MAX PLANCK WORKSHOP ON HOMOTOPY THEORY, MANIFOLDS, AND FIELD THEORIES

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1. June 15: Charles Rezk: Elliptic cohomology and elliptic curves

[Good morning everybody, I welcome you here at the Max Planck institute. Most of you know are coming from HIM and so I wanted to steal five minutes of Charles' time. I want to talk about research in Germany outside universities. There are four societies, ordered from basic to applied. Max Planck is 1.6 billion euro per year, but all subjects, law, history. That's about the budget of Stanford. It's like one small university. The Fraunfhofer society does basic research looking for applications, 1.7 billion. The Helmholtz society works directly with industry, with budget 3.4 billion. The Leibniz association has budget 1.4, Oberwolfach is part of this. This is a mix of basic and applied. That's research in Germany. Max Planck, each institute is built around people who run research institutes, with exceptions like here. The private grants are like five percent of our budget. Every state has roughly the same number of Max Planck institutes because the money is half and half federal and state.

We in Bonn are the only Max Planck institute that doesn't have groups, we want to do all of mathematics. We do our guest program. We have 20 grad students who do three to four years. They're funded here but might be advised elsewhere. We work closely with the big Bonn graduate school. We have 30 postdocs, with positions from one to five years. The rest is 50 people who do whatever they want to do. We'd prefer if they worked together. You should come here, relax from other duties, and do mathematics. You apply, we pick you, and then you come.

We have administration. You don't see them but they're really good. We have IT and people you may meet if you want money. We have a small but officient staff.

I wanted to show you this great picture, the institute was opened 30 years ago, this is a picture of Hirzebruch and Lüst, who was president of the Max Planck society at that time. One does admin for 2 or three years. I want to show the opening of the arbeitstagung in 1987, the index of the signature operator on loop space is on the board. This will be the fifth Felix Klein lecture but he promised to catch us up.]

I hope this talk will be understandable even if you haven't heard the previous ones. The talk on Wednesday will be on a completely different topic.

I'm interested in the following setup. I'll take R a commutative $\mathbb{S}$-algebra, so a spectrum with a highly structured multiplication, representing a cohomology theory if you want. I have the *unit spectrum* $gl_1(R)$, which is a (-1)-connected spectrum. You take the underlying space $\Omega^\infty R$, which projects to $\pi_0\Omega^\infty R = \pi_0 R$. I take inside of that $(\pi_0(R))^\times$, and the pullback is $GL_1(R)$. This has an ∞-loop space structure built from the multiplication on R. If I take $[X, GL_1 R]$, this is the units of $R^0(X)$.

The question is what we can say about the homotopy type of $gl_1(R)$. As a *space* it's $GL_1(R)$, which is basically the same as $\Omega^\infty R$, but this has all this other structure. One famous example is the sphere spectrum.

Think about, say you have E, F spectra. You can take $[F,E]_{sp}$, or you can take $[\Omega^\infty F, \Omega^\infty E]_{Top_*}$, and there's a functor Ω^∞ between these. Among the maps $\Omega^\infty F \to \Omega^\infty E$ are the H-space maps, and this is where this has to land. These are exactly the ones that induce cohomology operations which are Abelian group homomorphisms. The question is, what can you say about this functor? The answer usually is that there's nothing in particular to say. It's not generally surjective or injective. An easy example where it's not injective is when these are Eilenberg–MacLane spectra, suspended appropriately. In special cases, though, you can say more.

Example 1.1. I'll suppose that F is 0-connected and E is rational, that is, $\pi_* E = \pi_* E \otimes \mathbb{Q}$. Then $\Omega^\infty : [F,E] \to [\Omega^\infty F, \Omega^\infty E]$ is injective and has a canonical retraction. This is not deep. It's easy to compute stable maps for a rational spectrum. So $[F,E]$, the stable maps are homomorphisms of graded Abelian groups $\pi_* F \to \pi_* E$, because a rational spectrum is a product of the rational Eilenberg–MacLane pieces. I'll define

$$r : [\Omega^\infty F, \Omega^\infty E] \to [F,E]$$

where $f \mapsto [\pi_* f]$. You know if you have a stable map, well, it's a tautology that it's a retraction. Also, in fact, the image of this inclusion lands in H-maps, and in this case that's an equivalence $[F,E] \to [\Omega^\infty F, \Omega^\infty E]_H$.

We can apply this to R a commutative $\mathbb{S}$-algebra, say rational. I can take, I can use the map $s : GL_1(R) \to \Omega^\infty(R)$. This was defined as a subspace. I wanted to have pointed maps, the cartoon version of what it does is take x to $x-1$. The space $GL_1(R)$ is not necessarily connected, so I'll replace it with its 0-connected cover, its base point component.

Let me go back and say that r is an idempotent $\mathcal{E}$ on the set of unstable maps with these hypotheses, with image $[F,E]$.

I'll take s and apply this idempotent $\mathcal{E}s : gl_1(R)_{\langle 0\rangle} \to R$. This is an isomorphism of π_* for $* \geq 1$.

Let me do this again but in a more complicated way.

So I have this idempotent again $\mathcal{E}$ on $[\Omega^\infty F, \Omega^\infty E]$, with the first connected and the second rational. So I get $f : F^0(X) \to E^0(X)$. I want to compute $\mathcal{E}f$.

Proposition 1.1. *If X is connected and finite dimensional then*

$$\mathcal{E}f)x = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} Cr_n f(x,\ldots,x).$$

Now Cr_n is the cross-effect, if $f : A \to B$ is a function between Abelian groups, then $Cr_1 f(x) = f(1) - f(0)$, $Cr_2 f(x_1,x_2) = f(x_1+x_2) - f(x_1) - f(x_2) + f(0)$. In general

$$Cr_m f(x_1,\ldots,x_m) = (-1)^{m-|I|} \sum_{I\subset\{1,\ldots m\}} f(\sum_{i\in I} x_i)$$

Now $Cr_n f : (\Omega^\infty F)^{\wedge m} \to \Omega^\infty E$.

If I have a function between Abelian groups, I can factor $f : A \to B$ through $\mathbb{Z}[A] \to B$ which is a homomorphism, taking a to $[a]$. I only need to do this for maps $\delta : A \to \mathbb{Z}[A]$ and because B is rational I can do this for $\mathbb{Q}[A]$. I will write down the right hand side explicitly, $\mathcal{E}\delta(x) = \sum \frac{(-1)^{n-1}}{n} Cr_m \delta(x, \dots, x)$, but the Cr term looks like $([x] - 1)^n$ so the sum looks like "log([x])." In practice, I only need to apply this to f for which large cross effects vanish. If cross-effects vanish for all $n \geq N$, that's the same as saying that $\mathbb{Q}[A] \to B$ factors through $\mathbb{Q}[A]/I^{N+1}$. Now you can use the property of the natural log that you know, that it takes products to sums. Then you can show directly that $\mathcal{E}\delta(x + y) = (\mathcal{E}\delta)(x) + (\mathcal{E}\delta)(y)$.

I wanted to show this so that later I can do something similar, linearize a function if I can make some series converge.

Let me apply this now to the case I was interested in, GL_1, I'll apply this to the map s on the other board. You'll get an answer that's not much of a surprise in retrospect.

I have $s : GL_1(R) \to \Omega^\infty R$. I want to look at $(\mathcal{E}s)(x)$, and this is only a formula for connected spaces, and this turns out to be given by the series

$$(\mathcal{E}s)(x) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n}(x-1)^n = \log x$$

which is an operation $R^0(x)^\times \to R^0(X)$.

Somehow this looks like a curiosity, it wouldn't work if it wasn't rational. Surprisingly, you can do something.

Now I want to look at K-theory, p-adic K-theory. I'll actually be able to read off something interesting.

Example 1.2. Now I want to have $E = F = K_P^\wedge$. Then

$$[K_p, K_p] \xrightarrow{\Omega^\infty} [\Omega^\infty K_p, \Omega^\infty K_p]_H \subset [\Omega^\infty K_p, \Omega^\infty K_p],$$

which is maybe easiest to see by a computation. These are, respectively, $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ and $\mathbb{Z}_p[[\mathbb{Z}_]]$ which is the limit of $\mathbb{Z}_p[[\mathbb{Z}/p^n]]$ and this is the obvious map (which is not a ring map) induced by the inclusion of $\mathbb{Z}_p^\times$ into $\mathbb{Z}_p$.

The element $[\lambda]$ is the Adams operations (ring operations) $\psi^\lambda : \Omega^\infty K_p \to \Omega^\infty K_p$ and these are also the $[\lambda]$ on the left hand side.

What I want to concentrate is the idea that there is a retraction.

It's getting near to where I should take a break. Let me construct one retraction and then we'll see, maybe take a break. The first thing I'm going to use to get things set up for what I'm going to say later, the first method is to use Bott periodicity for p-adic K-theory, $\Omega^\infty K_p \xrightarrow{\beta} \Omega^2\Omega^\infty K_p$. So I also have $\Omega^2 f$ which is $\beta^{-1}(\Omega^2 f)\beta$, which is already an H-map. The idea is that I'll take an unstable map and turn it into a stable map by iterating ω, sending it off to ∞.

The first method, given f an H-map, I'm going to take $\lim_k \omega^{(p-1)p^k}(f) = \mathcal{E}f$.

I claim that this is idempotent and exhibits a retraction. The proof is easy once you know how to compute everything.

I know all I need to do is compute ω on Adams operations, because everything is a linear combination of these. It turns out that if you do the calculation, you only need to look at what Adams operations do on the 2-sphere. It turns out that $\omega\psi^\lambda = \lambda\psi^\lambda$. So $\lim \lambda^{(p-1)p^k}\psi^\lambda$. This turns out to be ψ^λ if p doesn't divide λ and 0 if it does.

2. Charles Rezk: Lecture 2

The idea of method two is to make use of transfers. If I have a finite covering map $Y \xrightarrow{f} X$, then stably there is a map the other way $\Sigma^\infty_+ X \to \Sigma^\infty_+ Y$ which is $K^0(Y) \xrightarrow{\tau_f} K^0(X)$. Stable maps have to commute with all transfers. However, unstable maps need not. So you can determine which H-maps are stable maps just by seeing where they commute with transfers.

Proposition 2.1. *A map $f \in [\Omega^\infty K_p, \Omega^\infty K_p]_H$ is Ω^∞ if and only if $f \circ \tau_p = \tau_p \circ f$ where τ_p is transfer with respect to $X \times EC_p \to X \times BC_p$:*

$$\begin{array}{ccc} K^0_p(X \times EC_p) & \xrightarrow{f} & K^0_p(X \times EC_p) \\ \downarrow{\scriptstyle \tau_p} & & \downarrow{\scriptstyle \tau_p} \\ K^0_p(X \times BC_p) & \xrightarrow{f} & K^0_p(X \times BC_p). \end{array}$$

The proof is a calculation, since we know about K-theory, the K-theory of $X \times BC_p$ is an extension of $K^0_p(X)$ by T mod $T^p - 1$. The transfer in K-theory is easy to compute. You get that $\tau_p(x) = xN$ where N is the regular representation.

So for $f = \psi^\lambda$, then $\tau_p \psi^\lambda(x) = \psi^\lambda(x)N$; on the other hand $\psi^\lambda \tau_p(x) = \psi^\lambda(xN) = \psi^\lambda(x)(1 + T^\lambda + \cdots + T^{\lambda(p-1)})$, which is either px if $p|\lambda$ or $\psi^\lambda(x)N$ if p doesn't divide λ.

I've been describing things in terms of idempotents. Let me describe something I can do there that picks out the infinite loop maps. We can produce an idempotent $\mathcal{E}$ on $[\Omega^\infty K_p, \Omega^\infty K_p]$ with image the stable maps by saying $(\mathcal{E}f)(x) = f(x) - \frac{1}{p}\langle f(\tau_p(x)), c\rangle$. This formula is only good mod torsion. I just should check this on the universal example, which is torsion free. What is $\langle\ , c\rangle$? Remember that $f(\tau_p(x)) = K^0_p(x)[T]/(T^p - 1)$, and this sends $T \mapsto \zeta_p$, so we land in $K^0_p(X) \otimes \mathbb{Z}(\zeta_p)$. If you want to check a formula like this, you check it on Adams operations, and I've given you all the ingredients.

Let's now apply this to GL_1. This is a construction originally due to Tom Dieck in 1989. I rediscovered it and it sometimes gets my name attached but it was known.

Applied to $GL_1(K_p)$, I'm interested in maps of stable

$$[gl_1(K_p), K_p] \xrightarrow{\Omega^\infty} [GL_1(K_p), \Omega^\infty K_p]_H \to [GL_1(K_p), \Omega^\infty K_p].$$

The structure on $GL_1(K_p)$ is multiplicative, not additive.

There's a theorem of Adams and Priddy that says if I take the 4-connected cover of $gl_1(K_p)$, that's equivalent to the 4-connected cover of K_p. This is not something that usually happens. That meas that I can feed this thing into the machine because it's not very far from K. I will just replace this in my commutative diagram. This is also proved by Madsen–[unintelligible]. Start with $f \in [GL_1(K_p), \Omega^\infty K_p]_H$, this is an Ω^∞ loop if and only if

$$\begin{array}{ccc} K^0_p(X)^\times & \xrightarrow{f} & K^0_p(X) \\ \downarrow{\scriptstyle P_p} & & \downarrow{\scriptstyle \tau_p} \\ K^0_p(X \times BC_p)^\times & \xrightarrow{f} & K^0_p(X \times BC_p). \end{array}$$

Here P_p is a *power operation*, something you define without looking at the unit. This uses the product E_∞ structure. This gives a reliable way to check whether or not a map is an infinite loop map.

tom Dieck gave the formula $\ell : K_p^0(X)^\times \to K_p^0(X)$ defined by

$$\ell(x) = \frac{1}{p} \log \frac{x^p}{\psi^p(x)} = \sum_{n \geq 1} \frac{(-1)^n p^{n-1}}{n} \left(\frac{\theta^p(x)}{x^p} \right)^n$$

is representend by a spectrum map $\ell_1 : gl_1(K_p) \to K_p$ which exhibits the equivalence on 3-connected covers. I'll leave this as an exercise using the above. Use the fact that $P_p(x) = \psi^p(x) - \theta^p(x)N \in K_p^0(X \times BC_p)$.

[more hints.]

Adams and Priddy did this in a different way, using completely different methods. Other people wrote down infinite loop maps. [Unintelligible].

I defined a more general construct for rational things. In fact, the following is true.

Proposition 2.2. *Let F be a spectrum and E be p-adic K-theory. Then the map $[F, K_p] \to [\Omega^\infty F, \Omega^\infty K_p]$ is injective with image equal to the image of an idempotent $\mathcal{E}$. It's computed on finite dimensional X up to torsion by*

$$\mathcal{E}f)(x) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \left(Cr_n f(x, \dots, x) - \frac{1}{p} \langle Cr_n f(\pi_1^* \tau_p(x), \dots, \pi_n^* \tau_p(x)), c^{\times m} \rangle \right).$$

The π_i are projections from $X \times BC_p^{\times n} \to X \times BC_p$. So $\langle\ , c^{\times n} \rangle$ corresponds to some sort of evaluation, I'm not going to write it down, it goes $K^0(X \times BC_p^n) \to K^0(X)$.

Once you've proved this, it turns out that tom Dieck's formula,

Proposition 2.3. *Remember I have the shift map $s : GL_1(K_p) \to \Omega^\infty(K_p)$, and $\mathcal{E}s = \ell_1$, tom Dieck's map.*

In fact, K_p is not crucial, you really need the target to be a K_1-local ring spectrum. Let me state a generalization, using Bousfield–Kuhn. If you liked method one, using a limit involving Bott periodicity, that silly trick is actually much more general and lives at the level of homotopy. I start with a space V and a map $\Sigma^d V \xrightarrow{\alpha} V$, of pointed spaces if you like. We'll say d is positive, and we'll make a functor $\phi_{V,\alpha}$ from $Top_* \to Sp$. It's a slightly startling construction.

So a spectrum is a sequence of spaces connected by maps. I can use, given the space X, I should tell you that E is spaces $underlineE_k$ and maps $\underline{E}_k \to \Omega \underline{E}_{k+1}$. I'll tell you $\underline{E}Skd \coloneqq Map_*(V, X)$ and the map to $\Omega^d \underline{E}_{kd+d} = \Omega^d Map_*(V, X) = Map_*(\Sigma^d V, X)$ is $\circ\alpha$. We can define $Tel_{V,\alpha} : Sp \to Sp$ by $Tel_{V,\alpha}(X) = \operatorname{hocolim} \mathcal{F}(\Sigma^{dk} \Sigma^\infty V, X)$. We find that the stable telescope $Tel_{V,\alpha} \cong \phi_{V,\alpha} \circ \Omega^\infty$.

We can use this idea to factor $Sp \xrightarrow{L_{K(n)}} Sp$, through an inverse limit of telescopes

$$\begin{tikzcd} Sp \arrow[rr, "L_{K(n)}"] \arrow[dr, "\Omega^\infty"'] & & Sp \\ & Top_* \arrow[ur, "\phi_n"'] & \end{tikzcd}$$

where ϕ_n is the Bousfield–Kuhn functor.

Now if $E = L_{K(n)}E$, we get get a map

$$r : [\Omega^\infty F, \Omega^\infty E]_H \xrightarrow{\phi} [L_{K(n)}F, L_{K(n)}E]_{sp} \cong [F,E]_{sp}$$

is a retraction of Ω^∞. [missed a little.]

Let me just write down an example for Morava E-theory. Then we'll call it a day. There's a general formula. Let's take E to be a Morava E-theory, an example of a $K(n)$-local spectrum.

