MATRIX: HIGHER STRUCTURES IN GEOMETRY AND PHYSICS

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1. June 6: Emily Riehl: Weak complicial sets I

Weak complicial sets. I sort of accidentally wrote lecture notes while preparing, and I'll post them at `www.math.jhu.edu/~eriehl/wcs.pdf`

What are weak complicial sets? One of the problems in higher category theory is defining a higher category. I'm interested in ∞, n-category, where you have weak composition in every dimension and things are invertible above a certain n. There are many "models" for $\infty, 1$-categories. There's one model, quasicategories, with thousands of pages of how to do math with them.

The models above 1 are complicated and that has been an obstacle for working with these, and that's why I'm excited about the model that I'm talking about. I'm happy that I've finally had the chance to work with these in preparing for these lectures.

I'll start with quasi-categories and then think about how to expand it to $\infty, 2$ and then we can see how to do this in n and that's enough for one hour.

Definition 1.1. A <u>quasi-category</u> is a simplicial set A such that any inner horn has a filler.

I'll write this

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & A \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

for $n \geq 2$ and $0 < k < n$.

The idea is tha this gives an $\infty, 1$-category. The set of objects is the set of 0-cells. The morphisms are the 1-cells, which have source and target. We think about a 2-simplex in A as a *witness* that the first face h is (equivalent to) the composite gf; then the role of the three simplices

$$\begin{tikzcd} & 1 \arrow[dr, "j"] \arrow[dd, "\ell"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "g"] \arrow[dr, "k"'] & & 3 \\ & 2 \arrow[ur, "h"'] & \end{tikzcd}$$

so this witnesses that $h(gf) \cong hk \cong \ell \cong jf \cong (hg)f$

The homotopy category has objects A_0 and morphisms A_1, modulo the relation $f \cong g$ if there is a degenerate one simplex so that g is the composite of f with the degenerate simplex.

Why is this $(\infty, 1)$? Well, we'll see that 2-simplices are invertible, equivalences up to a 3-simplex, and 3-simplices are equivalences up to a 4-simplex and so one all the way up.

I'll show the first, start with a degenerate 2-simplex, I'll build a $\Lambda^1[3]$ horn. I start from the 2-simplex α

$$\begin{tikzcd}
1 \arrow[r, equal] & 2 \arrow[d, equal] \\
0 \arrow[u, "f"] \arrow[ur, "g"] \arrow[r, "f"'] & 3
\end{tikzcd}$$

and then I can write on the other side degenerate cells

$$\begin{tikzcd}
1 \arrow[r, equal] \arrow[dr, equal] & 2 \arrow[d, equal] \\
0 \arrow[u, "f"] \arrow[r, "f"'] & 3
\end{tikzcd}$$

and realize this as a horn that shows that there is a filler in the bottom side of the first diagram which shows that there is a 1-sided inverse. I can run this argument in reverse and see that there is an inverse on the other side as well.

In general, I can build a $\Lambda^1[3]$-horn in A as

$$\begin{tikzcd}
1 \arrow[r, "g"] \arrow[dr, "g"] & 2 \arrow[d, equal] \\
0 \arrow[u, "f"] \arrow[r, "h"'] & 3
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[d, equal] \\
0 \arrow[u] \arrow[ur, dotted] \arrow[r, "h"'] & 3
\end{tikzcd}$$

and once I get this filler I can turn this composition into the kind of one I had at the beginning.

So then the question is how can we model an $infty, 2$-category? The challenge is that the 2-simplices need to play dual roles. The perspective as a witness to composition and the perspective as [unintelligible] are not compatible.

So I'll mark 2-simplices as "thin" when they witness composition of 1-simplices. All the data for the 1-category structure is thin, so the 2-simplices are thin in the quasicategory case.

So how do I compose non-thin 2-simplices? If I have a non-invertible 2-simplex $\alpha : k\ell \to g$ and another $\beta : gh \to f$, well, first ,I'll find a thin 2-simplex witnessing a composition ℓh and so then I can get a $\Lambda^2[3]$-horn, and a filler will give me a "composite" of α and β.

Now let me introduce a definition to make this all precise.

I have an ordinary simplicial set, and then I'll need some special subset. Let me make this formal.

Definition 1.2. A *stratified* simpliciial set is a simplicial set with a specified collection of *marked* or *thin* simplices in positive dimensions, including all degeneracies.

A map is a simplicial map which preserves markings.

There is a forgetful functor from stratified simplicial sets to simplicial sets, which I'll call $(-)^{\#}$ and $(-)^{\flat}$ and $X^{\#}$ is marking everything in dimension at least 1 and $X^{\flat}$ is marking only degeneracies.

We have special inclusions of stratified sets $U \hookrightarrow V$, something is a *regular* inclusion if the markings in U are created in V, a simplex in U is thin if and only if it is marked in V. The other thing is entire, we say the inclusion is entire if this is an identity on underlying simplicial sets. So for example, all monomorphisms in stratified simplicial sets are generated under pushout and transfinite composition, coproduct if you will, by two classes, boundary inclusions of simplicial sets, $\{\partial\Delta[n] \hookrightarrow_r \Delta[n]\}^{\flat} \cup \{\Delta[n] \hookrightarrow_e \Delta[n]_t : n > 0\}$ where $\Delta[n]_t$ means the top simplex is marked as thin.

[some discussion of history]

Definition 1.3. A *weak coplicial set* is a stratified simplicial set that admits extensions along two classes of maps. [the way a quasicategory admits fillings along inner horns].

(1) *complicial horn inclusions* $\Lambda^k[n] \hookrightarrow_r \Delta^k[n]$, $n \geq 1$, $0 \leq k \leq n$ where a nondegenerate m-simplex in $\Delta^k[n]$, the k-*admissible* n-*simplex*, is thin if and only if it contains $\{k-1, k, k+1\} \cap [n]$. Thin faces include:
   - the top n-simplex.
   - the codimension 1-simplices except the $k-1$, k, and $k+1$.
   - the two simplex $[k-1, k, k+1]$ for $0 < k < n$
   - the one-simplex in the case $k = 0$ and $k = n$

   These should parameterize admissible composites existing.
(2) *complicial thinness extensions* $\Delta^k[n]' \hookrightarrow_e \Delta^k[n]''$. These are both the n-simplex, and both of these contain the k-admissible n-simplex. But in $\Delta^k[n]'$, I mark the $k-1$st and $k+1$st faces and in the codomain I mark all codimension 1 faces. This should mean that composites of thin faces are thin.

Let's see what this looks like in low dimensions.

Let's look at $\Lambda^1[2] \hookrightarrow \Delta^1[2]$

$$\begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "k"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd}$$

and the thinness gives the extra condition that if 01 and 12 are thin, so is 02.

Okay, and for $\Lambda^0[2] \hookrightarrow \Delta^0[2]$

$$\begin{tikzcd} & 1 \arrow[dr, "fe^{-1}"] \arrow[d, phantom, "\cong"] & \\ 0 \arrow[ur, "e\cong"] \arrow[rr, "f"'] & {} & 2 \end{tikzcd}$$

And for $\Lambda^2[3] \hookrightarrow \Lambda^2[3]$

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[d] \\
0 \arrow[u] \arrow[ur] \arrow[r] \arrow[u, phantom, shift left=4, "\Uparrow"] & 3 \arrow[ul, phantom, "\Uparrow"]
\end{tikzcd}$$

with fillers of both triangles, then I get a composite in the bottom half here:

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr] & 2 \arrow[d] \arrow[dl, phantom, "\cong"] \\
0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\Uparrow"] & 3
\end{tikzcd}$$

where the top half is filled with an equivalence. The thinness criterion says that if the first two 2-simplices were thin, so is the composite.

What about the outer one? [missed]

Then for $\Lambda^2[4] \hookrightarrow \Delta^2[4]$ I get a parameterization of composable 3-simplices:

$$\begin{tikzcd}
& 1 \arrow[d] \arrow[dr] & \\
0 \arrow[ur] \arrow[r, bend left] \arrow[dr] \arrow[rr, bend right] & 2 \arrow[d] \arrow[r, bend left] & 4 \\
& 3 \arrow[ur] &
\end{tikzcd}$$

and i fill this with

$$\begin{tikzcd}
& 1 \arrow[dd] \arrow[dr] & \\
0 \arrow[ur] \arrow[rr] \arrow[dr] & & 4 \\
& 3 \arrow[ur] &
\end{tikzcd}$$

Maybe you believe that this models some kind of (∞, ∞)-category, I have uniqueness up to higher dimensional simplices, I have these in all dimension, I have non-thin simplices in all dimension.

I can also specialize and get (∞, n) in all n. Let me say something about that.

