Q}$ and mod out, $\lim \mathcal{O}_K/p^n = \mathcal{O}_p$, the p-adic integers. This is a good candidate (well its spec is) for a neighborhood of $\operatorname{spec} \mathbb{F}_p$. What is $\operatorname{spec} \mathcal{O}_p$? It's just $\{(O), (\mathfrak{p}\mathcal{O}_\mathfrak{p})\}$. Our two residue fields are K_p and $\mathbb{F}_q$. If we remove $\operatorname{spec} \mathbb{F}_p$ you get $\operatorname{spec} K_p$ as desired. If $K = \mathbb{Q}$ then $\mathcal{O}_\mathfrak{p} = \mathbb{Z}_p$ and $K_p = \mathbb{Q}_p$.

I hope I can convince you somewhat that this is a good candidate for a tubular neighborhood of S^1 in M. What is a tubular neighborhood? It's $S^1 \times D^2$, so the last equation should remind you that $V \backslash S^1$ you get something that is homotopy equivalent to the boundary of V which is just a torus. If you take $\pi_1(\partial V)$ and map it to $\pi_1(V) \to 1$, this is a surjection with kernel a meridian and image a longitude. So $\pi_1(\partial V)$ is $\langle \alpha, \beta \rangle$, where the only relation is that they commute.

What's the number field analogue? It goes like this. We have $\pi_1(\operatorname{spec} K_p)$ mapping to $\operatorname{spec} \mathbb{F}_p$, which was generated by $\langle \sigma \rangle$, the Frobenius element. The kernel is some inertia group I_{K_p}. I want a particular element of this group that corresponds to β. Previously I told you that $\operatorname{spec} K_\mathfrak{p}$, well, you should take the separable closure of $K_\mathfrak{p}$. Below we have K and its maximal unramified extension $K_\mathfrak{p}^{un}$. Then you can further extend to the maximal tame extension, defined so that at each prime, does this make sense? All the exponents you get, they should be prime to p. Of course an unramified extension is tame since all the e_i are 1. So $K_\mathfrak{p}^{un} = K_\mathfrak{p} p(\zeta_n) | (p, n) = 1$. Then you have $K_\mathfrak{p}^t = K_\mathfrak{p}(\sqrt[n]{\pi})$ for π a prime element in K_p and $(p, n) = 1$. Then $\pi_1(\operatorname{spec} K_p)$ projects down to $\pi_1^t(\operatorname{spec} K_p) \coloneqq Gal(K_p^{/} K)$, where $\tau(\sqrt[n]{\pi}) = \zeta_n \sqrt[n]{\pi}$ and $\tau \zeta_n = \zeta_n$.

Let me also write down the fundamental group. So $\pi_1(K_p)$ is generated by σ and τ subject to $\sigma \tau \sigma^{-1} = \tau^p$. So how does this correspond to the torus? This is in some sense a reason why the boundary of this tubular neighborhood looks like a surface. If we didn't have p, er, if $p = -1$ then this would be the Klein bottle. So someone suggested I should think of this as a Klein bottle because it's unoriented. There's some Poincaré duality that is 2-dimensional, but with a twist, so this is an analogue of that.

I should also say, I didn't define it, but the étale cohomology of $\mathbb{Z}$ and arbitrary number rings, satisfies some three-dimensional duality, and this also suggests that actually number rings are three-dimensional, at least up to higher étale cohomology, which may have 2-torsion. Okay, thanks, time's up.

2. Mikhail Kapranov: Arithmetic topology, Part II

[In the second half of the seminar, Mikhail Kapranov will tell us more about this fascinating connection between topology and number theory.]

I will also be elementary, maybe even more elementary. Let me summarize the first hour.

(1) spec($\mathbb{F}$) is not a point, but something like the classifying space of $Gal(\overline{\mathbb{F}}/\mathbb{F},1)$. So
(2) spec($\mathbb{F}_q$) is something like a circle, because a circle, the coverings are classified by natural numbers and so are the extensions.
(3) spec($\mathbb{Q}_p$) is something like a Riemann surface.
(4) spec($\mathbb{Z}$) is something like a three-manifold, not fully compact.

That's basically the point of view. So a prime number in $\mathbb{Z}$ is something like a circle in a 3-manifold.

[Why is #3 true?]

Let's look instead at $\operatorname{spec}\mathbb{F}_p[t]$, well $\operatorname{spec}\mathbb{F}[t]$ where $\mathbb{F}$ is algebraically closed, this is like $\mathbb{A}^1_{\mathbb{F}}$, this should be 2-dimensional. So then you have a circle fibration, you have like $\operatorname{spec}\mathbb{F}_p$ and over it you have a fibration with 2-dimensional fibers, an affine line over something should be 2-dimensional.