I'll write down the formula for the idempotent

Proposition 2.4. *The Bousfield–Kuhn idempotent $\mathcal{E}$ on $[\Omega^\infty F, \Omega^\infty E]_H$ is computed by*

$$(\mathcal{E}f)(x) = \sum_{r=0}^{n} (-1)^r p^{\binom{r}{2}-r} \sum_{[\alpha]} \langle f(\tau_p^r(x)), \alpha\rangle$$

where $\tau_{p^r} : E^0 \to E^0(X \times B(\mathbb{Z}/p)^\times r)$ is [missed] and $\alpha : (\mathbb{Z}_p)^\times n \to (\mathbb{Z}/p)^\times r$, with $\langle\ ,\alpha\rangle$ is a character map evaluated at α, a map $E^0(X\times B(\mathbb{Z}/p)^\times r) \to E^0(X)\otimes_{E_0} D$.

From this, you can determine whether something is an infinite loop map by checking that it commutes with τ_{p^r} transfers.

Finally, the motivation was gl_1. There's a version that I can apply to non-H-maps, so I can apply it to the shift map from $GL_1(E)$ back to $\Omega^\infty E$ to get $\ell_n : gl_1(E) \to E$. Let me do the $n = 2$ case. Then

$$\ell_2(x) = \frac{1}{p}\log\frac{x^p N_2(x)}{N_1(x)}.$$

I'll remind you that Morava E-theory has power operations where $E^0(X) \xrightarrow{\psi_r} E^0(X)\otimes^s_{E_0} A_r$. So N_1 is

$$E^0(x) \xrightarrow{\psi_1} E^0(x)\otimes^s_{E_0} A_1 \xrightarrow{\text{norm}} E^0(X).$$

Since I didn't talk about Hecke operators, I'll say this is the multiplicative version of the first Hecke operator. N_2 is even easier. In this case there's a distinguished subgroup of order p^2 which is the p-torsion. So

$$N_2 : E^0(X) \xrightarrow{\psi_2} E^0(X)\otimes^s_{A_0} A_2 \xrightarrow{\text{id}\otimes\pi} E^0(X)$$

where $\pi : A_2 \to A_0$ classifies $G[p] \subset G$. It turns out that $X^p N_2(x) \equiv N_1(x) \pmod p$.

On homotopy groups, where $*$ is positive, let's say even, we get

$$f \mapsto f - T_y f + p^{k-1}f.$$

Then $T_p f$ is $\frac{1}{p}$ times the additive version, the trace of ψ_1. The Hecke operators show up in modular forms. If I put in an Eisenstein series I get 0. This is related to calculating the string orientation of tmf. Since I'm out of time, I can't give you a punchline. The punchline is that there's something suspicious that indicates that there's something equivariant going on.

[question]

There's a $K(2)$ local spectrum, you can compute, up to torsion, you can write it as $gl_1(tmf) \xrightarrow{\ell_2} L_{K(2)}tmf$ and this factors through $tmf_p^\wedge$. The conjecture is that this happens because tmf has this fancy structure in the background, this equivariant structure, these operations that I've argued should exist.

[Let's thank Charles again and see you back at 3.]

3. Nathalie Wahl: three points of view on string topology

[So before we start the afternoon session, let me say that everyone who registered has a folder with the list of abstracts, please pick that up if you did register. Please fill out a sheet so you can get reimbursement if there is one in there.]

I will start by saying what I mean by this. I'll talk about string topology on manifolds. I mean the space of maps from S^1 into M, LM. Then this goes back to Chas–Sullivan, what they did is to construct a product, when we have based loops, you get a product by concatenation of loops. If we go to chains on the loops, take $C_p(LM) \otimes C_q(LM)$. You should think of your p-chain as a family of loops, your q-chain as a family of loops. Think of the chains of basepoints in the manifold, and where they intersect you can take a product. When I do this, this should give me a map to $C_{p+q-n}(LM)$, where I lose the dimension of the manifold because I am taking an intersection. There is a product here, some sort of graded product. There is also a circle action by rotating loops. What they show is that together these two things form a Batalin–Vilkovisky structure, they satisfy a relation with too many terms to write down.

The question I will address today is, are there more interesting operations on chains or the homology of LM? Charles has been talking about operations. I'll be mixing, mixing some things from the homology and things that we are taking the homology of.

So inspiration from string theory, string theorists think that there should be operations parameterized by the moduli space (chains on) the moduli space of Riemann surfaces. You have your strings, they are circles, they evolve, they interact somehow, they form surfaces, you maybe get genus, and then they output. The Batalin–Vilkovisky structure have some number of inputs and one output in genus zero.

The first point of view, the Chas–Sullivan, I wanted to call this geometric, and that turns out to be very difficult, very tricky, you can ask Gabriel, who has just posted a paper on this approach, this physics approach, I don't know much about this, and then I'll say, let's do something more easy, let's do algebra.

There are several things that one could do, I'll assume M is 1-connected and work over a field and there's a model that says $C^*(LM)$ can be modeled by the Hochschild chains on $C^*(M)$ valued in themselves. We already know that the S^1 action we had here is modeled by Connes' B operator. The idea here is the algebraic structure of cochains on M should give structure on this Hochschild complex. Then you say okay, so what is this algebraic structure on cochains on M, so on $H^*(M)$, this satisfies Poincaré duality, and is what is called a Frobenius algebra and therefore cochains on M should be some homotopy version of Frobenius. Somehow making this precise is not so easy. I'm going to keep simplifying. I don't have a good way to give a homotopy Frobenius algebra. Over the rationals I can cheat and use [Lambrechts–Stanley] who say that $C^*(M)$ are quasi-isomorphic as a commutative differential graded algebra to a strict commutative Frobenius algebra. You can do this even if the manifold is not formal. It's an actual Frobenius algebra. So what do I want to do with this? I want operations on this Hochschild complex. With Craig Westerland I built a machine for doing this. We input in the machine a type of algebra and what comes out is a chain complex acting on $C_*(A, A)$ of such algebras.

If my input was "symmetric Frobenius algebras" (saying that the pairing is symmetric) the output is the chain complex of "Sullivan diagrams" which I will describe in a second. If my input is a "commutative Frobenius algebra," a slightly stronger condition, then the output was calculated by Klamt and it's called "loop diagrams" which I'll talk about in a little bit. I could also input an open conformal field theory and get as output a closed moduli space, and I don't know how to use it in string topology.

Let me explain Sullivan and loop diagrams and the harmonic compactification of moduli space.

So $SD(p,q)$ will be a space of p to q Sullivan diagrams. I hav p inputs and q outputs and this is an equivalence class of fat graphs of p disjoint circles with q other boundaries modulo edge collapse away from the p circles.

A fat graph is a graph with a cyclic order at the vertices. Maybe something like this: [picture]. You can fatten this. the outputs should have start points. This is an object of this sort. I make the equivalence relation, I can collapse edges but not loops.

Sullivan diagrams are equivalence classes of such graphs, and the topology is the metric on the p circles. The rest of the graph is, the metric is just on the circles.

What do these have to do with moduli space?

Theorem 3.1. *(Egas–Kupers) Metric fat graphs on p circles with q outputs have a quotient to $SD(p,q)$, the domain models moduli space of Riemann surfaces with $p+q$ marked boundary components (this is what fatgraphs were invented for); this is a special version of that old theorem where we only take certain fat graphs, and the space on the right $SD(p,q)$ is a model for Bödigheimer's harmonic compactification of moduli space, unimodular compactification.*

We're starting to know more and more about this quotient map π, on homology it's an isomorphism on the Batalin–Vilkovisky components, genus zero with one output, and on the other hand, kills all stable classes in the homology of moduli space. We also know more things, there ar classes in the compactification that are not in the moduli space.

Going to the Sullivan diagrams, they actually model some specific compactification, this is a compactification where you allow, say you have a surface with genus, you allow some handle to go to a very thin one, or you allow an output to become a point, as long as you can think of water having to go through.

I owe you a definition of loop diagrams. I remember when I went to algebra, I took the simplest model I could find. Some might say I took too simple a model, and yet I come out with something having to do with moduli space.

I said if we started with a Frobenius algebra I get loop diagrams $LD(p,q)$, an element there is (Δ, P, w, ℓ), and what are these things? So Δ is a set of points on p circles (including the base points of each circle). Let me draw a picture. [picture]. Then P is a partition of Δ into a number of subsets. The w is a set of weights on each partition subset, and ℓ is a map from the disjoint union of q circles to the disjoint union of u circles modulo (Δ, P). These are my loop diagrams. The topology is that from the metric on S^1, the placement of these points, with the weight increasing if two points of the same partition subset collide. They become one point, if the partitions were different they join and if they were in the same subset the weight increases by 1. That's the topology on the space.

As I have been trying to suggest with the pictures, there is a map $SD(p,q) \to LD(p,q)$, we sort of know this from the algebras, and this takes a Sullivan diagram, and it remembers the points on the inputs where the graphs are attached, and the weight is the genus of the graph that is forgotten, and the output loops are given by the outputs of the string diagram. This map is not surjective at all. In particular you could start doing many things with loop diagrams. It seems to be quite close to being injective. On the components that it hits, these spaces seem to be quite close to each other.

I wanted to go back to string topology now. I said I would take three different poitns of view. I want to go back to string topology and the geometric approach.

I started by this geometric approach to string topology, we intersected these chains of loops and this is tricky to do in practice. There's a construction of Cohen–Jones, they have a construction of the Chas–Sullivan product and the construction is a product on the loop space. We start with $LM \times LM$, and we can look at this as $LM \times_M LM$, this maps from two disjoint circles into M accepting a map from the space of maps of two circles that agree at the basepoint into M. If I am in this subspace, then I can go to the loop space by taking the concatenation product.

$$Maps(\bigcirc\bigcirc, M) \hookleftarrow Maps(\infty, M) \to Maps(\bigcirc, M)$$

We need to reverse this first map, they use Pontryagin–Thom, and to do this they construct a tubular neighborhood of $LM \times_M LM$, a neighborhood that looks like a bundle, you go $LM \times LM$, collapse whatever is outside that neighborhood, and land in the Thom space of that bundle, and at the level of chains you get a map (at the level of homology an isomorphism) to $LM \times_M LM$, and then from there to LM. That's the Cohen–Jones construction.

$$LM \times LM \to Th(N(LM \times_M LM)) xrightarrow c_* LM \times_M LM \to LM.$$

Now I want to argue that loop diagrams are precisely the right kind of thing to do this construction. What do I mean? Suppose I want to generalize the Cohen–Jones construction? I started with maps from 2 loops to M and we are now thinking about p loops. We have some points on the loops, and then, this is $LM^p \times \Delta^{d_1-1} \times \cdots \times \Delta^{d_p-1}$, and then we do self-intersection, given by my condition, where I go to maps from my circles where, these are loops satisfying the intersections given by the partition at time t If they satisfy such an intersection, then I can use my map ℓ, the outputs, to go into LM^q, I just read off some new loops, these things we're assuming that the intersection is satisfied. This looks like what I've written for Cohen–Jones. I'm missing w, and the weights, there's a problem with this construction, these are to fix the non-tubular neighborhood. What we had in the original Chas–Sullivan situation we had a nice tubular neighborhood, we don't have a tubular neighborhood but a stranger object. What we had, let me do an example to show what goes wrong.

If I start with a coproduct diagram, label my outputs 1 and 2, I'm looking at LM, and I cross with Δ_1, which is the position of the non-basepoint attaching point, I'm looking at (γ, t), looking at the space where $\gamma(t) = \gamma(0)$. This is codimension n when $t \neq 0, 1$ but it's codimension 0 if $t = 0$ or 1. If we do this on S^1 instead of the Δ_1 factor, it looks like codimension n everywhere except at zer, where I have two fibers. The reason for this, I'll say this, well, let me finish this instead of trying to go too quickly on other things. At time $t = 0$, every loop satisfies $\gamma(0) = \gamma(0)$. A neighborhood of that looks like two copies of, well in general it's TM. I can use the tangent vector to push my two points apart, using a geodesic. I get a different point in the space, I can do this in two ways, I can do this in the beginning or the end of my loop. If I do something nontrivial, I have two choices. What happens in general, we can describe these as colimits of tubular neighborhoods, of bundles, and you can thicken up bundles that are too small. This is something I've been doing, joint with Nancy Hingston.

I'm at the end of my talk. Let me say a couple more things. I was supposed to do three points of view on string topology, Sullivan and then Poirier, and recently Poirier with Gabriel and Nathaniel Rounds, that's a slightly different model. Here I showed two points of view and get the same space. There's a model of Kaufman (sp?) and Penner giving strings interactions, and they're using arcs in surfaces, and this also gives a model for string topology. I should stop.

[Loop diagrams form a prop? Can you see that this acts on the negative tangent bundle over the loop space?]

I don't know.

[What if you took the entire prop?]

There's a bigger thing that comes out, a huge thing, and it's a bit, it's surprising that this huge thing, most of it collapses down and it becomes this nice simpler complex. As a homotopy theorist the big thing is okay, but it doesn't look too much like moduli space.

4. Thomas Niklaus: a universal description of global spectra (or cohomology theories on stacks)

[Tomorrow evening you're all invited to a reception. So that will be the only official dinner this week.]

It looks like a crowded audience, I'm happy to speak here. I want to report on a joint project with David Gepner. We wanted to understand things related to elliptic cohomology.

I'm teaching a course about ordinary homotopy and homology. I'll pass to stable homotopy theory. One question you have to answer is "why stable homotopy theory?" Sometimes this is painful. Why would you want to pass to spectra? I can think of two reasons you want to do this.

(1) There are lots of phenomena that stabilize, like the Freudenthal suspension theorem or the cohomological suspension isomorphism. So stabilization feels like it's leading to the core of something.
(2) Maybe you want to represent cohomology theories. If you have a cohomology theory it's represented by a spectrum. This lets you compute natural cohomology operations. In motivic homotopy theory this was used by Voevodsky.

One you formally invert the suspension functor on pointed spaces, this solves both 1 and 2. You have these two a priori independent motivations and this solves both of them.

I want to talk about this in an equivariant version. You could look at spaces with an action of the compact Lie group G. You have equivariant cohomology theories or the equivariant Freudenthal theorem. Now you have to invert more things. Stabilize with respect to $\wedge S^V$, where V is a G-representation. Again once you've done that, cohomology theories become representable, which are graded cohomology theories graded on representations, not integers. Those occur all over the place. The most important one is equivariant K-theory. In degree zero these are vector bundles, and that becomes representable.

I want to take the global setting where morally we encounter things that respect all Lie groups. Schwede has studied this but I will take a different approach. I will work with topological stacks, some kind of functor from spaces into groupoids. Instead of giving definitions, I'll give examples

Example 4.1.
$\mathbb{B}G$, the category of G-bundles.
$\underline{X}$, which sends X to continuous functions into X, which gives an embedding of spaces into stacks.
quotient stacks $[X/G]$ where X has a G-action. This gives a functor, not fully faithful, from G-spaces to stacks.
there are orbifolds (it's open whether they can always be written as $[X/G]$).

Just as G-spaces have a homotopy theory, stacks have a homotopy theory, and that's what I want to contemplate. This homotopy theory I'll just denote *Stacks*. Say it's model categories or infinity categories or homotopy categories you prefer, I'll say that's what I mean by *Stacks*. This happens to be equivalent (Gepner–Enriquez) to orbispaces, which is equivalent (Schwede) to global spaces. In other words, these are different ways of describing the same homotopy theory. This has an inclusion from all equivariant homotopy theories but it has more.

For example, what are the cohomology theories I will care about? K-theory will be one. I can just as before define K-theory groups for stacks. Because they restrict to equivariant K-theory they have stability phenomena.

(1) $\mathcal{X} \to \mathbb{B}G$ mean that $\Sigma^V \mathcal{X}$, for V a G-representation, which is the same as $S^V \wedge_{\mathbb{B}G} \mathcal{X}$, this is a suspension isomorphism. I'm smashing over $\mathbb{B}G$, it's a relative smash. Then this stabilizes restricted like this.
(2) The cohomology theories I have in mind restrict to $RO(G)$-graded theories on G-spaces.

I have the stabilization phenomena and cohomology theories, how do I represent them, as I asked in the beginning.

Theorem 4.1. *(N., Gepner)*

(1) *We can "stabilize relatively" $S^V \to \mathbb{B}G$ in Stacks to obtain a homotopy theory $Stab_{\mathcal{V}}(Stacks)$ and in a formal way we can stabilize, but in a relative way. I'll describe that later, and it's equivalent to stabilize with respect to all one point compactifications of $V \to \mathcal{X}$ where V is a vector bundle.*
(2) *An object $E \in Stab_{\mathcal{V}}$ is informally given by an assignment which assigns to a vector bundle V over $\mathcal{X}$ an ordinary spectrum $E^V(\mathcal{X})$. Whenever you have $\mathcal{Y} \to \mathcal{X}$ it produces a map $E^V(\mathcal{X}) \to E^{p^*V}(\mathcal{Y})$. Whenever you have $S^V \to S^W$ over $\mathcal{X}$ it will give you $E^V(\mathcal{X}) \to E^W(\mathcal{X})$. It will satisfy some axioms. It will be a functor, I'll write down the axioms later.*
(3) *Every cohomology theory on stacks is represented in $Stab_{\mathcal{V}}(Stacks)$. By a cohomology theory I mean something that assigns an Abelian group to each vector bundle over each stack. That will become representable by such a thing.*
(4) *$Stab_V(Stacks)$ is modelled by a model category called orbispectra Sp^{orb} which is Quillen equivalent to Schwede's global model structure. Now every*

guy that Stefan has produced gives us a cohomology theory of spectra. That's the thing we were really after, and why? Because:

(5) *There are objects related to TMF in $Stab_V(Stacks)$, which has the property, for example, built using ideas of Jacob, that, well, $TMF^0(\mathbb{B}T) = \Gamma(\mathcal{M}_{univ}, G^{top})$. Also, there is like inertia K-theory and its cousins (for orbifolds only) and I define the value $K^V_{inert}(\mathcal{X})$ to be $K^{\Lambda V}(\Lambda\mathcal{X})$. So we use the theorem twice, once to define K-theory of $\Lambda\mathcal{X}$ (the inertia groupoid) and then to say what we want about the left hand side that we have defined.*

The inertia group $[\Lambda([M/G])]$ is the disjoint union of $[Fix^h(M)]//G$, which is functors from $pt/\mathbb{Z}$ to $[M/G]$, this is a version of elliptic cohomology, K-theory of the loop space, some version of Tate K-theory.