Definition 1.4. A stratified simplicial set is n-*trivial* if all r-simplices with $r > n$ are thin.

There is a full subcategory of n-trivial stratified simplicial sets, which is both reflective and coreflective, via tr_n and core_n. So $\operatorname{tr}_n X$ is an idempotent monad, where $\operatorname{tr}_n X$ makes all higher simplices thin. The unit is $X \hookrightarrow_e \operatorname{tr}_n X$. Then $\operatorname{core}_n X$ is the regular subset spanned by the r-simplices whose faces above dimension n are thin. Then the counit is a regular inclusion, $\operatorname{core}_n X \hookrightarrow_r X$. These functors are adjoints, let me now draw a diagram explaining what this looks like in all dimensions.

At the bottom I have 0-trivial, which is like ordinary simplicial sets

$$\mathrm{Simp} \xrightarrow{(-)^{\#}} \mathrm{Strat}_{0,\mathrm{tr}} \hookrightarrow \mathrm{Strat}_{1,\mathrm{tr}} \hookrightarrow \cdots \hookrightarrow \mathrm{Strat}$$

and along each one of these there are sections tr_n and core_n. So an n-trivial weak complicial set is an (∞, n)-category. So if A is a weak complicial set, then $\operatorname{core}_n A$

is as well, but $\mathrm{tr}_n A$ may not be because I'm changing horns. There is an analogy here, the left adjoint to inclusion should be some free extension and the right adjoint should be the maximal subthing.

What are some examples? A 0-trivial weak complicial set is exactly a Kan complex. So when everything is marked, the thinness extension is for free, and the horn inclusions are exactly the horn inclusions, including outer.

A 1-trivial weak complicial set is a quasicategory. Conversely, a quasicategory admits markings, a 1-trivial thinness structure making it a weak complicial set. There's a canonical one, I'll talk about that in the third lecture. The final thing that's the preview for the afternoon, is that strict n-categories define strict n-trivial complicial sets (and there's a weak version).

2. Martin Markl: Operad-like structures, pasting schemes, and graph complexes I

Let me try to say roughly what operads and props are, so operads are suppos to model composable maps with several inputs and one output, while props model composable maps with several inputs and several outputs.

Let me try to arrange things into a sort of a table.

directed	non-directed
operads (symmetric or nonsymmetric) props (product and permutation category) dioperads half-props	cyclic operads modular operads (like correlation functions in physics)

and there are some structures that don't fit into these classes, wheeled props, and some exotic things that don't fit like hyperoperads, n-operads by Batanin, permutads

So for me these are determined by nested graphs having hereditary pasting schemes. These can be found in the paper that was stolen from me by Elsevier in 2008. This notion was developed by Borisov and Manin, and then by Ralph Kaufmann in his notion of Feynman category. There are other notions, like one by Berger, one by Batanin-Kock, and Barwick's operator categories. These were generalized by Batanin and me in the notion of an operadic category. These approaches are based on the notion of fibers and don't work in the nondirected case.

I'll work in Top or Sets or Vect. I'll ignoer units, signs, degrees, and symmetric group actions.

There will be two kinds of definitions, one will be unbiased (monadic), and the other will be biased.

Let me recall monads. A monad, also called a triple (namely in Montreal) on a category $\mathcal{C}$ is a functor $T : \mathcal{C} \to \mathcal{C}$ along with a transformation $\mu : T^2 \to T$ and another transformation $\nu : \mathbf{1} \to T$ such that this data forms a unital monoid in the category of endofunctors of $\mathcal{C}$.

An algebra over a monad is an object $A \in \mathcal{C}$ along with a map $\alpha : TA \to A$ which satisfies the following property (commutativity of the diagram.

$$\begin{tikzcd} T(TA) \arrow[r, "T(\alpha)"] \arrow[d, "\mu"] & T(A) \arrow[d, "\alpha"] \\ T(A) \arrow[r, "\alpha"] & A \end{tikzcd}$$

and while I said I wasn't going to do units, let me put it on the board.

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, equal, "\mathbf{1}"'] & & T(A) \arrow[dl, "\alpha"] \\ & A & \end{tikzcd}$$

So for example, take the free monoid monad, $T : \text{Sets} \to \text{Sets}$ where $TX = \bigsqcup_{n \geq 1} X^{\times n}$, where I'll draw this as

$$\bullet^{x_1} \text{------} \bullet^{x_2} \text{------} \cdots \text{------} \bullet^{x_n}$$

Then what does T^2X look like? It looks like a chain of things like this arranged in a line. I interpret T^2X as a nested graph, the outside level is this big vertices and the inside have their own list of bullets. Then I simply forget the nests. You may also ask how the transformation ν from X to TX looks in this language. This takes X to the graph which is $\bullet^x$. This is a very simple example of what we called a pasting scheme in my approach.

Then $\alpha : TX \to X$ is the same as an associative monoid, this is an unbiased definition. A biased definition, it's a map $\mu : X \times X \to X$ which is associative, $\mu(\mu \times \mathbf{1}) = \mu(\mathbf{1} \times \mu)$.

By the way, this is a nice example, motivating eample of a polynomial monad. I learned what these are a couple of months ago. Some monads in my talk will not be polynomial.

Let's do something a little closer to the theme of this meeting. First of all, let's start with classical operads. There are two versions, non-symmetric (or non-Σ) and Σ-operads. So non-Σ means you may not act by symmetric groups. There is a definition using $\circ_i$ operations $X(m) \times X(n) \to X(m+n-1)$ and the other based on $\mu : X(k) \times X(n_1) \times \cdots X(n_k) \to X(n_1 + \cdots + n_k)$. Sometimes the first approach is called a Markl operad and the second called a May operad. Under the presence of a unit, they are equivalent but in general they are not.

What about operads based on $\circ_i$ operations? Let me start with an unbiased definition. The category on which my monad will sit is collections, C_j where $X = \{X(n)\}$ for $n \geq 1$. Then $TX(n)$ is X-decorated (directed) rooted trees with n legs. I'm given a particular leg of a tree. Then each vertex has precisely one output edge. The vertices are decorated by elements of the collection, and this is precisely the number of incoming edges. [pictures]

Maybe I want planar, maybe I want to take a colimit if I'm in the Σ-category, this will not bother us during this talk.

What about T^2X? These are trees where the vertices are decorated by trees decorated by X. Again, μ, the structure operation forgets these nests.

So you can now ask what algebras over such a monad are. You'll decide that the structure of such an algebar is given by, in the monoid case it came from the map $X^2 \to X$, and here it's given by a graph with two vertices. [pictures]. The corresponding operation is $\circ_i : X(m) \times X(n) \to X(m+n-1)$. These are structure operations for a biased definition and they satisfy associativity, equivariance, and unitarity if I wish to have units.

I can also say that my operads are characterized by pasting schemes which are rooted directed trees.

I may also formulate more precisely the principle and what I mean by hereditarity. Assume I'm given a type $\mathbb{T}$ of graphs. In this case it means rooted directed trees. In general these will be potential pasting schemes. If I define TX as the set of decorated $\mathbb{T}$-graphs, then I should be able to come up with $T^2X \to X$, and a unit. You can easily see from the picture that the existence of μ says that if I am given a graph of type $\mathbb{T}$ with vertices decorated with graphs of type $\mathbb{T}$ then I get a graph of type $\mathbb{T}$ by expanding the vertices. For ν I should also get graphs with one vertex.

I have the following stupid example which illustrates that not every category of graphs has this property.

Take $\mathbb{T}$ to be graphs with at most two vertices. Then you can see that this kind of graphs is not hereditary. You can also ask how the monad describing May's operads looks. This is a monadic theory. You can define the monads very explicitly. I have a special assignment for Mark which is to decide if the monad is polynomial.

Let's move ahead. Now we may speed up a little bit. We have cyclic operads. So pasting schemes are trees, just trees, no directions, [pictures], something like this, which means, since I have no directions, I cannot distinguish between inputs and outputs, so I think of my operations as a blob or spider. Hereditarity of this type of trees is obvious. The biased definition is given by things with two vertices, ${}_i\circ_j : X(m) \times X(n) \to X(m+n-2)$. There are two versions, a Σ-version using abstract graphs and everything sits on collections so that $X(n)$ is a Σ_n-space, the symmetry of a graph with 1 vertex and n legs. Or I can do non-Σ, where $X(n)$ is a C_n-space, because there is cyclic symmetry for the single vertex.

Let's move, let me say what cyclic operads are good for, they describe algebras with a bilinear form.

Let me now move to a slight generalization. [unintelligible] moduli space of algebraic curves.