So p can be knotted, linked, et cetera. We can try to understand such phenomena in an elementary way. Let me discuss how we can say if two such circles can be linked.

Let me start with an elementary survey of the linking number in topology.

2.1. **Linking numbers.** If you have two circles in a sphere or other three-manifold, M^3 a compact oriented 3-manifold, and $\mathcal{C}$ and $\mathcal{D}$ two disjoint oriented circles.

There are several definitions, some better suited to arithmetic than others.

2.1.1. *via intersection.* Assume that $[\mathcal{C}]=[\mathcal{D}]=0$ in $H_1(M,\mathbb{Z})$. Then take a chain σ whose boundary is $\mathcal{C}$ and then $lk(\mathcal{C},\mathcal{D})=\sigma\bullet\mathcal{D}$. If you choose a different σ', then $\sigma-\sigma'$ is a 2-cycle and $((\sigma-\sigma')\bullet\mathcal{D})_{hom}=0$ by triviality.

It's important that this is symmetric, we can use τ such that $\partial\tau=\mathcal{D}$. We can also do a version of this modulo n. Then we need only the class of $[\mathcal{C}]=[\mathcal{D}]=0$ in $H_1(M,\mathbb{Z}/n)$, which is $H_1(M,\mathbb{Z})/n$. This generalizes to cycles of arbitrary dimension, dim M, dim $\mathcal{C}$, and dim $\mathcal{D}$ arbitrary except that $\dim\mathcal{C}+\dim\mathcal{D}+1=\dim M$. For $M=S^n$ this gives Alexander duality. For any K this gives a pairing $H_i(K)\otimes H_{n-1-i}(S^n-K)\to\mathbb{Z}$ or in otherwise $H^{n-1-i}(S^n-K)=H_i^{red}(K)$. This is the most common point of view but requires a working homology theory.

2.1.2. *via coverings.* Fix $\mathcal{C}$ and $\mathcal{D}$. Take a $\mathbb{Z}$-covering of the complement of $\mathcal{C}$, call it $\widetilde{S}\xrightarrow{p}S^3-\mathcal{C}$. This uniquely exists by the knowledge that $\pi_1^{ab}=H_1$. Then $\mathcal{C}$ and $\mathcal{D}$ don't meet and we can take the monodromy along $\mathcal{D}$ and that's the linking number. Similarly, there is a $\pmod n$ version, where we take a $\mathbb{Z}/n$ covering. There is a bonus, which is that $\widetilde{S}$ extensd to a compact manifold ramified along $\mathcal{C}$ like for Riemann surfaces.

Let me mention a particular case. For $n=2$, we have $lk_2(\mathcal{C},\mathcal{D})$ is 0 if the monodromy is trivial, that is, if $p^{-1}(\mathcal{D})$ is two circles. It's 1 if $p^{-1}(\mathcal{D})$ is one circle.

This is for a sphere. For a manifold, it's similar. We used the existence of the covering, which is equivalent to the fact that the knot is homologically trivial. For the modulo n version, the existence of $\widetilde{M}\xrightarrow{p;\mathbb{Z}/n}M$ ramified along $\mathcal{C}$ is equivalent to the fact that $[\mathcal{C}]$ vanishes in $H_1(M_1,\mathbb{Z}/n)$. Compare that a double cover of $S^2=\mathbb{CP}^1$ is ramified at an even number of points.

2.1.3. *via cup product.* Let me make a picture [picture]; so we take two clases γ,δ in $H^1(S^3-\mathcal{C}-\mathcal{D})$ which is by Alexander the same as $H_1(\mathcal{C}\cup\mathcal{D})$ generating the circles. Then $z\in H^2(S^3-\mathcal{C}-\mathcal{D})$ is the same by Poincaré–Lefschetz duality as

$H_1(S^3, \mathcal{C} \cup \mathcal{D})$. Then $\gamma \cup \delta = lk(\mathcal{C}, \mathcal{D})z$. Why is this symmetric where cup product is antisymmetric? Because we had to orient from $\mathcal{C}$ to $\mathcal{D}$ to get z.