Now I want to describe what this means, to relatively invert representation spheres. I want to talk about stabilizing with respect to a functor and how to make that precise.

The thing I want you to take away, when you want to go from stacks to a global setting, you need to localize in a relative setting.

I'll have to get more precise and thus more abstract. Now Pr^L will be presentable ∞-categories and left adjoint functors.

One important piece of structure is the tensor which makes this symmetric monoidal, and this corepresents Quillen bifunctors. A morphism from that product is the same as a Quillen bifunctor. If I fix one object it preserves colimits in the other variable, and vice versa.

The category of spaces is the unit. This is very useful. It was introduced by Jacob Lurie.

I want to take $\mathcal{C}$ a Cartesian closed presentable ∞-category, and $\mathcal{S}$ a set of objects or rather a class of objects in $\mathcal{C}$ generated by a set closed under smashing, retract, and equivalence. These are the objects you kind of want to invert. For example I can let $\mathcal{C}$ be spaces and $\mathcal{S}$ be S^1. I can let $\mathcal{C}$ be G-spaces and $\mathcal{S}$ representation spheres. For motivitic homotopy theory, you have $\mathcal{C}$ be motivic spaces and $\mathcal{S}$ is P^1.

$Mod\mathcal{C}$ is presentable ∞-categories tensored over $\mathcal{C}$. By this I mean you can tensor objects in M with objects in $\mathcal{C}$. An object tensored over G-spaces is a G-enriched model category, for instance.

Definition 4.1. I say M in $Mod\mathcal{C}$ is $\mathcal{S}$-stable if

(1) M is pointed and
(2) $\wedge s$ is an equivalence $M \to M$ for every $s \in \mathcal{S}$.

In the first case, where I let $\mathcal{C}$ be spaces and $\mathcal{S}$ the circle, then this is ordinary stability. For G-spaces I get G-stability. So we want to impose $\mathcal{S}$-stability for an appropriate class $\mathcal{S}$.

Theorem 4.2. *(Mostly Robalo, but also Lurie, Hovey, many others, Voevodsky)*

(1) *You can formally invert the inclusion $Mod\mathcal{C}_{S-stable}$ into $Mod\mathcal{C}$*
(2) *explicitly, if $\mathcal{S} = \{s\}$ then $Stab_{\mathcal{S}}(M) \cong \operatorname{colim}^{Mod\mathcal{C}}(M_* \xrightarrow{\wedge s} M_* \xrightarrow{\wedge s} M_* \to \cdots)$ if s satsifies cyclic invariance. so this is the same thing as sequences of objects $m_0, m_1, \ldots$, with equivalences $m_i \xrightarrow{\sim} \Omega^s m_{i+1}$.*

Let's look at our examples. We get spectra, as expected.

(3) *The stabilization of any module category M happens to be equivalent to the stabilization over $\mathcal{S}$ of $\mathcal{C}$ tensored over $\mathcal{C}$ with M. We use that this*

tensoring commutes with colimits. Now we have an endofunctor of stable categories, it's called smashing localization $Stab_{\mathcal{S}}(\quad) : Mod\mathcal{C} \to Mod\mathcal{C}$. *Then* $Stab_{\mathcal{S}}(\mathcal{C})$ *admits a unique symmetric monoidal structure such that* $\mathcal{C} \to Stab_{\mathcal{S}}(\mathcal{C})$ *admits a symmetric monoidal refinement. This gives precisely one structure on spaces with unit the sphere, which is something that Jacob does.*

(4) *If tensoring with s preserves compact objects and everything (*$\mathcal{C}$*) is compactly generated, then we can understand compact objects in the stabilization, which will be important for Brown representability. Then every compact object of* $Stab_{\mathcal{S}}(\mathcal{C})$ *is of the form* $\Sigma^{\infty}_{+} c \otimes S^{-n}$. *If you have a cohomology theory defined on compact spaces, you extend it to compact spectra and then use a version of Brown representability to prove that it's a cohomology theory. This is under some well-known assumptions on a triangulated category, satisfied in our setting. In the motivic setting this was proved by Spitzweck and [unintelligible].*

Now we want to move to the more complicated setting where we relatively stabilize things. Let me say that the setting is more complicated but all these things still hold true.

What is the setting here? $\mathcal{C}$ is locally Cartesian closed and should be thought of as stacks. I want to fix a class $\mathcal{F}$ of pointed morphisms, which you should think of as spherical fibrations.

Definition 4.2. Let $c \in \mathcal{C}$ and $M \in Mod\mathcal{C}$, then $M/C = M \otimes_{\mathcal{C}} \mathcal{C}/c$ where $\mathcal{C}/c$ is the slice category.

Example 4.2.
(1) If $\mathcal{C}$ is spaces and M is spectra then M/c is just the category of parameterized spectra over c.
(2) If $\mathcal{C}$ is G-spaces and M is G-spectra, then I can take $M/(G/H)$ and that turns out to be $Spectra^H$. That's why I call it the slice category.

Now I want to say what I mean for a category to be stable with respect to my class.

Definition 4.3. I say that $M \in Mod\mathcal{C}$ is $\mathcal{F}$-stable if
(1) M is pointed and
(2) $\wedge_S E : M/S \to M/S$ is an eqeuivalence for $E \to S$ in $\mathcal{F}$

For example, take $\mathcal{F} = \{s \to *\}$ for $s \in S$, and then we get the old stability.

There is now the following question. We inverted smashing with spheres in spaces and got spectra. What about all $S^V \to X$ for all vector bundles. We know that in parameterized spectra these and in fact all spherical fibrations are equivalences. This is a formal way of showing this thing we already knew.

Here's a lemma:

Lemma 4.1. *If* M *in* $Mod\mathcal{C}$ *is* $\mathcal{F}$*-stable then it is* $\overline{\mathcal{F}}$*-stable where* $\overline{\mathcal{F}}$ *is the saturation of* $\mathcal{F}$ *under*
(1) $A \to S, B \to S \in \overline{\mathcal{F}}$ *if and only if* $A \wedge_S B \to S \in \overline{\mathcal{F}}$
(2) *pullbacks and being a summand or factor is a smash product (I guess this latter is already in the first case).*
(3) *if* $\mathcal{C}$ *is an* ∞*-topos, then descent*

(4) *(something) which tells me that in equivariant homotopy theory (May–[unintelligble]) which tells you [missed]*

Proposition 4.1. *(N., Gepner)*

(1) $Mod\mathcal{C}_{Stab_{\mathcal{F}}} \to Mod\mathcal{C}$ *is universally invertible via* $Stab_{\mathcal{F}}$,
(2) $Stab_{\mathcal{F}}$ *is smashing, and*
(3) *there is a formula I might not get to write down, basically what I said at the beginning, you can write this as a colimit. This is important because this is what we use to describe the examples.*

Corollary 4.1. *Compact objects (let me specialize to stacks) in* $Stab_{\mathcal{V}}(Stacks)$ *(where* $\mathcal{V}$ *is one point compactifications of vector bundles over stacks with the* ∞ *section) is of the form*

$$(\Sigma^\infty_+ \mathcal{X})^{-V}$$

where $V \to \mathcal{X}$ *is a vector bundle.*

Corollary 4.2. *Brown representability.*

Corollary 4.3. *The Picard category, the objects which are invertible in* $Stab_{\mathcal{V}}(Stacks)$ *is just* $\mathbb{Z}$*, we run the usual object that says this has to be a retract of a sphere which is a sphere.*

I've only showed you this very formal part. We formally inverted representation spheres and got some Brown representability things. This can be easily modelled by a very nice model category that looks like orthogonal spectra, it's a functor category, and then we can write down an explicit equivalence to Stefan's model category and then import his examples. Or for the TMF or Tate K-theory example, this is what we do. For TMF we follow Jacob's ideas. This is how you can use this relative inversion. Sorry for going overtime.

5. June 16: Owen Gwilliam: Factorizing the index

[Welcome to the second day of the workshop.] Today I'll talk about how ideas from factorization algebras might connect up to the index theorem. In the eighties there was a frenzy of activity about this, and this is a first attempt to find a factorization algebra version of index theorems.

There are two parts, a kind of warmup and then a more sophisticated analogue.

The simplest version of the index theorem, I have two vector bundles over a closed manifold

$$\begin{tikzcd} V^0 \arrow[dr] & & V^1 \arrow[dl] \\ & X & \end{tikzcd}$$

There is an ellictic operator $\Gamma(X, V^0) \xrightarrow{P} \Gamma(X, V^1)$ and the index of P, which is the difference between the dimension of the kernel and cokernel of P is the same as the integral over X

$$\int_X Td(X) ch(P).$$

I'm going to do some factorization version of this. I need to introduce some ideas from the Batalin–Vilkovisky formalism. The input is a shifted symplectic vector space. I want to deal with dg modules over R and have a symplectic pairing of degree 1. I'll say the category $Prsymp_1(R)$ is the pullback of the following:

$$\begin{tikzcd}
Presymp_1(R) \arrow[r] \arrow[d] & Ch(R)/R[1] \arrow[d] \\
Ch(R) \arrow[r] & Ch(R) \times Ch(R)
\end{tikzcd}$$

The bottom horizontal map is $\wedge^2 \times R[1]$ and the right vertical map takes $M \to R[1]$ to $(M, R[1])$. So these are complexes with a pairing to $R[1]$. If you have $M, \omega : \wedge^2 M \to R[1]$ in $Presymp_1$ then I can get a bracket $\{\, , \}_\omega : M \otimes M \to Sym(M)$ which takes $m \otimes m'$ to $\omega(m, m')$ and you can extend by Leibniz to get a $Poisson_0$-algebra. Now BV quantization, you deform the differential, you take $d_{Sym\ M}$ and deform by Δ_ω. I won't say what it is, but you can ask me after the talk.

Let me summarize what this construction does, and let me say that doing this carefully is joint work with Haugseng–Scheimbauer.

Proposition 5.1. *There is a symmetric monoidal functor from $Presymp_1$ with $\oplus$ to Ch with tensor product. We have this for Sym, and the deformation of the differential goes along for the ride. When R is a field of characteristic zero, $\mathbf{k}$, then for M a symplectic object with finite dimensional cohomology, then $bvq(M, \omega)$ has one dimensional cohomology.*

Let me summarize that in a picture.

$$\begin{tikzcd}
dgVect^{fin} \arrow[d, "V \mapsto V \oplus V^*[1]"] \arrow[dr, "cq"] & \\
Symp_1^{fin} \arrow[r, "bvq"] \arrow[d] & Ch^{inv} \\
Presymp_1 \arrow[r, "bvq"] & Ch
\end{tikzcd}$$

Here $cq(V) \cong det(V)[\delta(V)]$ where there's this funny shift depending on the Betti numbers of V.

There's an action of $\mathbb{G}_m$ on V where λ takes v to λv. Then on the dual λ takes v^* to $\lambda^{-1} v^*$. You can check that $\mathbb{G}_m$ acts on $cq(v)$ by $\lambda^{\chi(v)}$.

So I've given you this functor and now I want to show how to use the functor to make a factorization algebra. So unlike the determinant, the bv functor makes sense on large complexes. If I look at the index theorem, I might have infinite dimensional kernel and cokernel.

Before, remember, I had $V^\bullet$ which has V^0 and V^1. I'm going to explain how to take V and do everything open set by open set on X. Let me define a cosheaf on X, I have the category of open sets on X, and there's a functor to $Presymp_1$, which puts in degree 0 and 1 the compactly supported sections over V^0 and V^1, and then in degree -1 and -2 the compactly supported sections of $V^0 \otimes Dens$ and $V^1 \otimes Dens$. Then when I pair I can integrate my densities. The differentials use P and P^*. Call this sheaf $\mathcal{E}^P$.

Now I can consider the following composition $bvq \circ \mathcal{E}^P$. this takes $U \sqcup U'$ to $bvq \circ \mathcal{E}(U \sqcup U') \cong bvq \circ \mathcal{E}(U) \otimes bvq \circ \mathcal{E}(U')$.

The upshot is that $bvq \circ \mathcal{E}^P$ is a factorization algebra on X. Then we know that $bvq \circ \mathcal{E}^p(X)$, for X closed, there's a natural action by $\mathbb{G}_m$ by $\lambda^{ind(P)}$, but this is the same as $H_*^{fact}(X, bvq \circ \mathcal{E})$, which I can write of the homotopy colimit over disks of $bvq \circ \mathcal{E}$. So this is a local to global object that recovers the index.

That was the first part of the talk. Thanks to BV quantization you can give a local construction that globalizes. So I want to pursue something more like Riemann–Roch. I'll do a complex geometry version, there are other versions.

What is an elliptic complex that lives on every complex d-dimensional manifold? We have a shy student in the front who didn't want to speak up. But there are tensor bundles over every complex d-fold, $T^{(m,n)} = (T^{(1,0)})^{\otimes m} \otimes (T^{1,0})^{*\otimes n}$. Consider $\Omega^{0,*}(X, T^{(m,n)})$, the Dolbeaut complex. I'll do the exact same procedure on this thing.

I want to construct a presymplectic vector space. Again I'll work with compactly supported sections. I want to work in a funny degree to match the conventions from physics. So $\mathcal{E}^{(m,n)}$ will be $\Omega_c^{0,*}(\quad, T^{(m,n)})[1] \oplus \Omega^{d,*}(\quad, T^(m,n)^*)[d]$. The d is my replacement for twisting by the density bundle. So I get

$$\begin{tikzcd}
-2 & -1 & 0 & 1 & 2\,. \\
\cdots \arrow[r, "\partial"] & \Omega^{d,d-1}(X, T^{m,n*}) \arrow[r, "\partial"] & \Omega^{d,d}(X, T^{m,n*}) & & \\
& \Omega^{0,0}(T^{m,n}) \arrow[r, "\bar\partial"] & \Omega^{0,1}(T^{m,n}) \arrow[r, "\bar\partial"] & \cdots &
\end{tikzcd}$$

What does BV quantization produce from this? I have the moduli space of closed complex d-folds $\mathcal{M}_{(d)}$, and sitting over it is C_d, the universal d-fold.

We have that $bvq \circ \mathcal{E}^{(m,n)}$ is a line bundle on $M_{(d)}$. Can I identify it? What's its Chern class? This is called an anomoly in physics?

Grothiendieck–Riemann–Roch tells us how to compute the Chern class of this line bundle, $ch(T_{c/M}^{(m,n)}) \wedge Td(T_{c/M})$. Our strategy to try to recover this kind of result is to follow Alessandro's suggestion. We could put a connection on it and compute the curvature, which represents the first Chern class. I can do a formal geometry version. To compute the curvature of a connection on this line bundle, let me fix a point in the moduli space and over it the line bundle. The formal neighborhood of this point x in the moduli space, thanks to Kodaira and Spencer, this is a dg Lie algebra, specifically $\Omega^{0,*}(X, T_X^{1,0}) =: \mathcal{T}_X$. If you continue using this Koszul duality dictionary, then a vector bundle is a module of $\mathcal{T}_X$, and c_1 corresponds to the action of $\mathcal{T}_X$ on this module. If you have an action, you can postcompose by trace to get a map to your base ring, and this composition is a 1-cocyle in the Chevalley–Eilenberg chains of the Lie algebra, which you can think of as a cohomology class in the Lie algebra cohomology of the Lie algebra. I won't explain this dictionary but I want to use it.

Since $\mathcal{T}_X$ is local on X, we can consider the problem for X a polydisk of dimension d. The action is via the Lie derivative, and so it's a local action. There's a local version of Lie algebra cohomology for these. The c_1 lives there. This is a version of Gelfan'd–Fuchs cohomology. We'll piggyback on others' computations of this. I won't describe this Lie algebra cohomology other than saying it's topological, so you can write down continuous cochains and you can look only at cochains with support on the small diagonal.

With Brian Williams, who is somewhere in here, we computed, and let me introduce some notation. Consider the universal bundle $EU(d)$ over $BU(d)$. I can look at the $2d$-skeleton and pull back to $P(d)$. For any complex d-fold, I can do a Borel construction, take $P(X) = P(d) \times^{U(d)} T_X$.

Theorem 5.1. *(Gwilliam–B. Williams)*
First,

- *there is a natural isomorphism* $H^k_{loc}(\mathcal{T}_X) \xrightarrow{\cong} H^{2d+k}_{sing}(P(X), \mathbb{C})$.
- *The more interesting part is that we actually identify the cocycle for bvq of this* $\mathcal{E}^{(m,n)}$ *thing.*
  For bvq $\circ\, \mathcal{E}^{(m,n)}$ *(a factorization algebra as before), if I just compute the first Chern class when* X *is a polydisk, well, let me say, for* X *contractible we have* $H^{2d+1}(P(d)) \cong H^{2d+2}(BU(d))$. *Then that maps under this isomorphism to*
  $$c_1(bvq \circ \mathcal{E}^{(m,n)}) \cong [Td \wedge ch(T^{m,n})] \in H^{2d+2}(BU(d)).$$

So there's a corollary of the statement that may or may not be familiar to you. It's quite punchy. You've certainly heard people assert in some cryptic way something like this.