So modular operads I have to tell you the pasting scheme, which is for marked connected graphs. [pictures].

A marking is a map g from the vertices to the natural numbers. These are genera. If I'm given such a graph Γ, I calculate the genus of Γ as the sum of the local genera along with the overall genus of the graph.

So what is the underlying category? Now I have some kind of modular collections $X(n,g)$, for $n \geq 1$ (not so important) and $g \geq 0$. Then TX is the set of X-decorated graphs. This means I decorate each vertex with an element of X so that the number of adjacent edges is n and the genus marking is g. [pictures].

To get a biased definition, I need to specify first of all, things which correspond to these graphs, still in my picture. Now my ${}_i\circ_j$ now goes $X(n,g_1) \times X(m,g_2) \to X(n+m, g_1+g_2)$.

The operation here is of the following form: $\xi_{ij} : X(m,g) \to X(m-2, g+1)$.

Let me mention the stable version of modular operad.

I don't know if I'll get to the non-Σ version in my talk, I'll tell you tomorrow maybe, it's unexpected and quite surprising.

Every X determines a cyclic operad $\Box X$, where $\Box X(n) = X(n,0)$, and this functor has a left adjoint which is the modular completion. Little is known, the examples that are known explicitly are the ones for commutative and for associative algebras. Is this exact? I don't think so. What about the modular envelope of Lie? This is related to graph complexes very strongly.

What is commutative? In sets, it has $\mathrm{Com}(n) = *$ and it's the terminal cyclic operad. It's not too difficult to describe that $\mathrm{Mod}(\mathrm{Com})(n,g) = *$, and this is also terminal in the modular category. We can describe this with surfaces. [picture] This is an analogue (too fancy, but, well) of the cobordism hypothesis. I'll stop here.

3. Emily Riehl: Weak complicial sets II

Thanks again, so this talk we'll be in the weeds the whole time, I apologize about that.

A weak complicial set is a simplicial set with some higher simplices marked as thin, and these model ∞, n-categories for any n, including ∞.

Definition 3.1. A *weak complicial set* is a stratified simplicial set A admitting extensions of two types.

(1) $\Lambda^k[n] \hookrightarrow_r \Delta^k[n]$, where a non-degenerate simplex in the codomain is thin if and only if it contains the three or maybe two vertices $\{k-1, k, k+1\} \cap [n]$. There are two cases depending on whether this is an inner or outer horn. If $0 < k < n$, this includes the top simplex, all faces except for three, and the two simplex spanned by $\{k-1, k, k+1\}$. These aren't the only thin things but they're important ones. This thing, called the k-admissible n-simplex, as a composite of the d^{k+1} and d^{k-1} faces along their common boundary. If I have a pair, padded with a bunch of thin simplices, then a composite exists.

    In the $k = 0$ or $k = n$ case, we still have the top simplex, all but two codimension one face, and either the edge 01 or $(n-1)n$; this is something about equivalences and let's not stress.

(2) If the stuff that is present in the horn is thin, then so is the codimension one face obtained by filling. The way to say all that is, some further stratification of the n-simplex, again, I'm not fond of the notation, but $\Delta^k[n]' \hookrightarrow_e \Delta^k[n]''$, where in $\Delta^k[n]'$ we mark the $k-1$ and $k+1$ codimension one face, and in $\Delta^k[n]''$ we mark all codimension one faces.

Now that I reviewed, I can say

Definition 3.2. A *strict complicial set* is a complicial set admitting unique extensions.

Now I want to inttroduce a source of examples of strict or weak complicial sets, from n-categories or ω-categories, and we'll get this from a nerve functor.

I'll be talking about the *Street nerve*. The precise definition will occupy essentially the entire hour. I'll start with an overview and then come back. It will be a functor from ω-categories to simplicial sets, N, and then I can choose a lift (we'll be interested in two different lifts) to Strat. This will be a generalization of the more familiar nerve of 1-categories.

How do I get such a thing? There is a well-known Kan construction. We combine a functor $\mathbf{\Delta} \xrightarrow{O} \omega - \mathrm{Cat}$, which takes $[n] \mapsto O_n$, the *nth oriental*.

The reason we care about this is on account of the following theorem, called the *Street–Roberts conjecture*, proved by Dominic Verity.

Theorem 3.1. *The Street nerve defines a functor* $\omega - \mathrm{Cat} \xrightarrow{N} \mathrm{Strat}$ *where* $\Delta[n] \to N\mathcal{C}$ *is thin if and only if* $\mathcal{O}_n \to \mathcal{C}$ *carries the top* n*-cell in* $\mathcal{O}_n$ *to an identity in* $\mathcal{C}$.

Moreover, N *is fully faithful with essential image the strict complicial sets.*

This theorem led to the development of this theory, Roberts introduced strict complicial sets because he suspected something like this would be true. So I'll introduce ω-categories, orientals (this is the weeds part), the nerve, and then other stractifications on $N\mathcal{C}$, which will take us to the question of saturation which will motivate part three.

Let's dive in. The definition of ω-categories is cute, it's due to Ross Street, it's a single-sorted definition, I'll do it backwards, start at the top and unpack.

Definition 3.3. An ω^+*-category* (don't ask about the +, I'll talk about it later) is a set C with $(*_n, t_n, s_n)$, composition, source, and target, for $n = 0, 1, 2, \ldots$ so that for all $m < n$, the tuple $(C, *_m, s_m, t_m, *_n, s_n, t_n)$ is a strict 2-category.

Definition 3.4. A 2*-category* is a set C with $*_0, s_0, t_0, *_1, s_1, t_1$ such that $(C, *_i, s_i, t_i)$ are 1-categories such that, well, I should be able to compose 2-cells horizontally.

If $t_0(a) = s_0(b)$, then $s_1(b) * s_1(a) = s_1(b *_0 a)$ and similarly $t_1 b_0 * t_1 a = t_1(b *_0 a)$.

I have globularity, $s_0 t_1 = s_0 s_1 = s_0 = s_1 s_0$ and similarly $t_0 t_1 = t_0 s_1 = t_0$ and there's another thing that you don't have to assume.

Then there are middle four interchange, where I can compose in multiple directions, you can write down the axiom.

$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5, "\Downarrow"] \arrow[r, phantom, shift right=2.5, "\Downarrow"] & \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5, "\Downarrow"] \arrow[r, phantom, shift right=2.5, "\Downarrow"] & \bullet \end{tikzcd}$$

Definition 3.5. A 1*-category* is a set C with $*$ and s and t so that, $tt = st = t$ and $ss = ts = s$ and $C \times_C C \xrightarrow{*} C$ with $s(a * b) = s(b)$ and $t(a * b) = t(a)$ with identity and associativity axioms.

An n*-cell* in an ω^+-category is an identity for $*_n$. A *cell* in an ω^+-category is an n-cell for some n. An ω-category is one containing only cells, and an n-category is one containing only cells of size at most n.

One example of an ω^+ category that's not an ω category is the free ω^+ category on an element, where I have two 0-cells, $(0, s)$ and $(0, t)$, I'll have two 1-cells called $(1, s)$ and $(1, t)$, and so on. I'll have a single element in the middle ω which doesn't have a dimension.

Call this $\mathbf{2}_\omega$, then a functor from $\mathbf{2}_\omega \to C$ is an element of C with unrestricted dimension.

What's cool about this is that I can give a nice description of ω_+ functors, elements of $[A, B]$ are $A \times \mathbf{2}_\omega \to B$.

Theorem 3.2. *(Street) There is an equivalence* $(\omega^+ - \mathrm{Cat}) - \mathrm{Cat} \to \Omega^+ - \mathrm{Cat}$ *and from* $(n - \mathrm{Cat}) - \mathrm{Cat} \to (1 + n) - \mathrm{Cat}$ *for all* $n = 0, \ldots, \omega$.

This is similar to proving that a Cat-enriched category is a 2-category.

Now I'll tell you about the orientals. The oriental $\mathcal{O}_0, \dots$, well, the oriental $\mathcal{O}_n$ is an n-category, and the idea is that it's the free n-category on the faces of an n-simplex, and the reason that's hard is because it's hard to make it precise, and saying it's free makes it a little bit hard too. In low dimensions, it's clear enough.