So now let me discuss the arithmetic analogue

2.2. **Legendre symbols as linking numbers.** Instead of M^3 we consider $\operatorname{spec}\mathbb{Z}$. Instead of $\mathcal{C}$ and $\mathcal{D}$ we consider p and q. We take $n = 2$. So first of all, what is the analogue of a double cover $\pi : \widetilde{M} \xrightarrow{2:1} M$ ramified along p? Suppose I have $X = \operatorname{spec}(A)$ and inside we have $Z \subset \operatorname{spec}(A)$ which is $\{f = 0\}$. Then naively, a double cover ramified along Z_f is $\operatorname{spec}(\mathcal{A}[\sqrt{f}])$. [Picture]

So we take $\widetilde{M} = \operatorname{spec}(\mathbb{Z}[\sqrt{p}])$ which projects to $M = \operatorname{spec}\mathbb{Z}$ inside of which we have $\operatorname{spec}\mathbb{F}_q = \mathcal{D}$. So we look at $\pi^{-1}(\operatorname{spec}\mathbb{F}_q)$. That's $\operatorname{spec}\mathbb{Z}[\sqrt{p}]/q$. That's $\operatorname{spec}\mathbb{F}_q[\sqrt{p}]$. Originally, this is $(\mathbb{Z}[y]/(y^2 = p))/q = \mathbb{F}_q[y]/(y^2 = p)$. So I should ask if this is one circle or two circles?

If it's one circle, it should cover twice and be $\operatorname{spec}\mathbb{F}_{q^2}$; otherwise it's two circles $\operatorname{spec}(\mathbb{F}_q + \mathbb{F}_q)$. We say respectively that the linking number of p and q is 1 or 0. But this is precisely, more or less, how the Legendre symbol is defined.

So one circle means $y^2 - p$ is irreducible in $\mathbb{F}_q[y]$ which means that p is not a quadratic residue $\pmod q$ (that is, $\left(\frac{p}{q}\right) = -1$); two circles means it is a quadratic residue ($\left(\frac{p}{q}\right) = 1$). But this is not symmetric. Gauss' reciprocity tells us that $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)(-1)^{\frac{p-1}{2}\frac{q-1}{2}}$ while linking numbers are symmetric.

I did something naively before which is why this is happening. When I adjoined a square root of p, it's not clear that things will only be ramified along p. Let me explain this.

So the explanation is that $\mathbb{Z}[\sqrt{p}]$ is not only ramified along p. How can we understand more precisely where it is ramified? Say you have $F(x,y) = 0 \subset \mathbb{C}^2$. Then $\operatorname{spec}(\mathbb{C}[x,y]/F(x,y))$ maps to $\operatorname{spec}\mathbb{C}[x] = \mathbb{A}^1$. The ramification locus is given by the set of x for which $\operatorname{Discr}_y F(x,y) = 0$. For any A (not just $\mathbb{A}^1$) and $F \in A[y]$, $\operatorname{spec}(A[y]/F \to \operatorname{spec}(A)$ is ramified along $\operatorname{Discr}_y F = 0$ in A. Now let's try to apply this to our polynomial $y^2 - p$.

In our case, $A = \mathbb{Z}$, $F(y) = y^2 - p$, and so the discriminant of F is $4p$. This *always* vanishes at 2. Moreover, if I do something, like $\operatorname{spec}(\mathbb{Z}[\sqrt{p}])$ may not be smooth. So far we have not distinguished by what happens at 2.

In elementary number theory, there's the concept of the "full ring of integers" and we need to desingularize, passing to all integer elements in $\mathbb{Q}(\sqrt{p})$, a bigger ring we'll call A_p. We remember from elementary number theory how this is defined. It actually depends on whether p is a prime of the form $4k+1$ or $4k+3$.

For $p = 4k+1$, the singular behavior happens but in $\operatorname{spec}A_p$, this behavior goes away.

For $p = 4k+3$, the behavior, the extra ramification, cannot be removed.

So the prime p should be thought of as homologically nontrivial $\pmod 2$.

For p and q both of form $4k+1$, we have $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$. One can similarly define, I shouldn't go there, modulo higher n, this corresponds to power residue symbols and this is sort of similar.

I should say at this point, one can approach this from various points of view. You can prove étale cohomology to prove reciprocity which as I understand it was the original motivation of Mazur. Let me discuss another subject of intuitive appeal, which is

2.3. **Higher linking numbers.** So you might have pairwise unlinked circles which together are linked.

The most algebraic approach is via Massey products. we have (R, d) an associative differential graded algebra. Suppose we have three elements in $H^1_d(R)$, α,β,γ, such that $\alpha\beta = \beta\gamma = 0$. Then choose 1-cocycles a, b, c, representing α, β, and γ. Then we can write $ab = d(u)$ and $bc = d(v)$. We can write $z = av + uc$, then $d(z) = 0$ by associativity. This 2-cocycle is denoted $[\alpha, \beta, \gamma]$, and is defined modulo some indeterminacy, modulo $\alpha H^1 + H^1\gamma$. When such things vanish, we can dig even deeper and get finer and finer homology classes.