Corollary 5.1. *Let's go to the simplest interesting case, with* $d = 1$ *and* $n = 0$, *so these are just tensor powers of the tangent bundle. Then the central chare of "free* $\beta\gamma$*" for* $T^{\otimes m}$ *is* $6m^2 + 6m + 1$. *For* $m = 1$ *you see that* $c = 13$, *for* $m = 0$ *the central charge is* 1. *If you try to do holomorphic bosonic string theory, where the target is* $\mathbb{C}^k$, *then the central charge is* $k - 13$, *which means you need the target to be* 13 *complex dimensional or* 26 *real dimensional. Physicists called this the holomorphic anomaly.*

I have just a few minutes. I got something that looks nice from the point of view of the index theorem. There's a consequence in the language of factorization algebras, which I'll sketch to finish off.

As we remarked before, since $\mathcal{E}$ has a $+1$ presymplectic pairing, the functor $Sym(\mathcal{E}())$ is a $Poisson_0$-algebra. Then $\mathcal{T}$ maps as a Lie algebra to $Sym(\mathcal{E}(-1)[1])$. This map sends a vector field v to $(\gamma, \beta) \mapsto \int (L_V \gamma) \wedge \beta$. Then $Sym(\mathcal{T}_c[1])$ maps to $Sym(\mathcal{E})$. I'm basically out of time. In words, both of these things are the associated graded of more interesting factorization algebras. On the right this looks like BV quantization. On the left, if I take the Chevalley Eilenberg chains, I get something that looks like this. You might ask if I can lift to a map of the quantizations? The obstruction to doing that is precisely this Chern class. If it does vanish, you can lift, but if it doesn't you could centrally extend and then do it. In the case of Riemann surfaces, you can recover Virasoro, you get a map from the Virasoro factorization algebra to a certain class of vertex algebra of higher dimension.

6. Claudia Scheimbauer: (Op)lax natural transformations for higher categories and two applications

This is joint with Theo Johnson-Freyd.

Thank you very much for giving me the opportunity to speak here. (Op)lax natural transformations, that means either lax or oplax. What is this? If I have two functors F and G from $\mathcal{B}$ to $\mathcal{C}$, I want a natural transformation. If I have $B_1 \xrightarrow{b} B_2$, I can apply both F and G to the 1-morphism, and we can require to have maps

$$\begin{tikzcd} F(B_1) \arrow[r, "F(b)"] \arrow[d, "\eta(B_1)"] & F(B_2) \arrow[d, "\eta(B_2)"] \\ G(B_1) \arrow[r, "G(b)"'] & G(B_2) \end{tikzcd}$$

If the target is at least a 2-category, we could rerquire the square to fill in with an isomorphism (called strong or pseudo) or with a map from $G(b)\eta(B_1) \to \eta(B_2)F(b)$ (lax), or the reverse (oplax).

For bicategories, this is already subtle. There is a bicategory of strong finctors, either lax or oplax transformations, and modifications, but these bicategories cannot be composed. These do not form a tricategory if you take lax or oplax here. So there's no interchange.

Why are we interested in such a thing? This will be both motivation and applications. The first application is relative or twisted field theories. Relative comes from Freed–Teleman and twisted from Stolz–Teichner. Take $\mathcal{B}$ to be a category of bordisms, and $\mathcal{C}$ to be some 2-category, probably a delooping of vector spaces. Take two such functors, symmetric monoidal, field theories, and ask for a natural transformation $T_0 \to T$. We often choose T_0 to be the trivial field theory. We call T the twist or if invertible the anomaly. In examples these are not strong. You have to weaken this a little bit. In the language of those examples, this is a projective functor; this implements the idea that to an object in the bordism category we get an element in $T(b)$. Another way to get at this is to do "boundary theories," you can go back and forth in a certain way. There was a nice paper of Fiorenza–Valentino to go back and forth.

Another reason to want lax or oplax. If you take topological bordisms, your category has adjoints. Then all strong natural transformations are invertible. So you won't get interesting examples of relative field theories unless you relax the conditions.

The second condition is what we could call Morita theory for (op)lax structures. You take an algebra (object in some higher category $\mathcal{C}$) and now morphisms, we normally require $\varphi(ab) = \varphi(a)\varphi(b)$. If I have a morphism $\varphi : A \to B$, I can consider the square

$$\begin{tikzcd} A \otimes A \arrow[r] \arrow[d] & A \arrow[d] \\ B \otimes B \arrow[r] & B \end{tikzcd}$$

and I can decide how to fill this. I can request this to be strong, lax, or oplax. We can play the same game with bimodules. For bimodules M and N, I can take

$$\begin{tikzcd} A \otimes M \otimes B \arrow[r] \arrow[d] & M \arrow[d] \\ A \otimes N \otimes B \arrow[r] & N. \end{tikzcd}$$

Now I can make a category with objects algebras, morphisms bimodules, and 2-morphisms morphisms of bimodules, and here I should again choose strong, lax, or oplax.

Now to preview the results,

(1) we build such a framework and then can give a definition in this setting and prove, interpret this element in $T(b)$ for oplax, or for lax

Theorem 6.1. *"lax trivially twisted field theories are untwisted"*

(2) We can also build the higher category $Alg_1^*(\mathcal{C})$ using strong, lax, or oplax for $*$. This will give E_d-algebras, we'll see that later.

The basic idea, let me call this "the oplax square." Given a (possibly symmetric) monoidal (∞, n)-category $\mathcal{C}$ we construct an again (possibly symmetric) monoidal (∞, n)-by (∞, n)-category $\mathcal{C}^\square$governing the desired diagrammatics. By $\mathcal{C}^\square$, I have two indices and get an (∞, n)-category if I fix either one of the two indices.

All you need to know about this for the purposes of this talk is that there is an (∞, n)-category $\mathcal{C}^\downarrow \coloneqq \mathcal{C}^\square_{\cdot,1}$ of "vertical" 1-arrows and an (∞, n)-category $\mathcal{C}^\rightarrow \coloneqq \mathcal{C}^\square_{1,\cdot}$ of "horizontal" 1-arrows. These have source and target maps to $\mathcal{C}$ that will let us build natural transformations.

For an (∞, n)-category $\mathcal{B}$ and an $(\infty, n+1)$-category $\mathcal{C}$, a *lax natural transformation* $\eta : F \to G$ between strong functors is a strong functor $\eta : \mathcal{B} \to \mathcal{C}^\downarrow$. An *oplax natural transformation* $\eta : F \to G$ between strong functors is a strong functor $\eta : \mathcal{B} \to \mathcal{C}^\rightarrow$, in both case so that $s \circ \eta = F$ and $t \circ \eta = G$.

Let's do examples. The lax case first. In this category $C^\downarrow$, the objects are vertical arrows in $\mathcal{C}$, a 1-morphism. Then in $\mathcal{C}^\square$, a one morphism between $C_1 \xrightarrow{c} C_2$ and $D_1 \xrightarrow{d} D_2$ is a pair of morphisms $a_i : C_i \to D_i$ and a 2-morphism $a_2 \circ c \Leftarrow d \circ a_1$. The two-morphisms are [picture].

$$\begin{tikzcd} C_1 \arrow[d] \arrow[r, bend left] \arrow[r, bend right] & D_1 \arrow[d] \\ C_2 \arrow[ur, Rightarrow] \arrow[r, bend left] \arrow[r, bend right] & D_2 \end{tikzcd}$$

So let's test the definition. Given F and G from $\mathcal{B}$ to $\mathcal{C}$, then $\eta : \mathcal{B} \to \mathcal{C}^\downarrow$, $s \circ \eta = F, t \circ \eta = G$. To β we get an object in $\mathcal{C}^\downarrow$, that is a morphsim in $\mathcal{C}$ between $F(B)$ and $G(B)$. I can look at what happens to the source and target of a one-morphism, and then there should be a 2-morphism in $\mathcal{C}^\downarrow$.

So for the first application, to relative or twisted QFTs, we have

$$\begin{tikzcd} \mathcal{B}ord_n \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathcal{C} \end{tikzcd}$$

Definition 6.1. Let $T : Bord_n \to \mathcal{C}$ be a symmetric monoidal functor. An (op)lax T-twisted field theory Z is a symmetric monoidal (op)lax transformation $\mathbf{1} \Leftarrow T$.

Theorem 6.2. *(Johnson-Freyd, S.) The lax trivially twisted theories*

$$\begin{tikzcd} \mathcal{B}ord_n \arrow[r, bend left, "\mathbf{1}"] \arrow[r, bend right, "\mathbf{1}"'] & \mathcal{C} \end{tikzcd}$$

are the same as untwisted field theories valued in the looping of $\mathcal{C}$, $\Omega\mathcal{C}$.

$$Bord \to \Omega\mathcal{C}$$

Okay so let's go back to our pictures on the top. We want the source to be $\mathbf{1}$ and the target to be b, so we have $\mathbf{1} \to T(b)$. The 1-morphism will be

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\mathbf{1}"] \arrow[d] & \mathbf{1} \arrow[d] \\
\mathbf{T(B_2)} \arrow[r, "T(b)"'] \arrow[ur, Rightarrow, "Z(b)"] & \mathbf{T(B_2)}
\end{tikzcd}$$

The two-morphisms I won't draw but you get something from $\mathbf{1}$ to $T(b)$.

For oplax, everything is switched. [pictures, missed some.]

Let me move on to the second application, the "even higher" Morita category of E_d-algebras. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Constructions by Haugseng using ∞-operads and by myself and Calaque using factorization algebras, lead respectively to unpointed and pointed bimodules.

What is the category that is constructed here? It's an (∞, d)-category, whose objects are E_d-algebras, morphisms bimodules of E_d-algebras, 2-morphisms bimodules of bimodules in a certain sense, and so on.

To give an example, for $d = 1$, we have algebras, really A_∞-algebras, the morphisms are bimodules, or actually homotopy bimodules. For $d = 2$, you take say, $\mathcal{C}$ to be some nice category of categories. For E_2 you get braided monoidal categories, then monoidal categories, and then categories. We'll see this later maybe in Chris Schommer-Pries' talk.

What can we do with this? Now we want to extend this to take $(d+1)$-morphisms, 1-morphisms of bimodules here. We'd add $d2$-morphisms, these should be 2-morphisms in $\mathcal{C}$. Now we can go all the way to $d+n$-morphisms in $\mathcal{C}$. Now here, we can choose if we want strong, lax, or oplax. We will fix this once and for all and take the same type of morphisms all the way down. This will give an $(\infty, d+n)$, in fact even symmetric monoidal, category $Alg_d^*(\mathcal{C})$. For this construction we'll use our oplax square which will let us use our morphisms.

Now the idea for $d = 1$, a *lax* morphism of algebras in an algebra object in $\mathcal{C}^{\downarrow}$. What does this mean? I have a map $A \otimes A \to B \otimes B$, and then both of these map to $A \to B$, and I have something in $\mathcal{C}^{\downarrow}$ between these, which is a lax morphism of algebras. If you do the oplax case you get the other direction.

Now we can play the game with a lax morphism of bimodules, it's a bimodule object in $\mathcal{C}^{\downarrow}$. You can check that that's exactly what you expect.

In the last two minutes, let me say some results.

Define $\mathcal{C}^*_{\vec{\mathcal{C}}}$ to be $\mathcal{C}^{\square}_{\cdot,\vec{\mathcal{C}}}$ for lax, $\mathcal{C}^{\square}_{\vec{\mathcal{C}},\cdot}$ for oplax, and $[\theta^{\vec{\mathcal{C}}}, \mathcal{C}]$ for strong.

Theorem 6.3. *(Johnson-Freyd, S.) For both constructions of $Alg_d()$, under some mild conditinos on $\mathcal{C}$, I get $(\vec{k}, \vec{\ell}) \mapsto Alg_d(\mathcal{C}^*_{\vec{\ell}})_{\vec{k}}$ is a symmetric monoidal $(\infty, n+d)$-category.*

7. Chris Schommer-Pries: Extended 3-dimensional topological field theories

I was first told about topological field theories because they gave manifold invariants. Here's one, it gives you 0 if it's not the 4-sphere and 1 if it is. But this is stupid, it's too hard to compute.

A field theory invariant is supposed to be local. If you cut your manifold into two pieces then you should be able to associate invariants to the two pieces and reconstruct the invariant to the union. You could think that you give complex numbers to the two halves and then multiply them. In dimension two this should be variations on the Euler characteristic, and in dimension three always the number 1. Maybe we should associate also something to the boundary where we glue the manifold together, maybe a vector space. To the first half maybe we get a vector and to the second half a covector, and then I can reconstruct the invariant of the manifold by pairing it to give a number.

This was done by Atiyah and Segal, one thing to do is axiomatize this in the following way.

We have the category $Bord_{(d-1,d)}$, where the objects are $d-1$-dimensional manifolds and the morphisms are d-dimensional bordisms. Then Z should be a functor, symmetric monoidal, to the category of vector spaces with $\otimes$. When we have disjoint union it goes to tensor product, the empty bordism gives me the trivial vector space, locality comes from this being a functor, a symmetric monoidal one.

Extended TFTs are a variation where you cut up the manifold along higher codimension surfaces as well. In order to do that, we had to find a home for something more complicated. You end up getting a higher category of bordisms $Bord_{d-k,\ldots,d-1,d}$ and a functor to a higher k-category of $nVect$. This is somehow a delooping of what we had before. We started with a number and passed to vector spaces. We had an endomorphism of the unit and that's like a loop. For $2-Vect$ we want a symmetric monoidal two-category where the endomorphisms of the unit is $Vect$. Then we would like to further deloop this.

What are some ways to deloop vector spaces?

Here's an example. There's this two-category Alg which is algebras, bimodules, and maps, which showed up in Claudia's talk. This is symmetric monoidal. The trivial algebra, a $\mathbf{k}-\mathbf{k}$ bimodule is just a vector space. You can look at extended (one layer down) TFTs with this as the target.

This is the same as linear categories, which are Abelian, $\mathbf{k}$-linear, satisfying finiteness conditions, right exact functors, and natural transformations. You want this to be interestingly monoidal and to do that, Deligne gives you something but it's only functorial for right exact things.

Let's look now at algebras in here, keep Claudia's machine going. Next you get TC, the symmetric monoidal 3-category of tensor categories, bimodule categories, functors, and transformations.

Here I'll consider the oriented case to make things simpler. You can think of this from the point of view of manifold invariants. These will be easier to compute. The unfortunate thing is that as far as manifold invariants go, they have not been so spectacular. Topologists are good at computing things and working with manifolds. It's really hard to come up with something to learn about manifolds that wasn't known in a simple way long ago. There are connections to quantum knot invariants but you can do all of that with diagrammatics.

Recently in the past ten years there has been a rennaissance in our ability not only to compute, to calculate, but also to *classify* TFTs. The surprising thing is when you classify them, you see a structure emerge, a structure you want to understand for reasons unrelated to manifolds. When you have higher bordisms, you get higher algebraic structure.

The easiest example is the statement that $Fun^{\otimes}(Bord_{(1,2)}, \mathcal{C}^{\otimes})$ is the same thing as commutative Frobenius algebras in $\mathcal{C}$. With these classification results, they proceed by combining two important basic ideas. The first is a differential topology ingredient, Morse theory, Cerf theory, that tells you how to decompose manifolds, and then something higher categorical, combinatorial, to classify functors between higher categories. Combine these and you get classification results.

We have a presentation of $Bord_{(1,2)}$ here: [picture]. The objects are generated by a circle. Then the morphisms are generated by the cup, cap, pants, and copants. Reading up, you get a multiplication from the pants. You get a unit from the cap. The relations say that this is a commutative multiplication, unital, and then eventually see this is the universal theory for Frobenius algebras. Then any surface gives us an equation in the language of commutative Frobenius algebras. We won't learn anything new about surfaces or commutative Frobenius algebras. Taking this perspective allows us to look here and see the kind of games we can play. One thing we can do is the process of dimensional reduction.

When I say classification, I haven't done a complete and total classification. Commutative Frobenius algebras are not classifiable in a precise sense. But we can learn some things. One thing you can always do, there are maps between the bordism categories. If I have a k-manifold, I can get a map from the d-bordism category to the $d + k$-bordism category by crossing with my k-manifold. We could consider $Bord_{0,1}$ and cross with the circle. We get a functor that goes

$$\begin{array}{c} Fun^{\otimes}(Bord_{(0,1)}, Vect) \\ \uparrow \\ Fun^{\otimes}(Bord_{(1,2)}, Vect) \end{array}$$

These are easy to understand, we have the objects + and − and then we have morphisms a left and right elbow and relations that say when we compose these we get the identity. This tells me that I should get two vector spaces and a way to pair them, and an element in the tensor product, and I can get the identity. I can represent the identity in the tensor product under the pairing. Then this should be finite dimensional and that forces the value on − to be the dual of +. So you get the elementary fact that a commutative Frobenius algebra is finite dimensional. That's one thing you can do.

There are different classification results. There's a famous one, the cobordism hypothesis. This is about fully local field theories. It says that functors from $Bord_{(0,\ldots,d)}$ to any symmetric monoidal category $\mathcal{C}$ is the same thing as the d-groupoid of so-called fully dualizable objects in $\mathcal{C}$ plus $SO(d)$-homotopy fixed point data (because we're working with oriented versions). You can see this directly by giving a generators and relations presentation for dimension 2. Now you get an invertible two-morphism that witnesses the thing that used to be the identity. But you have other morphisms, bordisms that connect and relate the two different morphisms, and then you have new relations. [picture] This is the same as saying the left and right elbow are adjoints. When you increase your dimension, you add more duality to your structure. We had duality at the level of objects and now you get it at the level of morphisms as well. If you want the oriented theory you get extra structure there. I won't give you a full list of generators.