Definition 3.6.
- $\mathcal{O}_0$ has a single 0-cell.
- $\mathcal{O}_1$ has two 0-cells and a single 1-cell.
- $\mathcal{O}_2$ has three 0-cells as generators but then it's free so there's another 1-cell, which is the composition $\{01, 12\}$, and a 2-morphism $02 \to \{01, 12\}$. Then we get a two-category with three 0-cells, three atomic 1-cells (and all compositions) and this has one atomic 2-cell.

$$\begin{tikzcd} & 1 \arrow[dr, "12"] & \\ 0 \arrow[ur, "01"] \arrow[rr, bend left, "{\{01,12\}}"] \arrow[rr, bend right, "02"'] & & 2 \end{tikzcd}$$

- I have four 0-cells for $\mathcal{O}_3$, the free category on this graph

$$\begin{tikzcd} 1 \arrow[r, "12"] & 2 \arrow[d, "23"] \arrow[r, phantom, "\Rrightarrow"] & 1 \arrow[r, "12"] \arrow[dr, "13"] & 2 \arrow[d, "23"] \\ 0 \arrow[u, "01"] \arrow[ur, "02"] \arrow[r, "03"'] \arrow[ur, phantom, "\Uparrow", near start] & 3 & 0 \arrow[u, "01"] \arrow[r, "03"'] \arrow[ur, phantom, "\Uparrow"] & 3 \end{tikzcd}$$

- In general, the atomic k-cells are k-dimensional faces of $\Delta[n]$. The source of a k-cell is a pasted composite of all of its odd faces.

  Precisely, a k-cell in $\mathcal{O}_n$ is a pair (M, P) where M and P are subsets of faces of $\Delta[n]$ which are *well-formed*, non-empty, and each *move M to P*.

  Here S a collection of faces is *well-formed* if it contains at most one vertex and if $x \neq y$ in S then x and y have no sources in common and no targets in common, this is a condition that only needs something for cells of the same dimension.

  Suppose S is a subset of faces of $\Delta[n]$, let S^- be the union of all sources of S and S^+ the union of all targets. Then S moves M to P if $M = (P \cup S^-) \backslash S^+$ and $P = (\cup S^+) \backslash S^-$.

  Let me tell you about sources, targets, and composition. If (M, P) ns an n-cell, then the source $s_k(M, P)$ are [missed].

  The composition $(M, P) * (N, Q) = (M \cup N \backslash N_n, Q \cup (P \backslash P_k))$.
- So the 4-cell in $\mathcal{O}_4$ has $M = \{01234, 0234, 0134, 012, 023, 034, 04, 0\}$ and in P I have $\{01234, 0123, 0134, 1234, 014, 012, 234, 01, 12, 23, 34, 4\}$

The point is maybe that this is nontrivial mathematics.

What about doing some of these things explicitly for the 2-simplex? I should rename things

$$\begin{tikzcd} & (1,1) \arrow[dr, "{(12,1)}"] & \\ (0,0) \arrow[ur, "{(01,?)}"] \arrow[rr, bend left, "{(\{01,12,\},?)}"] \arrow[rr, bend right, "{(02,0)}"'] & & (2,2) \end{tikzcd}$$

The free oriental is generated by the faces of the k-simplex in a reasonable sense. If I've defined something up to k there's a unique extension to dimension $k+1$. This can be used to show that the orientals define a functor from Δ to $\omega - \mathrm{Cat}$.

Now when I have this I get a nerve. Maybe I'll just remind us of the definition. If $\mathcal{C}$ is an ω-category, then $N\mathcal{C}_n$ is $\hom(\mathcal{O}_n, \mathcal{C})$, and it's relative to that nerve that we're talking about the Street nerve.

Let's talk about 1 and 2-categories in the Street nerve. We're thankfully out of the weeds now.

Let me list these as facts.

(1) If $\mathcal{C}$ is a 1-category, then $N\mathcal{C}$ is the usual nerve as a simplicial set, with only the degenerate 1-simplices and all $n \geq 1$ simplices marked.
(2) If $\mathcal{C}$ is a 2-category, then $N\mathcal{C}_0$ is the zero-cells in $\mathcal{C}$, then $N\mathcal{C}_1$ is the 1-cells in $\mathcal{C}$ (the 0-cells include as degeneracies) and $\mathcal{C}_2$ is a triple of objects

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, bend left, "g\circ f"] \arrow[rr, bend right, "h"'] \arrow[rr, phantom, "\Uparrow"] & & 2 \end{tikzcd}$$

The two-cell is marked if and only if α is an identity.
(3) What about $N\mathcal{C}_3$? This has [pictures] and it's 4-coskeletal.
(4) In general, if $\mathcal{C}$ is an n-category then $N\mathcal{C}$ is $(n+1)$-coskeletal.

What about weak complicial sets? So $N\mathcal{C}$ is a strict complicial set if the thin simplices are identities. Can we give a different stratification, though, making it into a weak, not strict, complicial set?

Let's think about that.

I want to explore thinness structures, so $\mathcal{C}$ is still strict but I'll obtain a weak complicial set. Let me explore this in the case when $\mathcal{C}$ is a 1-category and $N\mathcal{C}$ is the usual nerve. It seems reasonable that we'll get things that are 1-trivial, so let's mark all n-simplices for n greater than 1. So we want to choose a different set of 1-simplices. Suppose I have f thin in $\mathcal{C}$. We saw this morning that I can build a horn $\Lambda^0[2] \to N\mathcal{C}$, for the diagram

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, equal] \arrow[ur, phantom, shift right=6, "\cong"] & & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr, "f"] & \\ 0 \arrow[ur, "h"] \arrow[rr, equal] \arrow[ur, phantom, shift right=6, "\cong"] & & 2 \end{tikzcd}$$

So any marked 1-simplex is an *equivalence* where

Definition 3.7. A 1-simplex is an *equivalence* if there is a pair of thin 2-simplices

$$\begin{tikzcd} & 1 \arrow[dr, "f"] & \\ 0 \arrow[ur] \arrow[rr, equal] \arrow[ur, phantom, shift right=6, "\cong"] & & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur, "f"] \arrow[rr, equal] \arrow[ur, phantom, shift right=6, "\cong"] & & 2 \end{tikzcd}$$

Definition 3.8. A weak complicial set is 1-saturated if the map $(*)$ is an isomorphism:

$$\begin{tikzcd} A_0 \arrow[r, "s_0"] \arrow[d] & A_1 \\ \mathrm{th}_1 A \arrow[r, "(*)"] & \mathrm{eq}_1 A \arrow[u] \end{tikzcd}$$

4. June 7: Martin Markl: Operad-like structures, pasting schemes, and graph complexes II

Thank you very much. Remember that yesterday I gave you the assignment of doing pasting schemes for [unintelligible] operads. Ralph Kaufmann did the exercise, let me tell you the answer. Remember that my May's operads had operations $P(k) \otimes P(n_1) \otimes \cdots \otimes P(n_k) \to P(n_1 + \cdots + n_k)$. Remember that the pasting scheme or monad is basically the free object of this type. So your understanding, if you start with this picture, which is indeed [unintelligible], you need to iterate it. You have levels, or maybe boxes [pictures] and then you may put things in this place on the picture, but in every slot. It's not so simple, because you can put something on top like this, but then you should insert inputs, a similar thing, and so actually what you get is this kind of structure, but inside each box the same structure.

It's actually, though, not so complicated, the boxes are determined by the underlying planar tree, as long as you have the condition that, well, any vertex either has all input edges leaves or all input edges are used.

If I make no mistake, the underlying trees of these diagrams, you can reconstruct this kind of thing from such data. Another assignment, to Mark, is this monad polynomial or not?

So very good, the last thing which I mentioned is a property of modular completion. Just to recall the notation, to recall that we had modular operads and the forgetful functor $\square$ to cyclic operads, and left adjoint is something I called modular completion, nowadays it's called modular envelope, and if I take Mod(Com) it has the property that for each arity and genus it is a point, so it's terminal in modular operads, just as Com is in cyclic operads. I got a nice description in terms of oriented surfaces with n holes and of genus g.

The last thing that I will do regarding the modular envelope is the same thing for Ass. So I should say that Ass is the terminal non-Σ operad. So one would expect Mod(Ass) to be a terminal non-Σ modular operad. The description of Mod(Ass) was known for a long time, to physicists, to Ralph Kaufmann, but as far as I know the first combinatorial description was due to my student M. Doubek. So what is the description?

I should give you $\mathrm{Mod(Ass)}(n;g)$. This is the set of all disjoint decompositions

$$((\ldots))_1, ((\ldots))_2, \ldots, ((\ldots))_b$$

of $\{1, \ldots, n\}$ into cyclicly ordered subsets. I also admit empty blocks. Then there will be infinitely many of them. We have a kind of geometricity, $G := \frac{g-b+1}{2}$ is a natural number. Then this tells you there are only finitely many of these.

There is a better way to think about this, due to physicists, because this and the related (open-closed) string field theory is that this is the same as the isomorphism classes of surfaces of genus G with b holes, each of which has the appropriate number of teeth.