So more generally, we can write $[\alpha_1, \ldots, \alpha_n] \in H^2$ modulo some indeterminacy, defined when all $[\alpha_i, \alpha_{i+1}, \ldots \alpha_j]$ are 0 modulo indeterminacy.

We represent α_1 by $a_{0,1}$, α_n by $a_{n-1,n}$, then we have $a_{0,2}$, all the way through $a_{0,n-1}$ and $a_{1,n}$. Here we should have $d(a_{ij}) = \pm \sum_{i<k<j} a_{ik}a_{kj}$ and $[\alpha_1, \ldots, \alpha_n]$ is the class of $a_{0,n-1}a_{n-1,n} \pm a_{0,1}a_{1,n}$.

The pattern here is something like $(C^*_{\mathrm{Lie}}(\mathfrak{n}), d)$ where $\mathfrak{n}$ is the strictly upper triangular (nilpotent) Lie algebra. If you let a_{ij} be dual to z_{ij} then you get precisely that $d(a_{ij}) = \pm \sum a_{ik}a_{kj}$.

This can be applied to knot complements.

Suppose I have three knots in S^3, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, and their linking numbers are pairwise 0. Then $\alpha_i \in H^1(S^3 - C)$ is the same by Alexander duality as $\cup C_i$ and $\gamma_{ij} \in H^2$ is by Lefschetz duality a path from C_i to C_j. Let $R = C^*(S^3 - C)$. I'm cheating a little but it will take time to do correctly. Then $[\alpha_1, \alpha_2, \alpha_3] = \lambda\gamma_{13}$ and then λ is $lk(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$. In the literature, if $lk(C_i, C_j) =\neq 0$ then $lk(C_1, C_2, C_3)$ is defined module the gcd.

We can do this via unipotent coverings. $lk(\mathcal{C}_1, \mathcal{C}_2) = 0$ if and only if there is a covering $\Sigma \to S^3 - C_1 - C_2$ with Galois group $N_3(\mathbb{Z}) = \{\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}\}$. Look at the monodromy of $\mathcal{C}_3$, then a is $lk(\mathcal{C}_1, \mathcal{C}_3)$, the b is $lk(\mathcal{C}_2, \mathcal{C}_3)$, and c is $lk(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$.

You can also do this via Fox calculus.

This definition is such taht it can be generalized to the arithmetic case which has also been done in the classical literature.

I'm out of time, can I have like ten more minutes?

2.4. **Higher Legendre symbols.** These were defined by Redei in 1939. Suppose that p_1, p_2, and p_3 are all primes, let's assume all are $4k + 1$ for simplicity. Assume that $\left(\frac{p_i}{p_j}\right) = 1$ for all i, j. Then there exists a $[p_1, p_2, p_3] \in \{\pm 1\}$ defined via Heisenberg extension $\pmod 2$. One can go further and this was done by Morishita using Fox calculus. One can probably also use Massey products in étale cohomology.

I want to say a little more, how those symbols have been applied to the problem of understanding homology of étale coverings.

2.5. **2-class groups of quadratic fields and cohomology of ramified covers of S^3.** So for M^3 we have $\mathrm{spec}(A)$ for A a ring of integers in k and $H_1(M, \mathbb{Z})$ as $Cl^+(A)$, the divisors on A modulo princial divisors.

Look at the class group of the field $k = \mathbb{Q}[\sqrt{p_1 \cdots p_n}]$, If we take $Cl^+(A_k) \otimes \mathbb{Z}/2$ you get a vector space over $\mathbb{Z}/2$ of dimension $n - 1$.

This has a topological analogue that is a pleasant exercise. Take a two to one map $\widetilde{S}^3 \to S^3$. The dimension of $H_1(\widetilde{S}^3, \mathbb{Z}/2) = n-1$. What is the structure of the 2-primary part of $Cl^+(A_x)$ or of $H_1(\widetilde{S}^3, \mathbb{Z})$? How many $\mathbb{Z}/4$ or $\mathbb{Z}/8$ are there? This problem was studied by Redei who proved the following thing. The number of $\mathbb{Z}/2^a$ for $a \geq 2$ in $Cl^+(A_k)$ is $(n-1) - rk\|\left(\frac{p_i}{p_j}\right)\|$ adjusted appropriately on the diagonal so that the sum over the rows is 0. The same is true for $\widetilde{S}^3$, proven by Morishita. The number of $\mathbb{Z}/8$ is governed by triple symbols.