You can now classify these, we can look at

$$Fun^{\otimes}(Bord_{(0,1,2)}, Alg)$$

and this is a theorem in a paper which is now a book that this is equivalent to the two-groupoid of semisimple symmetric Frobenius algebras, the same kind that showed up in Nathalie's talk.

There's a relation between the bordism categoris $Bord_{(0,1,2)}$ and $Bord_{(1,2)}$, which is the endomorphisms of the unit object. This is true in their targets so you get a restriction map from semisimple symmetric Frobenius algebra to commutative Frobenius algebras (by taking the center). There are many commutative Frobenius algebras which are not semisimple, and if you take centers, that preserves semisimplicity. You can completely classify these semisimple symmetric Frobenius algebras. [missed the classification]. They're all sums of Euler theories.

Now we can move on to other classifications. This theorem lets you classify fully local theories. Recently, well, here's another theorem, due to Bruce Bartlett, Chris Douglas, myself, and Vicary, a higher analogue of the situation for two dimensions.

Theorem 7.1. *$Bord_{(1,2,3)}$ is free on an (anomaly free) modular tensor object.*

We give an explicit presentation of the Bordism category. It's the $(1, 2)$ presentation futher categorified. Identities become invertible morphisms. You get an automorphism of the cylinder for the Dehn twist, and then you have non-invertible generators which are handles of some kind. There is a list of relations, you apply Cerf theory and get a presentation that's bigger than this, you whittle away and the relations don't look like anything, but after whittling away there are only 33 relations, all of which have clear higher structural meaning. You could look at representations of this in a target like 2-vector spaces. You get a category, a multiplication (a monoidal thing) and then an associator, and then a relation is that it satisfies the triangle, pentagon, and hexagon, and twist relations, so it's a balanced braided monoidal category. Then you get things that tell you that these things are parts of adjunctions. Things are both right and left adjoints. Then you can express relations that say you are a rigid object, then ribbon category, then eventually anomaly free. These are all things that are familiar to category theory.

So now, do I want to add anything to that? So now I'm in a good position, we can look at functors, monoidal functors from $Bord_{(1,2,3)}$ into $2-Vect$, and these are like linear categories with the Deligne tensor product $\boxtimes$. Then these are anomaly-free modular tensor categories. This is again Bartlett–Douglas–Schommer-Pries–Vicary.

You can also look at functors from $Bord_{(0,1,2,3)}$ into TC. There there's a theorem as well, joint with Chris Douglas and Noah Snyder. We identify inside the 3-category the fully dualizable objects which are the so-called fusion tensor categories, which are the ones that are semisimple. There's a certain variation on this, the spherical ones, which show up in nature, these are $SO(3)$-homotopy fixed points. So these field theories contain spherical fusion categories, representation categories of finite quantum groups. People who study finite dimensional Hopf algebras see these a lot. So these also show up with Von Neumann algebras, people working with operator algebras. They also come around from these TFTs. As before we can do dimensional reduction, that tells you that the underlying category has to be semisimple. You also get the Drinfel'd center from spherical fusion categories to modular tensor categories. You can ask if this is surjective. The cokernel is a group, the Witt group, and there are many interesting TFTs which have non-zero class inside the Witt group. The most basic example is quantum Chern–Simons theory. For almost all choices of level they don't extend to the point.

This is just scratching the surface. Just like every surface gives you an equation in Frobenius algebras, every three-manifold gives you an equation in modular tensor categories and spherical fusion categories. The most basic example that we can extract from that is this basic three dimensional fact [picture]. This is a framed surface in $\mathbb{R}^3$. This is a manifold, one that if you put two loops in $SO(3)$, you get the identity, the Dirac belt trick is what this picture is supposed to be. This is a proof, a proof of something, a proof of a theorem of Etingof and [unintelligible]. If you take, if you map to the quadruple dual, you're canonically equivalent to the identity. This is invertible, that tells you you have a natural isomorphism between these two functors. What else can we show? What does the prime decomposition of 3-manifolds tell you about modular tensor categories. The manifold invariant picture is, I think, the wrong idea for the future of field theories. Instead we can use what we know about manifolds to find out about these algebraic things.

8. June 17: Hiro Tanaka: A (possibly non-Thom) ring spectrum of Lagrangian cobordisms, and the Fukaya category

[One announcement. There will be a barbecue at HIM, bring your own food and dirnk, at 7 tonight.]

Thank you to the organizers for letting me talk at the workshop and letting me be here for two months, I've gotten a lot of math done. I wanted to talk about a relationship between a topic that belongs in stable homotopy theory and a topic that belongs in symplectic geometry. There's a central object in this story which is an E_∞ ring spectrum that I don't know much about. I won't assume that people know any of the things in the title, except maybe E_∞ ring spectrum, but let me start by introducing Fukaya categories. I'll do it with an example.

Consider the manifold $M = \mathbb{C}$. One I'll call γ, a curve, and the other will be $\mathbb{R}$ which I'll call P. I want to play the following game. I'm going to construct a cochain complex. First I should tell you the underlying graded vector space. As a vector space, or R-module, it's generated by the intersection points between γ and P. Let me name them p and q. Now I should tell you the differential. What is the differential of the point p? It's a count of the number of points in a moduli space. You have to worry about signs but that's a technical detail. I'll count the number of maps $u : \mathbb{R} \times [0,1] \to M$ satisfying conditions. First, I want boundary conditions. I want $u(t,0)$ to be in γ. Likewise I want $u(t,1)$ to be inside P. I want $\lim_{t\to\pm\infty}$ to be a constant path in $\gamma \cap P$. There's no way I can count the moduli space of such things. Even smooth, there are infinitely many choices you can make. I'll look not just at strips of this form but strips satisfying a differential equatian $\bar\partial u = 0$. This has a natural $\bar\partial$ operator because of the complex planes. I claim that there ar three kinds of obvious maps. There are the obvious Whitney disks. There are constant maps that go just to p and just to q. This is translation invariant in the $\mathbb{R}$ variable. There's an $\mathbb{R}$-action. SO I count the number of points in the set after modding out by this action.

I should think of these fixed points as -1-dimensional and exclude them. There is one and exactly one strip, and so the differential of p is q. I claim that the differential of q is 0.

It turns out that this is a game we can play, this one is a little dumb but I promise it'll pay off later. In general we can fix a symplectic manifold (M, ω, J), with a compatible almost-complex structure J. Being symplectic means that I've chosen a 2-form ω which is closed and such that ω^n is a volume form. An almost complex structure J is an endomorphism of the tangent bundle of M such that $J^2 = -1$. What does it mean that they are compatible? It means that $\omega(\quad, J \quad)$ is a Riemannian metric.

This is a lot of structure. In our example you can take $M = \mathbb{C}$, $\omega = dxdy$, and $J = i$.

Now take two Lagrangians γ and P in M. In favorable circumstances, you can define a cochain complex $CF^*(\gamma, P)$, called the Floer cochain complex of γ and P.

Everything I've said in the example works here except for $\bar{\partial}$, which might not work because J might not be integrable. I can say $du \circ i = J \circ du$, and that's what I replace the Cauchy–Riemann equation with.

So I get a cochain complex generated by intersection points of Lagrangians where the differential is given by counting a moduli space of holomorphic strips between them.

What do I mean by Lagrangians? I mean half dimensional submanifolds where the ω restricts to 0.

Given (M, ω, J), the *Fukaya category of* M is the category with objects certain Lagrangians of M and morphisms between γ and P the Floer chains $CF^*(\gamma, P)$.

So some remarks. You can try to do Morse theory on the paths from γ to P. The gradient flow equation becomes the Cauchy–Riemann equation.

If $P = \gamma$, we have to make choices. In fact, the Fukaya category depends on choices for k-tuples of objects. We deform P by a suitable isotopy and then we compute the cochain complex with the deformed copy. I don't want to get into this.

[What's composition?]

You sometimes hope that the audience won't ask this question if you don't want to talk about Fukaya categories for half an hour. The infinite strip is holomorphically equivalent to a disk with two boundary points missing. The composition will be defined by playing the same game with more points on the boundary. If I have three Lagrangians, $CF(L_1, L_2) \otimes CF(L_0, L_1)$ needs to map to $CF(L_0, L_2)$. I take $q \otimes p$ to $a_r r$, so a_r is the number of holomorphic maps from the disk to M such that the three points that are missing map to p, q, and r counterclockwise, and so on. The other question is, "is this associative?" and it is not. It's associative up to homotopy. You count a higher number of marked points. Was that homotopy a choice? It was. Can I contract it? Yes, and so on. So this is an A_∞ category.

This is equivalent to a strict category for us. Also, the coefficient ring depends strongly on the geometry of M. I didn't justify why this is $\mathbb{Z}$-graded, and it's often $\mathbb{Z}/n\mathbb{Z}$-graded, this comes from the geometry of the situation.

Let me give some motivation. So this topic became more interesting to people after a Kontsevich ICM address.

Conjecture 8.1. (Homological mirror symmetry) For every complex variety X which these days people would restrict to being Fano or Calabi–Yau, there exists another variety $X^\vee$, possibly with decorations, such that the following holds.

The derived category of coherent sheaves $D^b Coh(X)$ is a nice invariant of your variety. On the other hand, you could also look at the Fukaya category of X if

X is projective, it gets a symplectic structure. So the claim is that $D^bCoh(X) \cong Fuk(X^\vee)$ and $Fuk(X) \cong D^bCoh(X^\vee)$.

This is verified in the cases we have checked. Why should you care about a conjecture like this? One example is geometric Langlands. One way to state geometric Langlands is that you can look at $D^bCoh(Loc_\Sigma G)$ and $\mathcal{D}Mod(Bun^L_\Sigma G)$. In some cases, this category of $\mathcal{D}$-modules should look like $Fuk(T^*Bun^L_\Sigma G)$. You'll see now that this looks a lot like a statement that would come out of the mirror symmetry conjecture. You expect an SYZ-fibration between them in order to find this duality and that's what happns in the examples we know.

Okay, cobordisms. Let's let L_0 and L_1 be Lagrangian submanifolds of M. Then a Lagrangian cobordism is a Lagrangian submanifold Q inside $M \times T^*\mathbb{R}$. You might like to think of a cobordism as sitting over an interval, and at the ends the things you want your cobordism to be between. I could put this in $M \times [0,1]$ but that's not symplectic, so that's why I replace $[0,1]$ with the cotangent bundle of $\mathbb{R}$.

There's the part of the real axis that lives to the left of 0. We can ask that to the left, we get a collared copy of L_0 and likewise between 1 and ∞, we can ask that it look like L_1.

Just consider $M \times T^*\mathbb{R}$ projecting to $T^*\mathbb{R}$. I'm asking that its image to the left of 0 and the right of $\mathbb{R}$ is the real axis with preimage L_0 over each point to the left and L_1 over each point to the right.

Now we unite Fukaya categories and Lagrangian cobordism. Fix M and consider $M \times \mathbb{C}$, really $M \times T^*\mathbb{R}$. What I can do is also fix a cobordism P, which is some mess but looks okay to the left and right because it's collared. Also fix a Lagrangian X inside M. This is an object in the Fukaya category of M. This cobordism realizes a morphism in the Fukaya category between the ends of the cobordism. We compute $CF(X \times \gamma, P)$. What are the generators? They live over p and q. What are the intersections about p? They're intersections between L_0 and X. So the cochain complex is a direct sum of $CF(X, L_0)$ and $CF(X, L_1)$. For reasons that I refuse to explain I shift the grading on the second complex. Now what about the differentials? If I have a J-holomorphic map $u \to M \times T^*\mathbb{R}$ and then project to $T^*\mathbb{R}$, I can look at the image there to help classify. There were three kinds, the ones that are degenerate at p, at q, and between them. The ones at p are in the differential of $CF(X, L_0)$. Similarly, you get the usual differential of $CF(X, L_1)$. Finally there's a third piece Ξ_P. This is the mapping cone of a linear map. The fact that the differential squares to zero implies that Ξ_P is a chain map.

In fact, this statement is true no matter what X we chose, so this is a natural transformation, defined by a cobordism, between the functors represented by the ends. That's just an object in the Fukaya category by the Yoneda embedding.

Theorem 8.1. *Let M be a symplectic manifold satisfying some conditions. Then there exists a functor from Lag, well, what is that, objects are Lagrangians in M with some conditions and morphisms are Lagrangian cobordisms with some conditions, to $Fun(Fuk(M), Ch)$ I can send L to $CF(\quad, L)$ and P to Ξ_P*

[missed some]

Theorem 8.2. *(Nadler, Tanaka). For all $\Lambda \subset M$, the category $Lag_\Lambda(M)$ is a stable ∞-category.*

But cobordism categorise usually don't have zero objects. The zero object here is the emepty manifold. This might seem weird because not everything can be made empty. But I can multiply by a function that pushes everything out to ∞.

Does this functor respect mapping cones? Yes, it's an exact functor. Let me give a geometric corollary.

It turns out that any compact cobordism in this category is an equivalence, not because it's an h-cobordism, but just because of topology on the space. This tells us about characteristic classes in terms of the Fukaya category. Even if you didn't follow anything in this talk, this is a pretty clean statement.

Let me be a little more specific. The cartoon picture I drew earlier of $Q \subset M \times [0,1]$, you'll never be able to do stable homotopy. If M is a point, you'd like to study $M \times \mathbb{R}^\infty$, not just M. The objects are Lagrangians inside of $M \times T^*\mathbb{R}^N$. If I have Lagrangians in the wrong place, I can stabilize to push things to higher dimensios. [missed some].

I want to make two claims about Lagrangian cobordisms.

Theorem 8.3. *Consider $M = *$. Then $Lag(*)4$ admits a symmetric monoidal structure. It's the direct prodect. Moreover, this respects finite limits and colimits in each variable and the unit is the Lagrangian which is a point.*

At first this looks dumb, but the fact that it preserves small limits and colimits is what makes it really interesting.

Let me state a theorem that says that you'll be linear over the category $Lag_*(pt)$. I want to set up a theory of modules or Lagrangian coboordisms linear over this ring. I can take these R-linear categories.

Theorem 8.4. *For ever module, there is an action respecting finit limits and colimits $Lag_{pt}(pt)Lag(U) \to Lag_\Lambda(M)$.*

Corollary 8.1. *Every $Lag_\Lambda M$ is linear over $Lag_{pt}pt$.*

Let me give an example. $Lag_{pt}(pt)$ has that $\mathbb{R}_\infty$ factor. It's not hard to see that $Fun(Fuk_{pt}(pt), chain)$is $chain_R$. Sitting inside the Lagrangians is $End(pt)$ and then that maps to R [missed how]. I'll end here because I'm already over time.

9. Arthur Bartels: June 18: The 3-category of conformal nets

This is joint with Chris Douglas and André Henriques. I'll start very easy, with the 2-category of rings. The objects are rings R, the 1-morphisms are bimodules ${}_RM_S$, and the 2-morphisms are $R-S$-bilinear maps. TO be complete, we also say that composition of 1-morphisms is the tensor product over R of ${}_TN_R \otimes_R {}_RM_S$ and the unit is ${}_RR_R$ as a bimodule.

Let's make this a little more interesting, and go to the 2-category VN_2 of von Neumann algebras. Its objects are now von Neumann algebras, that is subalgebras of the bounded operators on a Hilbert space $\mathcal{B}(H)$, closed in the ultra-weak topology. We'll need it later, so I'll give the tensor product of von Neumann algebras, their tensor product is, well, start with the algebraic tensor product. Pick Hilbert spaces for A and B and then $A \otimes B$ is in $\mathcal{B}(H \bar{\otimes} K)$ and you close with respect to the ultra-weak topology. Then 1-morphisms are bimodules ${}_AH_B$ where H is a Hilbert space where $A \subset \mathcal{B}(H)$ and $B^{op} \subset \mathcal{B}(H)$ and the actions on H commute. The 2-morphisms are bounded $A-B$ linear maps.

Composition is a little more complicated, ${}_AH_B \boxtimes_B {}_BK_C$, with $\boxtimes$ Connes' fusion. Usually this is not a completion of the tensor product. You can't take A for the unit since it's not a Hilbert space, but you can take L^2A, which is a canonical Hilbert space which is an A-A-bimodule and the left and right actions have nice properties. Often but not always, we can find that L^2A is a completion of A with respect to some inner product.

The question that started us working on this was asked by Stolz–Teichner. I'm not sure how much they still care about the question, their program is much further along, but the question was, are there interesting deloopings of VN_2? Before answering it, I want to come back to von Neumann algebras and point out a place where it's more complicated than the 2-category of rings.

Remark 9.1. Let A and B be von Neumann algebras. Then often there are dense embeddings $A \otimes_{alg} B \hookrightarrow C$ but $C \neq A \otimes B$. A priori this might look like the tensor product but it's not. In terms of this 2-category, this might mean that there are more 1-morphisms than we bargained for in the beginning. An $A \otimes B^{op}$-module gives you an $A-B$-bimodule, but the converse is not necessarily true because of this situation, it's only true for the algebraic tensor product and not for the completed one. For rings we just have the tensor product.

Now I'll describe conformal nets, these are the objects of our delooping.

Definition 9.1. A *conformal net* is triple $(\mathcal{A}, H, u)$. Here H is a Hilbert space with $u : Diff(S^1) \to PU(H)$. The u is $\mathbb{C}$-linear (antilinear) if φ is orientation preserving (reversing). We have $\mathcal{A}$ which sends every closed subinterval of our circle to a von Neumann algebra inside the operators on H.

There are various axioms about the positions of the intervals.