If you think of how these isomorphism classes are described, you'll get this structure. You can kind of move the holes around the surface, and also may rotate inside the holes which corresponds to the cyclic symmetry.

What about operadic composition? If I'm given another surface of the same type, my circle product ${}_i\circ_j$ connects these things by a ribbon, and the $\xi_{3,1}$ are obtained by putting a ribbon inside the same surface. The operations may change the number of holes and geometric genus in a surprising manner.

So it turns out that this is the terminal non-Σ modular operad. If I'm not mistaken, Ralph didn't identify this in its own right. There is a smiilar thing for open-closed but it will take us too far.

So let's move to PROPs. If you have $X^{\times m} \to X^{\times n}$ in a Cartesian category, this is determined by $X^{\times m} \to X$, so PROPs are determined by operads. So let's move to the category of chain complexes. A PROP (short for product and permutation category) is a collection $\{P(m,n)\}$ with structure operations that can be read off from the motivating example of PROPs, that is, End_V so that $\mathrm{End}_V(m,n) = \mathrm{Lin}(V^{\otimes m}, V^{\otimes n})$, and the structure operations are those that you have for such a collection, you can define

- vertical composition which is
$$\circ : P(m,n) \otimes P(n,k) \to P(m,k),$$
- and then you have horizontal composition,
$$\boxtimes : P(m_1,n_1) \otimes P(m_2,n_2) \to P(m_1+m_2, n_1+n_2)$$
- you have a unit in $P(1,1)$
- a symmetric group action $\Sigma_m \otimes P(m,n) \otimes \Sigma_n \to P(m,n)$ involving the permutation of outputs of multilinear maps.

So what are pasting schemes for props? They are directed graphs, I should probably say what these are, each edge is directed and there are no oriented cycles. The first condition means also that the collection adjacent to each vertex can be divided into outputs and inputs.

I tend to denote operations in a PROP by boxes. [pictures]

What are the graphs representing these operations? The $\circ$ operation is contraction. The horizontal composition is disjoint union. The identity e is the trivial directed graph. The symmetric group action is by relabelling the inputs or outputs.

Let me give you a very important warning at this point. The warning is that this nice correspondence between the biased and unbiased definitions works only when $P(0,0) = P(1,0) = P(0,1) = 0$. This is why I assumed that $m,n \geq 1$. If I admit these things I can get a graph which is a single dot, with something from $P(0,0)$.

Okay so these are PROPs.

Let me mention some modifications of PROPs. In principle this is simple, but very useful, this is properads, introduced by Vallette. Pasting schemes are connected directed graphs. So while on the one hand the structure or unbiased definition is easier, it means you don't have horizontal composition, and the biased definition is slightly more complicated, because you are not allowed to use some part of the structure. The biased definition involves operations that correspond to graphs which are kind of difficult to draw, but I will try [pictures].

All structures and props you come upon naturally are properadic, you have to think hard or come up with something unnatural to get a structure which is not properadic. [example]. There is another simplification called dioperads, due to Gan, which means that I restrict to simply connected directed graphs. The biased definition involves only one connection between the two things. [pictures].

It turns out that there are some interesting objects that exist over dioperads like Lie bialgebras or infinitesimal bialgebras. I'll talk about them tomorrow.

Finally, there is something which I called $\frac{1}{2}$-PROPs. Tomorrow we'll see why these are useful. The pasting schemes are like for dioperads with an extra conditino. They are connected directed simply connected graphs with the property that for any edge of Γ, either e is a unique outgoing edge or unique incoming edge of its vertex.

Of course, you're probably interested to know what has the structure of such a thing, so the example is a half-bialgebra with the compatibility that $\Delta\mu = 0$.

Tomorrow we will see why these things are important.

Let me finish this exposition by mentioning what is a wheeled PROP. I'm referring to my work with A. Voronov. Wheeled PROPs were introduced by me, Merkulov, and Shadrin. The pasting schemes are directed graphs possibly containing directed loops, which we called wheels. The biased definition also included graphs like this [pictures] where you connect the input and the output of the same thing. The resulting operation resembles self-gluing for modular operads.

Why are these things useful? It can be shown that solutions to a specific master equation are algebras over wheeled PROPs.

Okay? There are modifications, wheeled properads, where you assume these graphs are connected, there are wheeled operads where you have either one or no output edge. I'm not going to speak about these things, I'm going to present more, I won't say more exciting because what is the notion of excitement in mathematics, but something different.

So hyperoperads, that was the last item on my list. In all examples which we saw so far, operads were indexed by natural numbers or a couple of natural numbers. I can call these arity and genus. But in general arities may be very different objects. Let me start with an example and then give a general approach. The example is the planar rooted tree hyperoperad PRT. Say I'm given two planar rooted trees S and T. A map between them is a map of trees that preserves external flags. The map φ is obtained by contracting some subtrees of S. I have some subtree T_v of S which is contracted to v. I call this the fiber over v.

If the vertices of T are labeled from 1 to k, then I have fibers $T_1, \ldots, T_k$, and the PRT hyperoperad is a collection indexed by planar rooted trees, and the structure map, for any φ, I get a map $A(T) \times A(T_1) \times \cdots \times A(T_k) \to A(T_S)$.

What about examples? There is a terminal one, given by $\mathbf{1}(T) = *$. You also have the endomorphism PRT-operad. So $X = \{X(n)\}_{n \geq 1}$, and then $\mathrm{End}_X(T)$ is the set of maps from $X(T)$ into $X(n)$. What is $X(T)$? It's a product over vertices of T of $X(\mathrm{arity}(v))$, so pictures might help: [pictures].

If A is a PRT-operad, then an A-algebra is a PRT-morphism $A \to \mathrm{End}_X$ which is $\alpha_T : A(T) \to \mathrm{End}_X(T)$.

The theorem is that 1-algebras are Markl operads. So α_T are determined by trees with two vertices, and then this is given by specifying a map $X(m) \times X(n) \to X(m+n-1)$.

There is a general scheme that given a class of graphs of some type $\mathbb{T}$ satisfying a hereditarity property, then there exists a notion of $\mathbb{T}$-hyperoperads such that operad or PROP or whatever determined by $\mathbb{T}$-pasting schemes is the same as algebras over the terminal hyperoperad of this type.

SoI still have a couple of pages but of course I don't want to go over time. So I'll tell you a couple of things. This scheme can be formalized in a couple of ways. While my operad is given by a monad determined by this pasting scheme, this is determined by the + construction of the monad. So it's a reflection of something sometimes called the [unintelligible] principle.

There is another attempt to generalize the situation given by the notion of an operadic category which you can find in my paper with Batanin. This is a category $\mathcal{O}$ such that maps have fibers. This is a generalization of an operator category, I guess by Barwick, which assumes the fibers are pullbacks, but there are examples where they are not pullbacks.

Most examples which I gave are operads in an operadic category. But this can't include modular or cyclic operads. There is nothing like a completely unifying approach, which is maybe part of the reason that operads are not welcome in other parts of mathematics. Ralph, I think, will attempt something in this direction.

Tomorrow I'll speak about something using half-props and dioperads. Thank you very much.

5. Emily Riehl: Weak complicial sets III

The lecture notes are at `www.math.jhu.edu/~eriehl/wcs.pdf`. Today we'll talk abou the issue of saturation for weak complicial sets and then end with some stuff about the homotopy theory of them, due to Verity.

Saturation applies to general weak complicial sets. Let's motivate with specific examples from ω-categories.

Last time we saw that if $\mathcal{C}$ is a strict ω-category, we have the Street nerve $N\mathcal{C}$, and it's most convenient to think of this as a simplicial set (so we can choose different stratifications) where an n-simplex is $\mathcal{O}_n \to \mathcal{C}$ where $\mathcal{O}_n$ is the nth oriental. The thing to remember today is that it has a unique n-cell representing the n-simplex. We can obtain a complicial set in various ways. The Street–Roberts procedure gives a strict complicial set, meaning we have a unique filler for all of these complicial horns, where a simplex is thin if $\mathcal{O}_n \to \mathcal{C}$ carries the n-cell to an identity. One thing that's cool about weak complicial sets is that I can fix a single simplicial set and change the stratification to give a more refined or generous theory of equivalence. This particular stratification is minimal, it uses only the identities. But I could imagine something more flexible where I use equivalences instead or something like this. You could also go the other direction, where you have a simplicial set where n-simplices are simplicial cobordisms, and this is a Kan complex, and you can ask how to make this a weak complicial set. You could use the maximal stratification where everything is thin, so you have all cobordisms are equivalences. You could make thin simplices ones that are h-cobordisms, or that the thin things are equivalences are, say, quasi-invertible, or trivial (in having a collapse onto positive or negative faces). Those are four distinct structures, stratification structures, and this theory is quite flexible in that sense.