Now just one last thing I want to say, is that all of this can be understood in the context of the moduli space of representations, an Abelian analogue of local systems.

How can one see that something like this will actually appear? Conceptually, $\mathbb{Z}/2^n$ is $((\mathbb{Z}/2)[t]/t^n)^*$, the Witt vector. This is, if $M = \widetilde{S}^3$, consider local systems of dimension 1 of $\mathbb{F}_2$ over M, then $LS^{\dim 1}_{\mathbb{F}_2}(M)$ is a scheme over $\mathbb{F}_2$. This scheme has only one point but possibly infinitesimal structure. The tangent space at the trivial representation is $H^1(M, \mathbb{F}_2)$, and deformation theory tells us, these are first order deformations, that the second order deformations are given by the condition $\xi \cdot \xi = 0$. The third order via $[\xi, \xi, \xi] = 0$, and so on. These have to do with how many nilpotent curves can be put there. All this part is like the Abelian theory of local systems. Part of this is again, very common in number theory, but there are more questions. Thank you very much.

3. June 30: Francois Charette, Ana Ros Camacho, Hiro Tanaka: Introduction to Homological Mirror Symmetry via Examples

3.1. **Hiro.** Thanks for letting us study in triplets instead of in pairs. I want to state the theorem that we will prove if we have time.

Theorem 3.1. *There exists an equivalence of A_∞ categories — I'll put four categories on the board that you may not know. We have two hours to give some examples in these categories. The first category is the Fukaya category $Fuk_\lambda(\mathbb{CP}^1)$. The second is matrix factorizations on $\mathbb{C}^\times$ with respect to $W - \lambda$ for a function W which is more or less $\frac{1}{z} + z$. So the second category is $MF(\mathbb{C}^\times, W - \lambda)$. These two are equivalent. Then the category $D^bCoh(\mathbb{CP}^1)$ is equivalent to the Fukaya–Seidel category of $\mathbb{C}^\times$ with respect to W.*

What is our rough outline? François will talk about Fuk_λ, Ana is our matrix factorization artist, and if I have time I'll talk about the second equivalence.

Conveniently, someone wrote down a Hodge diamond. Let me give a brief history of the evolution of mirror symmetry.

0 It's kind of like saying a poem about love when you've never experienced it, but you can still recite it. So first, there are $N = 2$ supersymmetric conformal field theories. So you could think of this as vector spaces with operators $\mathcal{F}$, and there's a conjugation that you can do, and if you do that conjugation, you get a new field theory $\mathcal{F}^\vee$. Some of these arise from Calabi–Yau threefolds X. There's a mirror Calabi–Yau threefold $X^\vee$ so that their conformal field theories are mirror.

So there are topological sectors where you can extract information about the field theory, doing calculations, and see that these depend on information about the Calabi–Yaus.

a So for example, you can figure out their Betti numbers and see that $h^{p,q}(X) = h^{n-p,q}(X^\vee)$ where n is the complex dimension of X. This is the origin of the word "mirror symmetry."

$$\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}$$

b Counting rational curves on X turns out to correspond to computing periods of variations of Hodge structures on $X^\vee$.

c More generally, $H^*(X, \bigwedge^* TX) \cong H^*(X^\vee)$ with the quantum product.

1 The next stage was Kontsevich. For every Calabi–Yau manifold X, there is a Calabi–Yau manifold $X^\vee$ such that $D^bCoh(X)$ and $DFuk(X^\vee)$ are equivalent, the computations of mirror symmetry are a reflection of this.

Complex subvarieties of X are interchanged with Lagrangians because of the mirror symmetry, this tells you about (a). Then the other two calculations have to do with Hochschild cohomology, $HH^*(D^bCoh(X)$ is $H^*(X, \bigwedge^* TX)$, and (at least in some cases), $HH^*(DFuk(X^\vee))$ is $H^*(X^\vee)$ with the quantum product.

2

Conjecture 3.1. (Kontsevich) For any Fano (hyperspace of projective space), Calabi–Yau, or general type X, there is a mirror $X^\vee$ and $W : X^\vee \to \mathbb{C}$, such that

$$Fuk_\lambda(X) \cong MF(X^\vee, W - \lambda)$$

and

$$D^bCoh(X) \cong FukSei(X^\vee, W).$$

Because our categories are nice, you want to find generators for the categories, and then compute the endomorphism algebras of them, and then that's enough to show that they are equivalent.