(1) If $I \subset J$ then $\mathcal{A}(I) \subset \mathcal{A}(J)$.
(2) If $I \cap J$ is at most two points, then $\mathcal{A}(I)$ and $\mathcal{A}(J)$ commute.
(3) If $I \cap J$ is a point, then $\mathcal{A}(I \cup J)$ is generated by $\mathcal{A}(I)$ and $\mathcal{A}(J)$.
(4) If $I \cap J$ is empty, then $\mathcal{A}(I) \vee \mathcal{A}(J) = \mathcal{A}(I) \otimes \mathcal{A}(J)$.
(5) $u_\varphi \mathcal{A}(I) u_\varphi^* = \mathcal{A}(\varphi(I))$ and if φ is the identity on a neighborhood of the complement of I, then $u_\varphi \in \mathcal{A}(I)$.
(6) $H \cong L^2(\mathcal{A}([0,\pi]))$

Example 9.1. Let G be a compact simple, simply connected Lie group. Then there is a loop group $\mathcal{L}(G)$, maps from S^1 to G, and there is a construction of an action on a Hilbert space $H_{0,k}$, a projective action. Once you've done this hard work, you can define the algebras fairly easily. You takes $\mathcal{L}_I(G)$ for the interval I, where $\varphi|_{S\setminus I}$ is the unit element. Then $\mathcal{A}(I) = \overline{\mathcal{L}_I(G)}$ in $\mathcal{B}(H_{0,k})$.

Okay, so before we come to the 3-category, let's talk about representations of $\mathcal{A}$. A representation is an alternative for the Hilbert space H. So it's a Hilbert space K and we ask that $\mathcal{A}(I)$ sit inside $\mathcal{B}(K)$ and the only condition is that this preserves inclusions in $\mathcal{B}(K)$. In particular we don't require a vacuum axiom. Wassermann defined in a beautiful way a tensor product for these representations, so think of one of these representations, and another one, think of it in the circle, split the circle in the right and left halves, and there is a canonical diffeomorphism, reflection, between them, and this gives us a diffeomorphism $\mathcal{A}(I_r)^{op} \cong \mathcal{A}(I_\ell)$, and we can take the fusion product over this algebra. We forget the two intervals we used for fusion, and in the middle we have $K_1 \boxtimes K_2$.

There's the following result about this representation category from Kawahigashi–Longo–Müger, that if $\mu(\mathcal{A}) < \infty$ for the net, then the representation category with $\boxtimes$ is a modular tensor category. I won't define μ but we can define the finiteness condition. Take the defining representation of the net, split the circle in four intervals, and then group I_0 and I_2 to the left and I_1 and I_3 to the right, which means we have to use some reflection, and then we obtain a bimodule, and the condition is that this is dualizable in the category of von Neumann algebras. Then KLM proves that this is a modular tensor category.

Of course, Reshetikhin–Turaev produced field theories from such things, a $1-3$ topological field theory. $Bord_1^3 \to VN_2$.

To summarize,

Remark 9.2. If $\mu(\mathcal{A}) = 1$, then the module above is invertible, then $Rep(\mathcal{A})$ is trivial, just contains $\{H\}$. So this is not injective.

Assume for a moment that $\mathcal{A}(I)$ are factors, that $Z(\mathcal{A}(I)) = \mathbb{C}$. Then I can cut the cirle into the left and right halves and get $\mathcal{A}(I_\ell)$ and $\mathcal{A}_{I_r}$, then I take the commutant of the right side $\mathcal{A}(I_r)'$, which must contain $\mathcal{A}(I_\ell)$. Then I get a Jones index and that's the number that I'd call $\mu(K)$. Then the sum over simple K of $\mu(K)$ is $\mu(A)$, well [missed].

Theorem 9.1. *(B., Douglas, Henriques) There is a symmetric monoidal* 3*-category* CN_3 *whose objects are conformal nets with* $\mu(A) < \infty$ *and all objects are fully dualizable. This deloops* VN_2*.*

Remark 9.3. Here what we mean is an internal dicategory in SymCAT, that's a notion that André and Chris invented. This might seem empty. To prove this, we had to solve lots of analytical problems.

Remark 9.4. So I want to define the 1-morphisms. The 2-morphisms are closely related to the representations. The 1-morphisms should end up being von Neumann algebras if I take the trivial object. Let me restate an alternative definition of a conformal net. $\mathcal{A}$ is a functor from the category of intervals to the category of von Neumann algebras. The objects are closed compact oriented smooth intervals and the morphisms are embeddings. And the von Neumann algebras also do embeddings. This gets rid of the additional data we have. We no longer have the Hilbert space. We still talk about the action of the diffeomorphism group. We still have axioms, locality, strong additivity, inner covariance, and vacuum.

Definition 9.2. A *defect*, well, replace the category of intervals with a slightly bigger category, that of bicoloured intervals, red and green. The interval has at most one color where the color changes, and there we ask for a local coordinate near that point. Morphisms respect the colors and the local coordinate. An $\mathcal{A}-\mathcal{B}$ defect is a functor $D : Int_{\bullet\bullet}$ so that on red intervals the functor coincides with $\mathcal{A}$, on green intervals with $\mathcal{B}$, and for two colors it should satisfy the corresponding axioms, locality, strong additivity as long as don't split at the local coordinate, and the vacuum axiom.

Briefly, if $\mathcal{A} = \mathbb{C} = \mathcal{B}$, then what happens when I evaluate D on a large bicoloured interval, that's $\mathcal{A}$ of most of the red stuff and then D of a little bit around the bicolor, and then $\mathcal{B}$ of most of the green, this always happens. By strong additivity, I can chop into pieces. Then if the nets are trivial these big one-colored pieces don't contribute. Then all of the information is about the value on this interval, and it's independent of the choice of bicolored interval and is a von Neumann algebra. I'll stop here.

10. Haynes Miller: Localization in Homotopy Theory

[Welcome to the oberseminar. If Haynes were talking in our workshop I wouldn't introduce him. But this is also a colloquium. He graduated in Princeton and had some stints, most of us know him at MIT, where he's been for almost thirty years. In topology he's very well known (summary of research and teaching). I saw that one of your first papers is a localization theorem in homological algebra from 1978 and now we're hearing a continuation of that.]

Topologists, this is a good time to take a nap or check your email because I'll be doing a very gentle introduction.

Algebraic topology is a meso-scale enterprise. We're interested in finite complexes, things that are high but not infinite dimensional. For instance, we're interested in knowing $\pi_k(S^n)$. When $n > 1$ we'll never know more than a finite number of these groups. Understanding maps between these complexes is a motivating goal. There are systematic phenomena that relate these homotopy groups, often called localization. There's something preliminary you can do, called stabilization. If I have a pointed space X, one thing I can do is embed it into the cone on the space, this is contractible. I can take the cofiber of that, and that's the suspension of X. This is a sequence that's set up so that the connecting map

$$\overline{H}_{n+1}(\Sigma X) \cong \overline{H}_n(X).$$

This destroys cup products in cohomology but it remembers the location of the cells. We want to start by inverting the suspension operator. The most elementary way to do this is the Spanier–Whitehead category. We start by defining the stable maps from a finite complex to a pointed space as the direct limit of what happens when I suspend both sides

$$\lim[\Sigma^k X, \Sigma^k Y].$$

I can enlarge this by formally appending desuspended objects. I want to regard (X, n) as the formal n-fold suspension of X. This is a very primitive way of inverting suspension.

This is fine working with finite complexes, but you do want to consider infinite complexes at some point. You need a further process of adjoining colimits, and you get to the stable homotopy category which I'll call $\mathcal{S}$. The basic domain of activity is the stable homotopy category. It has many standard things defined on it, homology, homotopy, oh, I should say, an object here is called a spectrum, that's a terrible choice of word. I can define $\pi_n(X)$ as $\{S^n, X\}$. That's a homology theory, called stable homotopy theory. You can define homology, which respects this limiting process. This is a non-Abelian derived category. It's got a triangulation, it's an additive category, a triangulated category. It has a smash product $\frac{X\times Y}{X\vee Y} = X \wedge Y$. I just want to work stably. You've eliminated the fundamental group, and then you can often analyze things one prime at a time.

The next localization I want to do is localizing one prime at a time. In the stable world, it's quite easy to localize a spectrum. I should smash with a spectrum $S_{(p)}$ whose homology is $\mathbb{Z}_{(p)}$ concentrated in degree 0. This is a good notion for the localization. This is an example of a much more general sort of localization due to

Pete Bousfield (in the literature he's A. K. but his friends call him Pete). If I have any homology theory E, I can look for maps out of X into something else that are E isomorphisms, and look for something terminal, and I can find that, it's $L_E X$. So that's what I got for $\pi_* \otimes \mathbb{Z}_p$.

I'm going to talk about several localization theorems in different avatars. One of them is a theorem of Serre. One way to say part of what he proved is that if you invert all primes, if you tensor with $\mathbb{Q}$ you get $\pi_*(S^0) \otimes \mathbb{Q}$ is $\mathbb{Q}$ in dimension 0 and that's all. If I take some spectrum and multiply by p and keep doing that, $X \xrightarrow{p} X \to \cdots$, the homotopy colimit is $p^{-1}X$. Each of these maps is an isomorphism in rational homology. so $X \to L_{H\mathbb{Q}}X$ is universal. A nice way to say Serre's theorem is that $p^{-1}X \to L_{H\mathbb{Q}}X$ is a weak equivalence.

The other theorem I want to remind you of is Nishida's. What other things could I invert, he asked? The answer is nothing. If I take the positive dimensional part of the ring $\pi_{>0}S^0$, that's nil. I might be able to invert and get something more, but Nishida's theorem is kind of a no-go result for that.

Chromatic homotopy theory involves itself with generalizations of these results.

The first thing you learn about in homotopy theory is the Hopf map to S^2 from S^3, which stabilizes to a map $\eta \in \pi_1(S^0)$. This is the first positive dimensional homotopy class. Stably, this is order 2, you have $2\eta = 0$. This has relatives at an odd prime, for any prime, the first p-torsion comes at $S^{2p-3} \xrightarrow{\alpha_1} S^0$, and if you precompose by multiplication by p you get zero, so the cokernel, I hope you'll let me say it's S^{2p-3}/p. So then the map descends to a map from this to S^0 killed by multiplying by p. Then you can take the kernel of that which is S^{-1}/p and then you get a map v_1 from S^{2p-3}/p to S^{-1}/p. This turns out to be non-nilpotent. I can keep iterating

$$S^0/p \xrightarrow{v} S^{-2p-2}/p \xrightarrow{v_1} \cdots \to S^{-k(2p-2)}/p.$$

These maps are essential, you can see this by using K-theory. That's been important historically. I can come into this on the bottom cell and out on the top cell and get a map $S^0 \to S^{-k(2p-2)+1}$, whose name is α_k. It's also nonzero, and the point is that the whole motivation for constructing self-maps of finite complexes is that it gives you a way to get infinite families in stable homotopy.

Another thing you can do is take the colimit of the mapping telescope of the diagram I wrote up there, which I can call $v_1^{-1}S^0/p$. What can we say about that spectrum? It has no homology, all of the maps induce zero on homology. But the maps are isomorphisms in K-theory. There is a K-theory isomorphism from S^0/p. I can understand this in terms of the universal K-theory isomorphism and I get a factorization $v_1^{-1}S^0/p \to L_k(S^0/p)$. The theorem about this is that this map is an equivalence, an isomorphism of stable homotopy. This is a theorem of Pete Bousfield. It uses a calculation that I did for p odd and [unintelligible] did it for two. I wish I could tell you $\pi_*(S^0/p)$, but I'll tell you $v_1^{-1}\pi_*(S^0/p) = \pi_*(v_1^{-1}S^0/p)$. So two homotopy classes are $\iota : S^0 \to S^0/p$ and $\iota\alpha_1 : S^{2p-3} \to S^0/p$. And that's about all., you get the calculation is $\mathbb{F}_p[v_1^{\pm 1}]\langle \iota, \iota\alpha_1 \rangle$. That's analagous to Serre's computation. I want to give an expression analogous to the way I expressed Serre's theorem.

This process of looking for self-maps and constructing maps of spheres out of them has a long history. It's hard work. Larry Smith, Toda, Mark Behrens, [unintelligible], Mike Hill, have done constructions like this. I can take the mapping cone of the self-map and see if it has a self-map. As long as $p \geq 5$ it does and there's one of degree $2(p^2-1)$ called v_2 and that's due to Larry Smith. It seemed like these are very special spectra that have self-maps like this.

Then the central event in chromatic homotopy theory revolutionized our understanding. This was Mike Hopkins, Jeff Smith, and Ethan Devinatz. You might ask how Larry Smith knew his things were nontrivial. You can use complex cobordism which K-theory comes from. He used MU to detect this, the basic ingredient in the work of Hopkins and Smith (building on Devinatz).

(1) $v:\Sigma^? X \to X$ for X finite. If $MU_*(v)$ is nilpotent, then v is nilpotent. But they said a lot more. I have to introduce you to another cast of characters,

    $K(n)_*$ is "Morava K-theory." Jack Morava realized that $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ with $|v_n| = 2(p^n-1)$. If $n=1$ this is essentially $\pmod p$ K-theory.

    So finite spectra are filtered by how many Morava K-theories are zero on them. To say that X is type n means that it's finite and $K(i)_*(X)=0$ for $i<n$. Ravenel proved that if $K(i)_*(X)=0$ then $K_*(i-1)_*X=0$.

    Let me tell you a second thing these guys showed.

(2) Any type n spectrum admits a self-map ϕ from some suspension $\Sigma^? X \to X$ given by multiplication by $v_n^{p^k}$ in the $K(n)$ homology. Let's let that sink in. This is a positive dimensional element. Any type n spectrum admits this symmetry, but there's no condition on X except that the first non-vanishing K-theory is in dimension n.

    You might say this is vacuous because there are no type n spectra. But

(3) For all n there is X_n which is type n and not type $n+1$. Moreover,

(4) This self-map is canonical essentially. Suppose I have any two type n spectra and any map between them, and I take any two of these self-maps

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_X"'] & Y \arrow[d, "\phi_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

    and up to taking appropriate powers of ϕ, this diagram commutes.

There's nothing special about the higher Moore space. You might have to go to v_2^p but you always have this symmetry.

Take a type n spectrum, this is a type 2 spectrum, and I can localize it by inverting the self map. This doesn't depend on which power of *phi* you invert. There's a map $X \to \phi_X^{-1}X$. This is the canonical localization of a type n spectrum. You'd like to know what the homotopy type of that is.

But let me go further. Let's take the constructive perspective. You want a map that's an isomorphism in $K(n)$ theory to something but do it in a constructive way. I wand to map into X a giant sum $\bigvee K$, a giant sum over all finite$K(n)$-acyclic spectra. I'll call the result from coning these off X_1, and then I can do that again and get X_2. In the end we get the finite localization of X, $L_n^f X$. These theorems show that these two constructions coincide on type n spectra.

Now I want to look at the other side of Serre's theorem. There is a map from X to $L_n^f X$ which is a $K(n)$-isomorphism, I get a map

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & L_n^f X \arrow[dl] \\ & L_{K(n)} X & \end{tikzcd}$$

but that's not quite right since you know all previous $K(n)$, we have the same situation from being type n so it's better to do

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & L_n^f X \arrow[dl] \\ & L_{K(n)\vee\cdots\vee K(0)} X & \end{tikzcd}$$

and I'll call that $L_n(X)$. The corollary of the calculation is that $L_1^f X \to L_1 X$ is an equivalence. In general this is the Ravenel telescope conjecture. Would you like to change this to a question, Doug?

[Let's call it a question, I don't believe it.]

[If we knew it was true, what would we learn?]

We use Adams operations for L_1. To do L_1^f, my calculation was quite complicated and this would give a much quicker way.

All right. So the nilpotence theorem, Hopkins and Smith, focuses on whether a map induces zero on MU-homology. To do a better job you should take homology operations in account. When you do that you get the Adams spectral sequence. I'll do this in the case that the spectrum is MU since this is the one that sees chromatic periodicities most easily. So $E = MU$ and $E_2 = Ext^{s,t}_{MU_*MU}(MU_*, MU_*(X))$ and this converges to $\pi_{t-s}(X)$. Along the bottom row, say for X a sphere, you get just a $\mathbb{Z}$. There are higher ext groups that contribute. Let me write down what, and I should say, studying this spectral sequence, Ravenel and Wilson and I used this to make many calculations about stable elements. When X is a sphere and a prime is 2, just for an example, you get

$$\begin{array}{cccccc} & \eta & \bullet & \bullet & \bullet & \\ \mathbb{Z} & & & & & \\ \hline 0 & 1 & 3 & 5 & 7 & 9 \end{array}$$

In fact we know that η^4 is zero but this thing doesn't see this, it's not nilpotent here. Doug and Steve and I discovered that the generators of all these cyclic groups [unintelligible]. We didn't find out what you get by inverting η in the E_2 term which is not very topological since it's nilpotent eventually. This was asked in 1967 and I can give the answer now, what you see is what you get. If I'm going to invert η, I want $\eta^{-1}E_2$, I might as well begin in homological dimension 0. These won't be in the E_2 itself, but in the localizaton. I can call them v_1^2, v_2, v_1^4, $v_1^2 v_2$, and so on.

Theorem 10.1. *(Michael Andrews, M.)* $\eta^{-1}E_2(S) = \mathbb{F}_2[\eta^{\pm}, v_1^2, v_2]/v_2^2 = 0$. *This is not on the zero line in MU, this is not chromatic.*

We know how η^4 is 0. The d_3 takes v_1^2 to η^3 in the localization. This looks familiar if you've thought about the relationship between KO and KU.

Now we know we have one worldsheet and close by are other worldsheets controlled by the ground field. Now we have motivic homotopy theory, and motivically, things happen quite a bit differently. Their work lets me play with motivic homotopy theory. All of the groundwork has been done by these people so I can play games.