It would take a long time to get that example up and running so I'll focus on the categorical example today.

From the perspective of the Street nerve, the question is, can I mark more simplices and still get a weak complicial set. So the answer is "yes" and the solution we'll focus on today is to construct a saturation of a weak complicial set.

The first major task is to define saturation in all dimensions, but we'll start in low dimensions and then generalize. So let's suppose, you might think of the nerve of a category but it's more general. Suppose A is any weak complicial set. Consider, well, which edges, at dimension one, which edges can we make thin? Consider a thin one simplex.

$$\begin{tikzcd} & 1 & \\ 0 \arrow[ur, "f"] \arrow[rr, equal] & & 2 \end{tikzcd}$$

and this provides us with a filler with the horn $\Lambda^0[2] \to \Delta[2]$

$$\begin{tikzcd} & 1 \arrow[dr, red] \arrow[d, phantom, "\cong"] & \\ 0 \arrow[ur, "f"] \arrow[rr, equal] & {} & 2 \end{tikzcd}$$

and we get the dual picture as well.

$$\begin{tikzcd} & 1 \arrow[dr, "f"] \arrow[d, phantom, "\cong"] & \\ 0 \arrow[ur, red] \arrow[rr, equal] & {} & 2 \end{tikzcd}$$

Definition 5.1. We say that a 1-simplex f is an *equivalence if there exist fillings*

$$\begin{tikzcd} & 1 \arrow[dr, red] \arrow[d, phantom, "\cong"] & \\ 0 \arrow[ur, "f"] \arrow[rr, equal] & {} & 2 \end{tikzcd}$$

$$\begin{tikzcd} & 1 \arrow[dr, "f"] \arrow[d, phantom, "\cong"] & \\ 0 \arrow[ur, red] \arrow[rr, equal] & {} & 2 \end{tikzcd}$$

Theorem 5.1. *Every thin* 1*-simplex in a weak complicial set is an equivalence.*

A weak complicial set is 1-*saturated* if every equivalence is thin.

Then given a 1-category, $N\mathcal{C}$ is saturated if it's given the 1-trivial stratification an isomorphisms are marked. This is the maximal thing making $N\mathcal{C}$ a weak complicial set.

We needed to know the thin 2-simplices to do this so we have to induct down.

So now suppose A is a weak complicial set and consider a thin 2-simplex $\alpha : f \to g$. From such a gadget I'll build an admissible horn $\Lambda^1[3] \to \Delta^1[3]$ which then has a filler. This has a thin filler.

$$\begin{tikzcd} 1 \arrow[r, equal] & 2 \arrow[d, equal] & \cong & 1 \arrow[r, equal] \arrow[dr, equal] & 2 \arrow[d, equal] \\ 0 \arrow[u, "g"] \arrow[ur, "f"] \arrow[r, "g"'] \arrow[ur, phantom, "\cong\ \alpha", near end, shift left=2] \arrow[ur, phantom, "\cong\ \beta"', near start, shift right=2] & 3 & & 0 \arrow[u, "g"] \arrow[r, "g"'] \arrow[ur, phantom, "\cong", shift left=2] \arrow[ur, phantom, "\cong"', shift right=2] & 3 \end{tikzcd}$$

We can also do the other handed version.

We call α an equivalence if there is a pair of thin 3 simplices as in this diagram and its other handed version.

So if α is thin then α is an equivalence, has liftings like this, and we say A is 2-saturated if all 2-equivalences are thin.

Consider a 2-simplex in a weak complicial set. [pictures]

An example, if $\mathcal{C}$ is a strict 2-category, make $N\mathcal{C}$ 2-trivial. If we mark the 2-identities, then this is 1-saturated but not 2-saturated. If we marke the 2-isomorphisms and the 1-equivalences then $N\mathcal{C}$ is saturated and this is the maximal stratification making $N\mathcal{C}$ a weak complicial set.

Now let me redefine this, first in dimension 1 and then in all dimensions.

If f is a 1-equivalence, we can build an admissible horn.

$$\begin{tikzcd}
 & 1 \arrow[dd, "f"] \arrow[dr, equal] & \\
0 \arrow[ur, "e"] \arrow[dr, equal] \arrow[rr, dotted, "h\cong"] & & 3 \\
 & 2 \arrow[ur, "g"'] &
\end{tikzcd}$$

and this defines a stratified map from $\Delta[3]_{\text{eq}} \to A$ where the domain is the three-simplex with stratification 1-trivial (marking the two and three simplices) but also the edges [02] and [13]. This is like 2 of 6 hypothesis. A map into A picks out an e, an f, and a g I have a three simplex where everything other than e, f, g and h is thin.

$$\begin{tikzcd}
1 \arrow[r, "f"] & 2 \arrow[d, "g"] \\
0 \arrow[u, "e"] \arrow[ur, "\cong"] \arrow[r, "h"', "\cong"] & 3
\end{tikzcd}
\quad \cong \quad
\begin{tikzcd}
1 \arrow[r, "f"] \arrow[dr, "\cong"] & 2 \arrow[d, "g"] \\
0 \arrow[u, "e"] \arrow[r, "h"', "\cong"] & 3
\end{tikzcd}$$

Then the other four are as well, this is a lifting property

$$\begin{tikzcd}
\Delta[3]_{\text{eq}} \arrow[r] \arrow[d] & A \\
\Delta[3]^{\#} \arrow[ur, dotted] &
\end{tikzcd}$$

and then we say that f is a 1-equivalence if there is $\Delta[3]_{\text{eq}}$ with f as 12 edge.

Okay, so $\boldsymbol{\Delta}$ sits inside $\boldsymbol{\Delta}_+$ which has the empty ordinal as well.

I now have the direct sum $\oplus$ which takes $[n]$ and $[m]$ to $[n+m+1]$ and then there's something called Day convolution. If I have presheaves here then I get a join. So if I look at augmented simplicial sets, I get

$$\mathrm{Simp}_+ \times \mathrm{Simp}_+ \xrightarrow{\star} \mathrm{Simp}_+$$

where $\Delta[n] \star \Delta[m] \cong \Delta[n+m+1]$. I can restrict this to $\mathrm{Simp} \times \mathrm{Simp} \to \mathrm{Simp}$, and I'm interested in a stratification of the join.

Definition 5.2. The join lifts to

$$\begin{tikzcd}
\mathrm{Strat} \times \mathrm{Strat} \arrow[r, "\star"] \arrow[d] & \mathrm{Strat} \arrow[d] \\
\mathrm{Simp} \times \mathrm{Simp} \arrow[r, "\star"] & \mathrm{Simp}
\end{tikzcd}$$

where $\Delta[n] \to A \star B$ given by $\Delta[k] \to A$ and $\Delta[n-k-1] \to B$ for some k is thin in $A \star B$ if and only if one of its components is thin in A or B.

Definition 5.3. A weak complicial set is *saturated* if it admits extensions

$$\begin{tikzcd}
\Delta[3]_{\mathrm{eq}} \star \Delta[n]^k \arrow[r] \arrow[d] & A \\
\Delta[3]^{\#} \star \Delta[n]^k \arrow[ur, dotted] &
\end{tikzcd}$$

Equivalently one could use $\Delta[n] \star \Delta[3]_{\mathrm{eq}} \to \Delta[n] \star \Delta[3]^{\#}$ or a two-sided version.

Let me draw a picture in a 4-simplex $\Delta[3]_{\mathrm{eq}} \star \Delta[0]$ [pictures].

The thin things are simplices whose intersection with $\Delta[3]_{\mathrm{eq}}$ is [02] or [13].

This is characterizing the equivalences whose edge from 0 to 1 is an equivalence. This is sufficient by a robustness argument and you have to do something more in higher dimensions but [unintelligible] still works.

Let me end with some remarks.

Remark 5.1. Saturation is an inductive definition. If I start with an n-trivial stratified simplicial set, then there exists a unique expansion of its stratification that's saturated. If I'm not n-trivial for any n, then there may be multiple saturated stratifications on a fixed simplicial set.

So this notion is most interesting when I'm not n-trivial for any n. For the case where I am, a category theorist thinks maybe the n-trivial unique one is the best choice, you want equivalences to be thin.

Eugenia Cheng has [missed] example.

The one thing that's not terribly well-understood, the last thing I want to admit, it's not entirely understood how stratifying an existing weak complicial set interacts with the axioms, there's something delicate there. Dom is convinced that you can start with one and stratify it minimally, then that's a complicial set as well; that's not totally proven yet. But if you can replace the underlying simplicial set then there's no problem, you can fibrantly replace, and that's what I'll end talking about.