3.2. **François.** All right, so um, I will talk about the Fukaya category. Any talk about this that wants to be precise is impossible. I'll give the spirit behind it. I'll mostly focus on the generator and we'll try to do computations with that. Once you reduce it to something computational your life becomes easy. So what is the setting? Let (M,ω) be a compact symplectic manifold and consider L, closed Lagrangians. Why would we care about Lagrangians?

Theorem 3.2. *(Gromov)*

- *In* $(\mathbb{C}^n, \omega_0 = \sum dx_i \wedge dy_i)$*, any closed Lagrangian satisfies* $H^1_{dR}(R) \neq 0$*, so thot* $\beta_1(L) > 0$.
- *There is an exotic symplectic structure on* $\mathbb{C}^n$ *so that* S^n *is Lagrangian. For* $n > 1$*, this shows that Lagrangians can distinguish certain symplectic structures*

Theorem 3.3. *(Chekanov) Under small physical perturbations* φ*, (we should think of a symplectic manifold as where we do mechanics), that is, a Hamiltonian, then* $\#L \cap \varphi(L) \geq \sum \beta_i(L)$*. You might call this the Arnold conjecture.*

The topological bound is $\chi(L)$. When you study Lagrangians, something symplectic and not purely topological happens.

Now we do a big leap to the present and study $Fuk(M,\omega)$. Let me give two references, one to Auroux, *a beginner's introduction to Fukaya categories* and if this is too easy, SEidel's book *Fukaya categories and...*

Let me hide what I can. The objects are Lagrangians. We'll need some technical restriction on the objects. The morphism space should be thought as, well, $HF(L_0, L_1)$, the Floer homology groups. I'll define only $HF(L_0, L_0)$. What one should know is that these admit a product structure like composition, you have a product $HF(L_0, L_1) \otimes HF(L_1, L_2) \to HF(L_0, L_2)$. What do we do with this category? It's not triangulated so we can't speak about generators.

Now I'll use the same notation but work at the chain level, $Mor_{Fuk}(L_0, L_1) = CF(L_0, L_1)$. You get an A_∞ product on the chain level. There's one little difference that I won't write. Let me attribute this to Oh. $\mu^2 = \lambda(L_0) - \lambda(L_1)$. So we restrict to the objects where $\lambda = 0$. That's the only case where experts don't argue. If you restrict to this then everyone's happy. So we restrict to $\mathcal{F}_\lambda(M)$, where the Lagrangians have fixed obstruction number λ.

Now we use some theorems.

Theorem 3.4. *(Kontsevich–Seidel, Auroux, Sheridan)—I'll give a baby version. For* $\mathbb{CP}^n$*,* $Fuk_\lambda(\mathbb{CP}^n, \omega_{FS})$ *is trivial unless* $\lambda = (n+1)e^{\frac{2\pi i k}{k+1}}$ *for* $k = 0, \ldots, n$*. Here, trivial means that* $HF(L,L) = 0$*.*

This is not obvious. For $n = 1$ this is $\lambda = \pm 2$.

Theorem 3.5. *For* $\mathbb{CP}^n$*, when nonzero,* $Fuk_\lambda(\mathbb{CP}^n)$ *is generated (under triangulated closure) by one Lagrangian, the Clifford torus.*

Let me write $CF(L_1)$, not as a complex but as a vector space, as $\mathbb{C}\langle L_0 \pitchfork L_1 \rangle$, when transversal. There are technicalities when these are not transversal.

Now we move to the theorem that we aim to prove, the endomorphisms of S^1, well, $\mathbb{CP}^1$ is a sphere and so Lagrangians are circles. The technical conditions make this the equator. In this case, you can see why this is necessary; you can make two circles that are not equators disjoint by a physical motion.

Now let me talk about the superpotential. This is undegraduate complex analysis. This will be a function encoding holomorphic disks with boundary on L. There are two, the top and the bottom. Let me define a moduli space $\mathcal{M}(A)$ and $\mathcal{M}(B)$, where these (A and B) are in $\pi_2(M, L)$. Then

$$\mathcal{M}(A) = \{u : (D^2, S^1) \to (M, L) : [u] = A, \bar{\partial} u = 0\}$$

and likewise for B.