Motivically, η is non-nilpotent. So cool, there's another place where η is non-nilpotent. In fact, Hu, Kriz, and Ormsby made a motivic version of this E_2 page, and they determined that they get $E_2(\mathbb{S}_{top}) \otimes \mathbb{Z}_p[\tau]$. That's interesting. You can use the calculation that we did here to find that motivically, there's another, there's a d_3 where $d_3(v_1)^2 = \eta^3\tau$. Up there you hit a unit and kill the spectral sequence. Now it hits something that isn't a unit. If I localize, $\pi_*(\eta^{-1}\mathbb{S}_{mot})$, this homotopy is now known, it's what's left over, $\mathbb{F}_2[\eta^{\pm}, v_1^4, v_2] = /v_2^2$. This was a conjecture of Isaksen and Guillou, and it's easy from this point of view.

This new element is nonchromatic, but maybe we can try to play the same game we were playing motivically. So maybe the next game is to cone it off and see what we get. Take the motivic sphere and cone off η and ask if it has an interesting self-map. Michael Andrews alone showed that there is a map. This is bigraded, it's degree 20 topologically and 12 in weight, and this is non-nilpotent, call this w_1^4, where $\eta = w_0$. Andrew sketches that I could take motivic homotopy over $\mathbb{R}$ and base change to $\mathbb{C}$ and that has an underlying spectrum. Michael has a sketch of an object X_n and a self-map in the real version such that if I realize in the underlying spectrum, the real points are a v_n-self map. One case of doing that gives you η itself, anyway, and presumably when I do this base change I get self-maps over $\mathbb{C}$. We don't know that they're non-nilpotent, but there's a sketch of a way to do that. I think that what's going to happen, there's the chromatic sequence and I'd suggest the name technicolor for these, that come from the existence of this subfield $\mathbb{R}$ in the complex numbers, this technicolor world seems to exist motivically.

11. Stephan Stolz: Twisted field theories from factorization algebras

[Welcome back. Unfortunately I don't get to say anything about Stephan, but one piece of advice, if you want to have fun in research, pick the right people to work with.]

This is joint with Bill Dwyer and Peter Teichner. Here are two particular models for field theories, it's like the wild west as far as saying what a field theory is. There are the functorial field theories, Atiyah, Segal, Kontsevich, Lurie making them local. Another language, more recent, is that of factorization algebras. I'm not very good with the history of that, trying to define vertex operator algebras in a coordinate free way, Beilinson and Drinfel'd started this, and most of what I know about this I know from Costello and Gwilliam's book. Work on the topological version has been done by John Francis and David Ayala. I want to talk about a way to go from a factorization algebra and get a twisted functorial field theory. I like this because functorial field theories are a little hard to construct, so it's good to have a mechanism to create examples in functorial field theories. This talk, though, won't be example heavy, because there's a lot to be done and I don't understand the examples so well yet.

I'll start by talking about factorization algebras, then twisted field theories, and then I'll relate the two.

11.1. **factorization algebras.** There are many ways to write down a formal definition. I'd like to think that if you have a quantum field theory, you might want to look at observables in an open part of spacetime. I start with a classical field theory. You have a manifold of a fixed dimension d, you have fields which are sections of a bundle. You have an action functional and you try to get at the Euler–Lagrange equations. Much more mysterious is the passage to quantum field theory. But you should have a manifold with some fixed geometry and have something that associates to your classical field theory a quantum field theory. In particular you should be able to look at an open set and get a vector space of observables. So I'll collect in the letter $\mathcal{G}$ a fixed dimension and a fixed geometry, like Riemannian, conformal, and so forth, it could involve spin or string structures or whatever.

Then I can talk about $\mathcal{G}$-manifolds and a $\mathcal{G}$-*prefactorization algebra* or prefactorization algebra $\mathcal{A}$ for $\mathcal{G}$-manifolds.

Definition 11.1. This consists of the following data.

(1) whenever I have a $\mathcal{G}$-manifold U I associate to it a cochain complex $\mathcal{A}(U)$, always over $\mathbb{C}$. You should think of this as the space of quantum observables in spacetime U. Suppose you have an open subset, you should be able to restrict. More generally, suppose you have a bunch of open sets $U_1, \ldots, U_n$ inside U, a structure-preserving embedding with disjoint image. Then you should get a map $\mathcal{A}(U_1) \otimes \cdots \otimes \mathcal{A}(U_n) \to \mathcal{A}(U)$.

   You want this to be compatible with composition. If these open subsets have inside them other open subsets. You could either look at the disjoint union of all of them inside U, or you could include Vs into U_is and then go further to U.

Suppose you gave up the disjoint condition. You would get an algebra of observables $\mathcal{A}(U) \otimes \mathcal{A}(U) \to \mathcal{A}(U)$. You can sort of multiply functions on your phase space. However, the fact is, for quantum observables, you can't keep multiply these variables because of the Heisenberg uncertainty principle. But they don't commute so you can't do this, unless the observations are in disjoint regions.

This is sort of like a cosheaf. In sheaves you get restriction maps that are contravariant. These are covariant, this feels more like a cosheaf. Now you want to include conditions that are more like sheaf conditions.

A prefactorization algebra is a *factorization algebra* if

(1) whenever U is the disjoint union of the U_i, the map $\mathcal{A}(U_1) \otimes \cdots \otimes \mathcal{A}(U)$ is a weak equivalence.
(2) you want to take a covering U_i in U for $i \in I$, and the kind of cover is not the usual cover, this is a Weiss cover, what you use when you study embedding spaces. What does a Weiss cover mean? A usual cover, a point in U is in some U_i. Here any finite subset of U is in some U_i.

   Here's one example. A Weiss cover of $[0,1]$, it's not so easy to come up with an interesting one. Take the U_i, take all of the interval, but exclude one point $\frac{1}{i}$. This is a Weiss cover. Another example, suppose that U is a Riemannian manifold. Take disjoint unions of balls of radius smaller than ϵ. That's a Weiss cover.

   Here comes the condition. The requirement to be a factorization algebra is that, well, $\mathcal{A}(U)$ gets a map from $\bigoplus \mathcal{A}(U_i)$. What about the intersections, you get a map $\bigoplus \mathcal{A}(U_i \cap U_j)$. There are two ways of getting to $\bigoplus \mathcal{A}(U_i)$.

If you follow that, you should get the same map to $\mathcal{A}(U)$. You can repeat this taking triple intersections $\bigoplus \mathcal{A}(U_i \cap U_j \cap U_k)$ and you get three maps. This is a simplicial object in chain complexes. The requirement is that if you take the hocolim or total complex, that the map from this to $\mathcal{A}(U)$ is a weak equivalence.

11.2. **Twisted functorial field theories.**

Definition 11.2. A T-twisted functorial $\mathcal{G}$-field theory is, I'll draw a diagram. You define in the usual way the bordism category of manifolds with $\mathcal{G}$-structure and you look at the category of algebras. These things are symmetric monoidal 2-categories. *Alg* is the 2-category that Arthur Barthels mentioned, where the objects are algebras, the morphisms are bimodules and the 2-morphisms are intertwiners.

Then T is a morphism and T_0 is the boring one that gives $\mathbb{C}$ for every object and $\mathbb{C}$ for every morphism. A T-twisted field theory is a natural transformation from T to T_0. It's much easier to understand that if I unpack it.

$$\begin{tikzcd} \mathcal{G}Bord \arrow[r, bend left, "T"{name=U}] \arrow[r, bend right, "T_0"'{name=D}] & Alg \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$$

More explicitly, on objects, I have closed $d-1$-dimensional manifold with a collar, with a $\mathcal{G}$-structure on it. I call this original boundary the "core" of Y; the other boundary I call the "end." The geometric structure is only on the interior.

So $T(Y)$ is an algebra and $T_0(Y)$ is $\mathbb{C}$ and so I get a map, which is a bimodule, so this is a right $T(Y)$-module.

So what do we do on morphisms? I have my core $\partial_C Y_0$ and my core $\partial_C Y_1$. Where do the bordisms come in? The collar of Y_0 sticks out of the bordism but the collar of Y_1 is contained within it. I hope it's clear how the composition would work, by gluing along collars. What do I associate to this thing here? If I look at T, I get $T(Y_1)$ and $T(Y_0)$, and I get a bimodule $T(\Sigma)$ between them. On the other hand I have $T_0(Y_1)$ and $T_0(Y_0)$ and I also have $E(Y_0)$ and $E(Y_1)$ already.

$$\begin{tikzcd} T(Y_1) \arrow[d, "E(Y_1)"] & T(Y_0)(\Sigma) \arrow[l, "T"'] \arrow[d, "E(Y_0)"] \arrow[dl, phantom, "\Nwarrow"] \\ T_0(Y_1) = \mathbb{C} & T_0(Y_0) = \mathbb{C} \arrow[l, "T_0(\Sigma)=\mathbb{C}"'] \end{tikzcd}$$

So going between these my 2-morphism is a map from $E(Y_0)_{T(Y_0)}$ to $E_(Y_1) \otimes_{T(Y_1)} T(\Sigma)_{T(Y_0)}$. Why do I want the map in this direction? If T is just T_0 this would be a map $E(Y_0)$ to $E(Y_1)$. So this is an enrichment, this is not just a vector space but a bimodule over the thing you're twisting by. If Y_0 and Y_1 are the empty set, you get a map from E to $T(\Sigma)$, you get a vector in the vector space. Examples of this are around for a long time, like determinant lines.

So that's a twisted functorial field theory.

11.3. **Results.**

Theorem 11.1. *Let $\mathcal{A}$ be a $\mathcal{G}$-factorization algebra. Then we can construct a twisted field theory $(T_{\mathcal{A}}, E_{\mathcal{A}})$, where you have to loosen up what you mean a little bit.*

How would you do this for a closed surface? You should get a chain complex, not a vector space. This should be understood in a derived sense. What I mean will be clear as I do my construction.

At the minimum instead of vector spaces I should talk about chain complexes. The outline of the construction. I'll concentrate for time reasons on the twist functor $T_{\mathcal{A}}$. I should start with an object of $\mathcal{G}Bord$ and associate an object. I'll do less than that. What is $T_{\mathcal{A}}$ on objects? I have an object Y in my category, with core $\partial_C Y$ and to that I should associate something more general than an algebra. This something will be a dg category. This means a category enriched in chain complexes. If this had one object, it would be a chain complex, one object's endomorphisms, this is an algebra. I pass to chain complexes and I have more than one object in this category. That's what I mean by derived.

I need to tell you what are the objects of this dg category. In the picture, the objects are neighborhoods, open neighborhoods, of the core, still collars of my core. Morphisms are $T_{\mathcal{A}}(Y)(U, U')$, I need to give you a chain complex of morphisms. Imagine that U' is a smaller neighborhood, then I can form the complement, $\mathcal{A}(int(U \backslash U'))$ is a chain complex. If U' is not contained in U then I get 0. The composition is the structure map. An even smaller one, the two regions, there's the part between $U'' and U'$ disjoint union the part between U' and U. That structure map gives me the modification.

What about our morphisms? In the bordism category we have bordisms, so here is a picture of a bordism [picture]. So what do I want to associate to that? In the twisted field theory case I should get a bimodule. So that's a bimodule in the derived sense, so $T_{\mathcal{A}}(\Sigma)$ is a bimodule with a left action of $T_{\mathcal{A}}(Y_1)$ and a right action of $T_{\mathcal{A}}(Y_0)$. I should say what I mean by a bimodule in this derived world. It's again a dg category with the following properties. The objects are the disjoint union of the objects of the $T_{\mathcal{A}}(Y_1)$ and $T_{\mathcal{A}}(Y_0)$. In a bimodule, I don't want any morphisms from U_1 in $T_{\mathcal{A}}(Y_1)$ to U_0 in $T_{\mathcal{A}}(Y_0)$. Thirdly, I want to say that $T_{\mathcal{A}}(Y_i)$ are full subcategories of $T_{\mathcal{A}}(\Sigma)$. The only interesting information is what goes from Y_0 to Y_1. If you had one object in each, this is a single chain complex and an action of endomorphisms on each side. What are the morphisms from U_0 to U_1?

I only have to tell you, what is $T_{\mathcal{A}}(\Sigma)(U_0, U_1)$? What I do is I take the thing between U_1 and U_0. Let me denote by $\widehat{U_0}$ the entire bordism, including the U_0 part of the collar. So I take the interior of $\widehat{U}_0 - U_1$. Then the definition in general, if $U_1 \subset \widehat{U}_0$, then I get $\mathcal{A}(U)$, with 0 otherwise. It's easy to see that this is a dg category with all the properties you need.

So now what is the most important thing you need to check? First you need to define composition on the right hand side. You need to tensor over the common algebra. On the left hand side you are gluing bordisms. Then that should correspond to the algebraic thing with bimodules. So my claim is that $T_{\mathcal{A}}(\Sigma_2 \cup_{Y_1} \Sigma_1) \cong T_{\mathcal{A}}(\Sigma_2) \otimes_{T_{\mathcal{A}}(Y_1)} T_{\mathcal{A}}(\Sigma_1)$.

[picture.] Now,

$$T_{\mathcal{A}}(\Sigma_2 \cup_{Y_1} \Sigma_1)(U_0, U_2) = \mathcal{A}(\underbrace{int(\widehat{U}_0 \backslash U_2)}_{U}).$$

I can calculate this on the Weiss cover $\{V_i \to U\}$ where V_i runs over objects U^i associated to Y_1. What is the V_i, you take $U\backslash\partial_e(U^i)$ It's easy to see that this is a Weiss cover in the one dimensional case but it's true in general. Then I get to write down, let me do the V_i in color here. What does V_i look like? [picture]. If you look at V_i, we know that $\mathcal{A}$ evaluated on the disjoint union is the tensor product, we get $\mathcal{A}(V_i) = \mathcal{A}(V_i^\ell) \otimes \mathcal{A}(V_i^r)$ Now let's look at the intersection, it's $\mathcal{A}(V_i^\ell) \otimes \mathcal{A}(V_i^r \cap V_j^\ell) \otimes \mathcal{A}(V_j^r)$.

Now I want to rewrite these things, the V_i^ℓ and V_i^r can be identified with $T_{\mathcal{A}}(\Sigma_1)(U_i, U_2) \otimes T_{\mathcal{A}}(\Sigma_2)(U_0, U_i)$; similarly, the left and right hand side, the thing in the middle is inside Y_1, so it's seen by this category $T_{\mathcal{A}}(Y_1)(U_j, U_i)$ and then you have a big hocolim which involves only the two dg categories that act on the left and on the right, and this is the definition of the tensor product of bimodules.

12. June 19: Gregory Arone: A branching rule for partition complexes

All right, thank you, I'd like to thank the organizers for making this happen and for inviting me. I should say something about the title. I feel like I have miscalculated with the title and abstract. I thought it was intended for beginning students, so I gave my title and abstract in accordance with that.

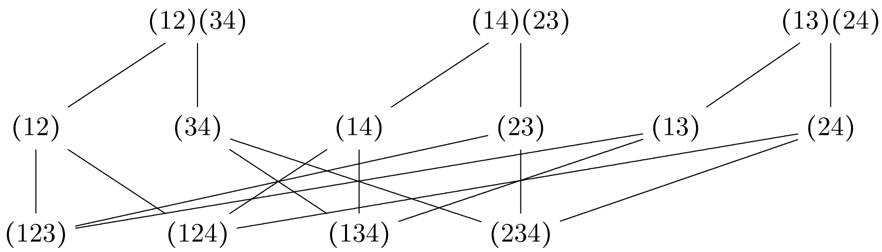
So I want to talk about some combinatorial gadget that arises here and there. I have a small thing to report about them and I thought I'd use the opportunity to talk about them in general.

The gadget is "partition complexes." Let me recall. That P_n is the poset of partitions of a standard set with n elements, ordered by refinement. So for example, so P_3 is something like this:

$$\begin{tikzcd}
 & (1)(2)(3) & \\
(12)(3) \arrow[ur] & (13)(2) \arrow[u] & (1)(23) \arrow[ul] \\
 & (1\ 2\ 3) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Well, I want to get rid of the initial and terminal element so that I have a nontrivial homotopy type. So then Π_n will be $|P_n|$, an $\Pi_n^\diamond$ is the unreduced suspension of Π_n, unreduced because it has no canonical basepoint. Now that I have take the unreduced suspension, then I can take the suspension, so $\Sigma\Pi_n^\diamond = S^1 \wedge \Pi_n^\diamond$.

For example, Π_2 is $\varnothing$ because there are no nontrivial partitions. The unreduced suspension is S^0. Π_3 is a set of three points. The unreduced suspension is the theta graph. Π_4 is a bit more complicated, it's the largest one we can try to draw,



So it's homotopy equivalent to a wedge of six copies of S^1.

Now there's a theorem which says that $\Pi_n \cong \bigvee_{(n-1)!} S^{n-3}$ and so let's define Π_1 as S^{-2}. Then the k-fold suspension should be a sphere of dimension $k-2$.

The symmetric group Σ_n acts on Π_n and $H_{n-3}(\Pi_n) = \mathbb{Z}^{(n-1)!}$ with Σ_n acting. This is closely related to the Lie operad and Lie algebras.

Then they arise in Goodwillie's calculus of functors [comments] because topological spaces are related to cocommutative coalgebra spectra, so by Koszul duality the Lie algebra arises. They also arise in cohomology of configuration spaces, related to the Poisson operad.

So the main result is about Π_n/G, for $G \subset \Sigma_n$. Suppose n is $n_1 + \cdots + n_k$, and consider the subgroup $\Sigma_{n_1} \times \cdots \Sigma_{n_k}$.