Before I do, I should talk about the ambient category— Strat is Cartesian closed, with product $\times = \circledast$ playing the role of the Gray tensor product (first introduced in 2-category theory). There's a strict product on 2-categories. This is probably too strict, and there's a 2-category which works with pseudofunctors and similar things, and replacing the Cartesian product with the Gray tensor product gives you that model. You also have that for bicategories.

I'll write $\mathrm{Hom}(A,B)$ for the right adjoint to this product. You should think of a vertex in this simplicial set as a pseudofunctor. There's an important fact

Lemma 5.1. *If I consider two sets of maps, one will be* $I = \{\partial\Delta[n] \hookrightarrow_r \Delta[n] | n \geq 0\} \cup \{\Delta[n] \hookrightarrow_e \Delta[n]_t | n \geq 1\}$ *and the anodyne extensions for* $J = \{\Lambda^k[n] \hookrightarrow_r \Delta^k[n] | n \geq 1, k \in [n]\} \cup \{\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'' | n \geq 1, k \in [n]\}$*, then for all* i *in* I *and all* j *in* J*, then the pushout product* $i \hat{\otimes} j$ *is a composite of pushouts of elements of* J*.*

Then $\mathrm{Hom}(X, A)$ where A is a weak complicial set, is itself a weak complicial set. So we have this exponential ideal kind of thing.

So there's a naive homotopy theory using the thin interval and the Gray tensor product or Hom. I want to talk about various model structures which can be built using Jeff Smith's theorem. I can be quite precise about the fibrant objects I want to specify. I want to give a whole family with different fibrant objects.

Definition 5.4. I want to use the same J, because this is the same meaning, but it might be a different class.

Let K be a set of monomorphisms in Strat, and I'll say a stratified simplicial set is K-fibrant if I can lift

$$\begin{tikzcd} X \arrow[r] \arrow[d, "k"] & A \\ Y \arrow[ur, dotted] & \end{tikzcd}$$

for all k in K. So let K be a set so that the following things hold.

1. $J \subset K$, so K is an extension of elementary anodyne inclusions.
2. (this and the next two are equivalent, so any or all of them hold) For each K-fibrant A and k in K, we have $\mathrm{Hom}(Y, A) \xrightarrow{k^*} \mathrm{Hom}(X, A)$ is a homotopy equivalence.
3. For each K-fibrant A and $k \in K$, we have $\mathrm{Hom}(Y, A) \xrightarrow{k^*} \mathrm{Hom}(X, A)$ is a trivial fibration (has the right lifting property against monomorphisms)
4. We have $i \hat{\otimes} k$ for $i \in I$ and $k \in K$ are in the class generated by K.

So I didn't name these but these are the K I want to consider.

Theorem 5.2. *(Verity) For any such* K*, there exists a model structure on the category of stratified simplicial sets so that*

1. *cofibrations are monomorphisms,*
2. *fibrant objects are exactly* K*-fibrant objects,*
3. *fibrations between fibrant objects have the right-lifting property with respect to* K*,*
4. *this is cofibrantly generated*
5. *weak equivalences are "*K*-equivalences" meaning* $X \xrightarrow{f} Y$ *is a* k*-equivalence if and only if* $\mathrm{Hom}(Y, A) \xrightarrow{f^*} (X, A)$ *is a homotopy equivalence for all fibrant* A*.*
6. *This is monoidal with respect to both* $\otimes$ *and* $\star$*.*

Let me conclude by giving you some examples. The proof, by the way, is Jeff Smith's theorem.

What's great about this result is it's pretty easy to apply if you want your own model structure. All you have to do is put J into K and then check the condition. There's the minimal one, where $K = J$, the elementary anodyne extensions, this is the model structure for weak complicial sets, fibrant objects are weak complicial sets.

We could let K be J together with something that asserts my stratification is n-trivial, throwing in the inclusions $\Delta[r] \hookrightarrow_e \Delta[r]_t$ for $r > n$, and now the fibrant objects are n-trivial weak complicial sets.

I should maybe say, well the last thing I can do is take these saturation-y maps. Take $\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^{\#} \star \Delta[n]$ for $n \geq -1$ for saturated weak complicial sets.

I can mix and match, get K-saturated and n-trivial. An instance of this, if I ask for 1-trivial saturated, these are naturally marked quasicategories. If you wanted a proof in the literature for Joyal's model structure for quasicategories, this is the one I'm aware of here.

6. Ralph Kaufmann: Feynman categories I

I do not take notes during slide talks.

7. June 8: Philip Hackney/Marcy Robertson: quasi-operads

I'd like to thank the organizers. This is part one of our three-part talk. We'll start with quasi-operads or dendroidal sets as one model for ∞-operads.

Some preliminaries that we'll use in all of our talks. We have a convention that for us a graph G consists of

(1) a directed graph
(2) connected, has half-edges, and no directed cycles.

Definition 7.1. A tree is a graph with a unique output (the root) plus

(1) A coloring function q from the edges of the tree to a set $\mathfrak{C}$
(2) Orderings, bijections $\mathrm{ord}^i_T : \{1, \ldots, n\} \to \mathrm{in}(T)$ ad $\mathrm{ord}^i_V : \{1, \ldots, k\} \to \mathrm{in}(v)$.

If I combine these, $q \in (v)$ gives a *profile* $c_1, \ldots, c_k$ for $c_i \in \mathfrak{C}$, and we'll call this $\underline{c}$.

Similarly you could write $q\,\mathrm{out}(v) = d$.

7.1. colored operads.

Definition 7.2. A *colored operad* P consists of the following data

(1) A set of colors $\mathfrak{C} = \mathrm{col}(P)$,
(2) for all $k \geq 0$ and for all $\underline{c}$ in k-profiles, and for all d, a set $P(\underline{c}, d)$, and
(3) an associative and unital composition given by $P(\underline{d}; d) \otimes P(c^1, d_1) \otimes \cdots \otimes P(\underline{c}^m, d_m) \to P(\underline{c}^1, \ldots, \underline{c}^m; d)$

We call the category of these Operad.

There are lots of examples, there are ones that encode operads and properads and maps between them and things.

We want now to talk about the free operad generated by a tree.

For any tree T, there exists a colored operad $\Omega(T)$ generated by T in the following way. The colors of $\Omega(T)$ are the edges of T. The operations are generated by vertices in the tree. [pictures]

Definition 7.3. The category Ω is the full subcategory of Operad whose objects are these $\Omega(T)$ for some tree T.

It's sort of a common, we're as guilty as anyone else, it's common to call Ω the category of trees, but it's actually a category of operads.

We'll actually write T for $\Omega(T)$ because we have too many Ω symbols.

7.2. **Coface maps and graph substitution.**

Definition 7.4. A *partially grafted corolla* P is a graph with two vertices u and v in which a nonempty finite number of outputs of u are inputs of v.

We'll use partially grafted corollas to make formal what we mean by graph substitutions.

Definition 7.5. Graph substitution $G\{H_v\}$ means plugging some H into a vertex v in G. We can only do this if:

(1) the inputs of H are in bijection with the inputs of v,
(2) the outputs of H are in bijection with the outputs of v, and
(3) the colorings have to match up.

[pictures]

So here's a little fact. Graph substitution induces maps between graphs. In particular, we'll get coface maps. Let T be a tree, if we have an internal edge of T (not the leaf or a root) with two vertices u and v, then we have a subtree H of T and a map $d^{uv} : H \to T = H\{P_{uv}\}$ where P is the appropriate partially grafted corolla.

[pictures]

Definition 7.6. Let u be a vertex of T where either

(1) all inputs are leaves, or
(2) there are no inputs of u

Then there is a similar way to write d^u.

Degeneracies I'll just write down a definition to save time.

Definition 7.7. A degeneracy map is a map $\sigma^v : H \to H\{|\}$.

Proposition 7.1. *(Moerdijk–Weiss) The category Ω is generated by the inner and outer coface maps and degeneracies and isomorphisms.*

In other words, any map in this category can be factored as all these. I won't talk too much about the isomorphisms.

7.3.

Definition 7.8. A *dendroidal set* is a functor $X : \Omega^{op} \to \mathrm{Set}$. The ones we use the most are the representable guys $\Omega[T] = \Omega(\quad, T)$

Let α be a coface map, either inner or outer, in Ω. Then the α-face of $\Omega[T]$ is the image of the induced map $\Omega[S] \to \Omega[T]$, and we'll write $\partial_\alpha[T]$.

Now these definitions sholud look familiar to people doing simplicial stuff.