So for the two-sphere there is one disk in each class. Now I will define a function $W : \{\rho : \pi_1(L) \to \mathbb{C}^\times\} \to \mathbb{C}$. So for the circle, the representations are precisely determined by $\mathbb{C}^\times$ itself. So I'll write z for $\rho([1])$. Then $z \mapsto \#\mathcal{M}(A)\rho([\partial A]) + \mathcal{M}(B)\rho([\partial B])$. This will be finite in general if L is nice enough. So ρ of the top disk gives us $1 \cdot z + 1 \cdot \frac{1}{z}$. If L is a torus, you'll get $(\mathbb{C}^\times)^n$.

Let's solve Hiro's exercise,

$$\frac{\partial W}{\partial z} = 1 - \frac{1}{z^2}$$

which has critical points ± 1 with critical values ± 2.

Let's compute some homology of this. So one model (not Sullivan model) for $HF(S^1, S^1)$, called Lagrangian quantum homology, due to Biran–Cornea, is based on ideas of Fukaya later treated by Oh.

Take a good model for homology and take a Morse–Smale function $f : L \to \mathbb{R}$.

So I'm out of time so I'll write the answer, $End(S^1) = \mathbb{C} \oplus \mathbb{C}$, that is, $\langle P \rangle oplus \langle Q \rangle$. Now Q is a unit, and for $\rho = +1$ we have $P^2 = Q$. This is evidently not the singular cohomology. I didn't have time to motivate this model. I wanted to introduce the model because it's easier than Floer homology. For $\mathbb{CP}^n$ one can do the same thing. There are many things that one can do for endomorphisms of one object using it. In the second part we'll see $\mathbb{C}[h]/h^2 - 1$ again in a different disguise.

3.3. **Ana.** So I come from a very different area, I'll explain in a bit how matrix factorizations work and do some calculations, and show a little bit how the magic goes.

To begin with, fix a ring $R = \mathbb{C}[t, t^{-1}]$ and take $W \in R$ and we'll define

Definition 3.1. The function W is called a *(super)potential* if $\dim(R/\partial_t W)$ is finite. This quotient is the Jacobian ring of W.

There are several choices about how to do this, if you are curious about generalizations ask me or Toby.

Definition 3.2. A *matrix factorization* of W is a pair (M, d_M) where M is a $\mathbb{Z}/2$-graded free finitely generated module over R and d^M is an odd morphism such that $(d^M)^2 = Wid_M$. I'll also call these *twisted differentials* and denote them by M. A morphism of matrix factorizations $M \to N$ is an R-linear map, well, $MF(R, W)$ has objects matrix factorizations and morphisms which are morphisms of matrix factorizations which are compatible with the twisted differentials modulo nulhomotopic morphisms.

So these were studied starting around 1980. This category is monoidal, admits a triangulated structure, and these can be related to certain things in physics, conformal field theories, D-branes, maximal Cohen–Macauly modules, path algebras, and a practical result for our purposes is

Proposition 3.1. *(Dyckerhoff 2009) The Hochschild cohomology of $MF(R,W)$ is isomorphic to the Jacobian ring, concentrated in even degree.*

This will give the first contact with Mirror symmetry.

So like François and Hiro, we will focus on $W = t+\frac{1}{t}$. Predicition 1 of homological mirror symmetry is that $End(S^1, p = 1)$ is $Jac(W)$. Then $\partial_t W = 1 - \frac{1}{t^2} = \frac{t^2-1}{t^2}$, and then $Jac(W) \cong \mathbb{C}[t]/\langle t^2 - 1\rangle$. So this matches.

Let's let S be

$$\left(R^{\oplus 2}, \begin{pmatrix} 0 & t \\ \frac{t}{t-1} & 0 \end{pmatrix}\right).$$

with $W = \frac{t^2}{t-1}$ (after the change of variables $t \mapsto t-1$).

Proposition 3.2. *(Dyckerhoff, 2009) S generates $MF(R, t+\frac{1}{t})$.*

The proof is long so I'll skip it but we can compute something nicer. We want to compute $End(S)$. I'm going to write down explicitly what the matrix factorization looks like

$$\begin{tikzcd} R \arrow[r, "t"] \arrow[d, "f_0"] \arrow[dr, "f_1"] & R \arrow[r, "\frac{t}{t-1}"] \arrow[d, "g_0"] \arrow[dr, "g_1"] & R \arrow[d, "f_0"] \\ R \arrow[r, "t"'] & R \arrow[r, "\frac{t}{t-1}"'] & R. \end{tikzcd}$$

Maybe I should have said that the moprhism space has a differential. $\delta_0 : hom^0 \to hom^1$ is given by

$$\delta_0 = (g_0 t - t f_0, \frac{t}{t-1} g_0 - f_0 \frac{t}{t-1})$$

and

$$\delta_1 = (\frac{t}{t-1} f_1 + g_1 t, t g_1 + f_1 \frac{t}{t-1}).$$

Let's compute hom^0, first we take the kernel of δ_0, this is (f_0, g_0) such that $g_0 t = t f_0$ so that $f_0 = g_0$, which is the image of $R \xrightarrow{\Delta} R \oplus R$ so is R. Then $Im\delta_1$ is the set of f_0, g_0 such that $f_0 = t(\frac{f_1}{t-1} + g_1)$ and $g_0 = t\frac{f_1}{t-1} + g_1$ and this is tR. So the homology is $\mathbb{C}$.