Theorem 12.1. *There is a $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariant equivalence*

$$\Pi_n \to \bigvee_{d|gcd(n_1,\dots,n_k)} \bigvee_{B(\frac{n_1}{d},\dots,\frac{n_k}{d})} (\Sigma_{n_1} \times \cdots \times \Sigma_{n_k})_+ \wedge_{\Sigma_d} S^{n-d} \wedge \Pi_d$$

where $B(m_1,\dots,m_k)$ is a set with $\frac{1}{m}\sum_{\ell|gcd(m_1,\dots,m_k)} \mu(\ell)\frac{m!}{m_1!\cdots m_\ell!}$.

You may recognize this from Witt's formula. If you let $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ act on $\mathbb{Z}/n\backslash\Sigma_n$. So $B(\frac{n_1}{d},\dots \frac{n_k}{d})$. You should be able to see what you get [explanation].

I should be careful about the action on the right hand side. In particular d divides n, so this sits inside $\Sigma_d^{\frac{n}{d}}$ which sits inside the product $\Sigma_{n_1} \times \cdots \Sigma_{n_k}$. The notation suggests that Π_d has a basepoint. So I should really say $S^{n-d-1} \wedge \Pi_d^{\diamond}$.

This breaks up as a Σ_d-equivariant homeomorphism, you can write $S^{n-d} = (S^{\frac{n}{d}-1})^{\wedge d}$ and thisi $S^{\frac{n}{d}-1} \wedge \widehat{(S^{\frac{n}{d}-1})}^{\wedge d-1}$ where it acts trivially on the left and by the standard representation on the right. So the one I split is one of the trivial ones.

So some consequences. First, about fixed points. Let G be inside Σ_n, say that G acts isotypically on $\{1,\dots,n\}$ if all the G-orbits are pairwise isomorphic as G-sets. It's like a representation that breaks up as a sum of identical representations.

Corollary 12.1. *If G does not act isotypically then the fixed point of the action are contractible.*

The partition behaves in a lot of ways like a contractible space. A special case is a transitive action.

Lemma 12.1. *If G acts transitively on $\{1,\dots,n\}$, then identify $\{1,\dots,n\}$ with G/H for some subset H of G. Then $\Pi_n^G \cong \{H < K < G\}$, the fixed points are described in terms of subset posets.*

You can get the formula for a general isotypical subset from this.

Maybe I'll mention a related result, joint with Katherine Lesh and Bill Dwyer. We're interested in p-subgroups of Σ_n. Then the only way the fixed points are not contractible is if p is an elementary Abelian group acting freely on $\{1,\ldots,n\}$.

This has a consequence about [unintelligible].

Corollary 12.2. *Let μ be a Mackey functor for Σ_n that takes values in p-local groups and satisfies a technical condition, projective relative to p-Sylow subgroups. Then the reduced Bredon homology of the suspension of Π_n with coefficients in the Mackey functor is 0 unless the degree is p^k; if the degree is p^k, there's some formula, $st(\mu(\Sigma p^k/(\mathbb{Z}/p)^k))$.*

So for the Mackey functor $\mu(\Sigma_n/G)$ is the p-local stable homotopy group $\pi^S(BG_{(p)})$. You can't calculate concretely but you can calculate the homology.

This has consequences related to the calculus of functors and should be an ingredient in a new proof of the Whitehead conjecture (Leuhn's theorem). Hopefully this can give a similar BU-version of the Whitehead conjecture. This seems to be related to some calculations of Charles Rezk about the Koszulity of the ring of operations and the E_n homology of [unintelligible] of Behrens.

Another obvious thing is about orbit spaces. I can't give a complete answer but this tells you something about orbit spaces, at least orbit spaces with respect to Young subgroups.

Corollary 12.3. *There is an equivalence between $\Pi_n/\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}$ and*

$$\bigvee_{d|gcd(n_1,\ldots,n_k)}\bigvee_{B(\frac{n_1}{d},\ldots,\frac{n_k}{d})} S^{n-d-1}\wedge\Pi_d)_{\Sigma_d}.$$

which reduces the problem to calculating $(S^{\ell d}\wedge\Pi_d^\diamond)_{\Sigma_d}$. If $\ell=0$, well, it's a theorem of Kozler that $(\Pi_d^\diamond)_{\Sigma_d}$ is contractible for $d>2$. For $\ell=1$, we have $(S^d\wedge\Pi_d)_{\Sigma_d}$ is always contractible. When $d=2$, then we are looking at $S^{2\ell}/\Sigma_2$ and this is the $(\ell+1)$-suspension of $\mathbb{RP}^{\ell-1}$. When $d=p$ is prime, this is, for ℓ odd, $(S^{\ell p}/\Sigma_p)_{(p)}$ and if ℓ is even, it will be the homotopy cofiber of $S^{\ell p}\to(S^{\ell p}/\Sigma_p)_{(p)}$. This is as far as I got. It's a nice problem to work out the homotopy type of this space.

One consequence is

Corollary 12.4.

$$\Pi_n/\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}$$

is homotopic to a wedge of spheres if and only if one of the following holds:

1. *the $gcd(n_1,\ldots,n_k)=1$. Then you only get a wedge of obvious spheres.*
2. *$k=2$ and $n_1=n_2=p$, then we're looking at $\Pi_{2p}/\Sigma_p\times\Sigma_p$.*

Maybe I should say something in five minutes about why this is true, which is related to the decomposition of the free Lie algebra.

Let's say we have V a free Abelian group and $Lie[V]$ the free Lie algebra generated by V. There is a connection between the free algebra, an isomorphism of graded Abelian groups between $Lie(V)$ and $\bigoplus_{n=1}^\infty Lie_n\otimes_{\mathbb{Z}[\Sigma_n]}V^{\otimes n}$, which is related to $\bigoplus H^{n-3}(\Pi_n)^\pm\otimes_{\mathbb{Z}[\Sigma_n]}V^{\otimes n}$ and on the other hand, there is a classic result, the algebraic Hilton–Milnor theorem, which says there is an isomorphism between $Lie[V_1\oplus\cdots\oplus V_k]$ and $\bigoplus_{n_1,\ldots n_k}\bigoplus B(n_1,\ldots,n_k)Lie[V_1^{\otimes n_1}\otimes\cdots V_k^{\otimes n_k}]$. I won't have time to explain it but it's not too hard. The homomorphism is easy to define from right to left. Take the formula and run it through here, it's an easy consequence, a branching rule for Lie representations. If you restrict to a subgroup of this form, you get

$$\bigoplus_{d|gcd(n_1,\dots,n_k)} \bigoplus_{B(\frac{n_1}{d},\dots,\frac{n_k}{d})} \mathbb{Z}[\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}] \otimes_{\mathbb{Z}[\Sigma_\ell]} Lie(d).$$

You have to check that this is a strong equivalence, respects fixed points, and so on.

13. John Francis: A proof of the bordism hypothesis

Thank you for the introduction and to the organizers for the invitation to speak here and spend this time in Bonn, which is a privilege.

I want to talk about the bordism hypothesis. Things here come via Baez–Dolan's original formulation, Costello, Hopkins–Lurie in the $n = 1$ case, and the formulation of Lurie (of many things). Everything I say will be joint with David Ayala. So what's the bordism hypothesis. I won't motivate it because then the talk will be over. I will say what's a little different that David and I can say about it.

Jacob's formulation says that for $\mathcal{X}$ a symmetric monoidal (∞, n)-category with adjoints (every k-morphism has a left and right adjoint) and duals, there is an equivalence between symmetric monoidal functors from the framed n-bordism category to $\mathcal{X}$ is equivalent to the underling space $\mathcal{X}^{\sim}$. This builds on papers with Nick Rozynblyum and Hiro Lee Tanaka. Some of the things are in preparation. There's factorization homology from higher categories (AFR2), a stratified homotopy hypothesis (AFR1), and local structures on stratified spaces (AFT). So this is the bordism hypothesis, this is some work built on it. Let me give the basic idea behind the proof and some of the steps.

The idea is a relationship between higher category theory and manifold theory. It's hard to prove directly but it's not supposed to be. It's not the starting point of the relationship between higher category theory and manifolds. They should be merged earlier and then it should be easy. There's a more basic relationship between category theory and differential topology.

One should understand the moduli space of stratifications on a manifold together with trace methods. This package is supposed to be related to the combinatorics of higher categories. This package together with the relationship, is given by a generalization of what we've been calling factorization homology. Once you've said the obvious things to ask about it, the bordism hypothesis is easy. This doesn't involve bordisms or adjoints, it has room to say very interesting things where the bordism hypothesis doesn't apply. Then maybe you can use this to talk about path integrals in physics and so on.

Today just the bordism hypothesis.

(1) The first, to say more formally what I mean by this basic relationship, there is a fully faithful functor from (∞, n)-categories to space-valued functors on compact vari-framed n-manifolds. Similarly, adjoint (∞, n)-categories embeds into functors to spaces from another solid-framed compact n-manifolds.
(2) The second step is to talk about pointed (∞, n)-categories, and that this has a fully faithful functor to functors to spaces from not-necessarily compact variframed manifolds, and likewise for adjoints. Now you have Euclidean space. If you have a pointed n-category, and you calculate $\int_{\mathbb{R}^k} \mathbb{C}$, you get $kEnd_{\mathbb{C}}(\mathbf{1})$.
(3) To show the tangle hypothesis, that $\mathcal{T}ang^{fr}_{n\subset n+k} \cong \mathbb{R}^k$.

(4) To show that $\mathcal{T}ang$ implies $\mathcal{B}ord$ because $\mathcal{B}ord_n^{fr} \cong \lim \Omega^k \mathbb{R}^k$.

So first, what is $c\mathcal{M}fd_n^{vfr}$?

Definition 13.1. Let M and K be stratified spaces. These are smoothly stratified spaces and satisfy some technical regularity conditions, see the AFT paper. A *constructible bundle* is a stratified map $M \to K$ satisfying the technical conditions that I'll never mention again, such that when you restrict to each stratum, you get a stratified fiber bundle.

This is a simple definition but it took a long time to realize that this was almost exactly what we needed.

The ∞-category of $c\mathcal{M}fd_n^{vfr}$ has as its objects M, compact stratified spaces of dimension at most n, with a *vari-framing.*

What's a vari-framing? You might imagine something like a framing on the interior and a framing on the boundary. You throw that idea out because it implies that you get a cylinder. But we'll use this idea. We have θ_M, the constructible tangent bundle. On every stratum, if we choose some point in M, then the stalk is the tangent space at x to the stratum of x. A variframing is an equivalence to the trivial bundle $\mathcal{E}_M^{dim}$, where the stalks vary in dimension, the stalk, for $x \in M$, is $\mathbb{R}^{dim\ M_p}$, where M_p is the stratum. This needs to be understood in a homotopy coherent way.

Intuitively, at every point you have a choice of flags in $\mathbb{R}^n$, determined by the dimensions of the strata which contain the point. In order to define this you need regularity in smooth families along strata. These issues are taken care of in AFT or AFR1.

Those are the objects, what are the morphisms? We stratify Δ_1 where $\{0\}$ is one stratum and its complement is the other stratum. We want a constructible bundle $\widetilde{M}$ with some data, an equivalence between $\theta_{\widetilde{M}}^{vert}$ and $\mathcal{E}_{\widetilde{M}}^{vdim}$, a morphism from the fiber over 0 with its restriction of the variframing, to the fiber over 1. It's hard to communicate how much thought went into that definition. The precursors were so much more complicated. It's simple compared to what it could have been.

I'll tell you the theorem and then break down the definition and draw some pictures.

Theorem 13.1. *(AFR2) There is a fully faithful functor called* factorization homology *from $Cat_{(\infty,n)}$ to $Fun(c\mathcal{M}fd_n^{vfr}, Spaces)$ where $\mathbb{D}^0 = *$ goes to $\int_{\mathbb{D}} \mathcal{C} = obj(\mathcal{C})$ the space of objects of $\mathcal{C}$. The variframed interval $\mathbb{D}^1$ goes to the space of 1-morphisms of $\mathcal{C}$. There is a hemispherical stratification of $\mathbb{D}^k = S^k\mathbb{D}^0$ with associated vari-framing, and that goes to the space of k-morphisms of $\mathcal{C}$.*

A really basic morphism is the following. [picture] and this, depending on our covention about variframing, gives a map $\int_{\mathbb{D}^1} \mathcal{C}$ to $\int_{\mathbb{D}^0} \mathcal{C}$, and this is the target map. If I chose the other picture I'd get the source. So a variframing is what lets you make these choices about source and target.

[picture]. Here is another constructible bundle. The underlying space looks like a product but it has a more refined stratification. This morphism gives composition. I'll draw one more example. Here's a constructible bundle over the 1-simplex. Over the open interval you get a product, and also over the point, and this gives $\int_{\mathbb{D}^0} \mathcal{C}$ to $\int_{S^1} \mathcal{C}$, and this is a version of the trace map from the objects of $\mathcal{C}$ to the Hochschild homology of $\mathcal{C}$.

I'll call these morphisms, respectively, closed, refinement, and creation morphisms.

I should say, I claimed this was an ∞-category. What's composition? Take two stratified intervals and glue them end to end. [pictures]

If you just glue naively, you get something that is not a constructible bundle. We can resolve the singularity using the blowups that we build into our definitions, we can retract the things that cause problems, like this: [picture].

The theorem, the hardest thing, is to say that this satisfies the Segal conditions. To do this, we concocted an entire theory of ∞-categories for sheaves on stratified spaces, because the simplex is topological and not just combinatorial. Almost everything else is easier than that.

So then you get values on more general manifolds. So now I'll describe the solid framed case.

In the solid framing, we don't want such a rigid notion.

Definition 13.2. The category of $c\mathcal{M}fd_n^{sfr}$ has objects as before but with solid n-framing. So instead of an isomorphism it's an injection into $\mathcal{E}_M^n$, the rank n trivial bundle.

Theorem 13.2. *(in preparation) If you have an ∞-category with adjoints, then factorization homology gives you a fully faithful embedding into functors from $c\mathcal{M}fd_n^{sfr}$ into spaces.*

It's illustrative to consider the difference in the 2-dimensional case [missed that discussion].

That's the end of part 1. In part 2, a similar thing is true if you consider pointed categories and not necessarily compact manifolds. It takes the same value as before on compact things, that doesn't depend on the basepoint, but $\int_{\mathbb{R}^k}\mathcal{C}$ is the k-endomorphisms of $\mathbf{1}$. We have a new cover, which is, let me draw a diagram, a limit diagram in $\mathcal{M}fd_n^{vfr}$,

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "emb"] \arrow[d] & \mathbb{D} \arrow[d] \\ \varnothing \arrow[r] & \partial\mathbb{D}^n \end{tikzcd}$$

and you get that factorization homology preserves the limit, and likewise with adjoints for the solid framed case.

That was part 2, and picking up speed, here's part 3. Define a functor $\mathcal{M}fd_{n+k}^{sfr}$ to Spaces, called $\mathcal{T}ang_{n\subset n+k}^{fr}$, and V goes to open embeddings of $\mathbb{R}^k \times M$ to V with a solid $n+k$-framing. Part of the theorem is that this is an $(\infty, n+k)$-category, not satisfying a completeness condition. You should think of this as codimension k tangles in V. [picture].

The theorem is that, mixing whether I think of things as a k-category or an $n+k$-category, we have an equivalence

Theorem 13.3. *There is an equivalence of pointed $(\infty, n+k)$-categories with adjoints $\mathcal{T}ang_{n\subset n+k}^{fr}$ and $\mathbb{R}^k$, where this is the functor corepresented by $\mathbb{R}^k$, maps out of $\mathbb{R}^k$.*

So why is this true? Well, we should check that $\mathcal{T}ang_{n\subset n+k}^{fr}(V) \cong Map_{\mathcal{M}fd_{n+k}^{sfr}}(\mathbb{R}^k, V)$. We said the left hand side was $\{\mathbb{R}^k \times M \hookrightarrow V\}$, and you can express something on the right hand side as a composite of a creation morphism and an open embedding $\{\mathbb{R}^k \to \mathbb{R}^k \times M \hookrightarrow V\}$. This first map is a choice of collar and that choice is contractible, so these are equivalent spaces. So that's step 3. Well, why does this imply the tangle hypothesis? If $\mathcal{C}$ is a pointed $(\infty, n+k)$-category with adjoints, then we can consider pointed functors $Fun_*(\mathcal{T}ang^{fr}_{n\subset n+k}, \mathcal{C})$, and this is the same thing by the previous theorem as $\int_{\mathbb{R}}^k \mathcal{C}$ which is the k-endomorphisms of the point in $\mathcal{C}$, which is one form of the tangle hypothesis.

So for step 4, for convenience I'll define $\mathcal{B}ord^{fr}_n$ as $\lim \Omega^k \mathcal{T}ang^{fr}_{n\subset n+k}$ [missed justification].

The proof of the bordism hypothesis, we want to calculate symmetric monoidal functors from $\mathcal{B}ord^{fr}_n$ to $\mathcal{X}$. We're mapping out of a sequential colimit, so this is $\lim_{\leftarrow} Fun^{\mathcal{E}_k}(\Omega^k \mathcal{T}ang^{fr}_{n\subset n+k}, \mathcal{X})$, and then we can write this as the limit of pointed functors

$$\lim_{\leftarrow} Fun_*(\mathcal{T}ang^{fr}_{n\subset n+k}, \mathcal{B}^k \mathcal{X})$$

and then applying the tangle hypothesis, we get $\lim_{\leftarrow}, k \to \infty k End_{\mathcal{B}^k\mathcal{X}}(\mathbf{1})$ which is $\lim \mathcal{X}^{\sim} \cong \mathcal{X}^{\sim}$.