Similarly, hom^1 is the kernel of δ^1 modulo the image of δ^0, which is (f_0, g_0) such that $t(\frac{f_1}{t-1} + g_1) = 0$ which is again R, and the image of δ^0 is pairs (f_1, g_1) such that $f_1 = t(g_0 - f_0)$ and $g_1 = \frac{t}{t-1}(g_0 - f_0)$ wihch is again tR. So we get again that the quotient is $\mathbb{C}$ and the endomorphisms of S, we have again the same calculation $\mathbb{C} \oplus \mathbb{C}$.

4. Hiro II

I have twenty minutes left. I can do one of two things. I can prove the equivalence of the other categories or I can summarize what actually happened and clarify, get our heads out of the trees and see the forest.

What we were supposed to have proven is that $Fuk_\lambda(\mathbb{CP}^1)$ is equivalent ot $MF(\mathbb{C}^\times)$ with respect to $W - \lambda$ where $R = \mathcal{O}(\mathbb{C}^\times) = \mathbb{C}[t, t^{-1}]$.

Once you pass to the homotopy category the right hand side has a triangulated structure. On the left hand side we should be taking a completion to match up with this, so that they are both $\mathbb{Z}/2\mathbb{Z}$-graded dg categories. Then the strategy is first to find generators. How do we do that? Ana stated a proposition or theorem of Toby's which is that this matrix factorization $S = \begin{tikzcd} R \arrow[d, bend right, "t"'] \\ R \arrow[u, bend right, "\frac{t}{t^2-1}"'] \end{tikzcd}$ where $R' = \mathbb{C}[t,(t-1)^{-1}]$ and this generates $MF(R', \frac{t^2}{t-1})$ and that's the same as $MF(R, t+\frac{1}{t})$. Then from François' talk, Abouzaid, [missed names] showed that equatorial circles in $\mathbb{CP}^1$ with $\rho = \pm 1$, generates $Fuk_{\pm 2}$. We stated these theorems, these are the objects we know we want to test.

Corollary 4.1. *The Fukaya category of* $\mathbb{CP}^1$ *is equivalent to* $DEnd_{Fuk_\lambda}(S^1)$.

This is a souped up version of the Mitchell embedding theorem for dg categories. Once I find a generator, I can take modules over the endomorphisms of the generator, This is usual mumbo-jumbo. Again, $MF(\mathbb{C}^*, W-\lambda) \cong DMod(End_{MF}(S))$. Once we show the right hand sides are equal we've got what we want. How do we show that these categories are equal? Maybe you get God-damned lucky and $End_{Fuk}(S^1) \cong End_{MF}(S)$ as $\mathbb{Z}/2$-graded A_∞-algebras. You might say that we didn't prove anything that looked so intimidating. François didn't have time to compute what he wanted to compute, but we got the cohomology of the right hand side. We didn't show an equivalence at the level of $\mathbb{Z}/2$-graded A_∞-algebras, we computed that they have the same homology. In this case we're saved.

Lemma 4.1. *Both of these algebras are formal.*

We can get an explicit map on the right hand side to realize formality. On the left hand side (the Fukaya category) it actually *isn't* formal in characteristic 2. So we're reduced to showing an isomorphism of graded algebras $H^*(End(S^1)) \cong H^*(End(S))$. For François, we heard that $H^*End_{Fuk}(S^1) \cong \mathbb{C}[h]/(h^2-1)$ where h is odd.

The claim is when you compute the *algebra* of $End_{MF}(S)$, you can just compose using the presentations of the kernel and image that Ana wrote down, and see you get the same thing.

Corollary 4.2. *Once I have this isomorphism* $H^*(End(S^1)) \cong H^*(End(S))$*, and so you get a lift to an equivalence at the level of dg algebras.*

You can tell that already there's a lot of nontrivial mathematics, but hopefully this illustrates some of the immense category of math that mirror symmetry brings together and hopes to relate. All of this is much more general. We did a great disservice to Toby's theore, for instance.