Definition 7.9. The *boundary* of $\Omega[T]$ is the $\partial[T] = \cup_\alpha \partial_\alpha[T]$. If I skip a coface map I get a *horn* $\Lambda^\beta[T] = \cup_{\alpha \neq \beta} \partial_\alpha[T]$, and if β is inner we call this an *inner horn.*

Now we can define a quasi-operad.

Definition 7.10. A dendroidal set X is a *quasi-operad* or *inner Kan* if for trees T and for all inner β, we have a lift

$$\begin{tikzcd} \Lambda^\beta[T] \arrow[r] \arrow[d] & X \\ \Omega[T] \arrow[ur, dotted] & \end{tikzcd}$$

If our trees are linear, we'd get a quasicategory.

Definition 7.11. A monomorphism of dendroidal sets $X \hookrightarrow Y$ is called *normal* if for all trees T in Ω and all $y \in Y(T)$, the set determined by T and Y, such that y is not contained in the image of $X(T)$, has a trivial stabilizer in the automorphism group: $\mathrm{Aut}(T)_y \leq \mathrm{Aut}(T)$.

These are our cofibrations of dendroidal sets.

The last thing I'll say is that there is a model structure here.

Theorem 7.1. *(Cisinski–Moerdijk)*

There is a model category structure on dSet *such that The inner Kan complexes are the fibrant objects and the normal monomorphisms are the cofibrations.*

I'll stop there.

8. Martin Markl: Graph cohomology

This will be my swan song. This is work with Voronov. I'll work in chain complexes, probably in a field of characteristic zero. As an example of something that is a graph complex, let me recall something that is 25 years old, namely the resolution of operad for associative algebras. This has a presentation $\Gamma(Y)$ modulo associativity, it's very simple and easy to see what I think it is, what is such an operad, it's the span of binary trees [pictures] and modulo the relation means that I can replace every right leaning edge with a left leaning edge. It's obvious that if I mod out by the ideal, I can find a representative for every arity of the left leaning type. Since I'm forgetting the symmetric group action, I get $\mathrm{Ass}(n) = \mathbf{k}$ in any arity.

So our task will be to describe or calculate the minimal model. I wish to resolve this operad in the category of operads in chain complexes. I want the smallest possible number of generators; I'll define minimality in a moment precisely. So I'll have the same generator, Y, in degree 0, and I'll need something in degree 1, Z, so that ∂Z is associativity. Then I have to think a little bit but if I think a little bit I get a generator of arity four in degree 2, and for any arity n, I'll get something of degree $n-2$, and a general form for the differential ∂ which acts on a generator, up to signs as decomposing this corolla into a directed tree with two vertices and a total of n inputs.

It turns out that this map, ρ, which sends Y to itself and other generators to 0, is a quasi-isomorphism, induces an isomorphism on homology. There's a nice way to see it, this operad is the span of cells of Stasheff's associahedron, which is contractible. You can see this without using anything fancy, Koszul duality or whatever. By minimal I mean that the differential has no internal part, the differential is quadratic or higher.

For me this is the motivating example for a graph cohomology. Why? This is the span of all trees, not necessarily binary or anything, I don't want to have bivalent vertices, and my differential expands each vertex. This is precisely the idea of a graph complex.

You may ask why anyone may be interested in this. It turns out that if I call this differential graded operad by A_∞, then A_∞-algebras are strong homotopy versions of associative algebras. This is a general rule, if I am given an algebra over an operad or PROP. If I find a cofibrant resolution for the original operad or PROP then algebras over the resolution will be strongly homotopy versions of the original type of algebra.

So let me formulate a (kind of ideological, rather than formal) definition:

Definition 8.1. A *graph complex* is the span of (decorated) graphs, with a differential given by expanding vertices.

It walks over vertices and expands each one according to some rule. More specifically, my graph complexes will be of the form $G = (\Gamma(E), \partial)$ with a boundary of this kind, and this will be a PROP-like free thing.

So maybe I can tell you something about the history and why these things are so interesting. So graph complexes describe many interesting things like automorphisms of free groups, moduli spaces of surfaces, Grothendieck–Teichmüller group, other things, I should mention some names: [unintelligible], Kontsevich, Penner, [unintelligible], more recently Willwacher, [unintelligible], and I apologize if I forgot anyone.

So I'll also have an example with a use for these ridiculous $\frac{1}{2}$-PROPs.

So a minimal model for P is a map ρ as follows:

$$P \overset{\rho}{\leftarrow} (\Gamma_P(E), \partial)$$

with $E(m,n)$ for $m+n \geq 3$ and $m,n \geq 1$; the differential ∂ should have no linear part and ρ should be a quasi-isomorphism.

If I write the dg thing as $\mathcal{M}_P$, then algebras over it will be strongly homotopy P-algebras.

Just recalling what you heard yesterday, the free PROP on such a collection is the direct sum

$$\bigoplus E\text{-decorated directed graphs}$$

Notice that this fellow is extremely big; there is a combinatorial explosion. For instance, $\Gamma_P(E)(1,1)$ is typically infinitely dimensional. It contains graphs like this: [pictures]

It is a huge object and therefore very difficult to work with and one needs to invent some more subtle methods of how to develop minimal models.

Let me again illustrate an example, strongly homotopy bialgebras. Here I mean Hopf algebra without unit, counit, or antipode, so this is a vector space V with associative product and coassociative coproduct and the compatibility $\Delta(ab) = \Delta(a)\Delta(b)$ [pictures].

Let me denote by B the PROP describing bialgebras, which can be generated as the free prop on the product and coproduct modulo associativity, coassociativity, and this compatibility. Actually, it's a simple exercise to describe the PROP explicitly.

Now I wish to construct a minimal model for the PROP. I start as in my motivating example. I take the free PROP, with two generators in dimension 0, and then to kill the axioms I shall have generators in degree 1 to kill the relations [pictures].

You can figure out the next step by brute force or by experimenting that you should have a 1 to 4 and 4 to 1 and 3 to 2 and 2 to 3 generator in degree 2, and maybe even the next step you can do by hand, but then you are stuck because of the combinatorial explosion. So the question is what one can say about this combinatorial explosion.

The strategy is the following ingenious idea. Consider the following family of PROPs B_ϵ depending on a parameter ϵ, where you have the same generators and the final relation depending on the parameter ϵ. So B_1 is B and B_0 describes these stupid things that I described yesterday.

You won't find such an algebra in nature but it makes sense to consider a prop for it.

You may notice that B_0 describes $\frac{1}{2}$-bialgebras. There is the following strategy. The first observation is that $\frac{1}{2}$-bialgebras are defined over $\frac{1}{2}$-PROPS. b_0 is the free $\frac{1}{2}$-PROP on the same generators modulo the same relations. Then $\frac{1}{2}$-bialgebras are algebras over this $\frac{1}{2}$-PROP.

(1) So the first step is to resolve B_0 in the category of $\frac{1}{2}$-PROPs, and then
(2) generate (using this resolution of b_0) a resolution of B_0 in PROPs.
(3) Then since B_1 is a perturbation of B_0, then the differential for B_1 we can expect to be a perturbation of the differential for the minimal model of B_0.

So the first step is fairly easy because b_0 is quadratic Koszul, I won't go into the theory but I'll tell you what it looks like. So b_0 has generators ξ_n^m for any m and $n \geq 1$. The degree of such a thing is $m+n-3$ is something like this. The differential ∂_0 is given by the summation of all possible $\frac{1}{2}$-PROPic decompositions. Rather than explain what I mean, I'll give an example [picture].

If steps 2 and 3 work, you already know the generators for the minimal model for B, which is a quite nontrivial result, I should say.

So to do the second step, I replace everything with analogs in PROPs. I replace b_0 with B_0 and take the free *PROP* now instead of $\frac{1}{2}$-PROP. I should be able to establish that ρ_0 is still a quasi-isomorphism from this "resolution" to B_0 as it was to b_0. So I have the categories of PROPs and $\frac{1}{2}$-PROPs, I have the forgetful functor $\square$ and it has a left adjoint F, and then this is F applied to my $\frac{1}{2}$-PROPic resolution. So if I prove that F is exact (meaning it preserves quasi-isomorphisms), then I have made step 2 work.

Let me prove it. Proofs are boring things in mathematics so I'll try to mae it short.

Theorem 8.1. *F is exact.*

Proof. Assume b is a $\frac{1}{2}$-PROP. How do you construct $F(b)$? By general facts it is a quotient of $\Gamma_P(b)$ by the ideal generated by the $\frac{1}{2}$-PROPic compositions. This is the sum of b-decorated directed graphs, modulo contraction of $\frac{1}{2}$-PROPic edges. [pictures]

So then I get a unique decomposition of sums of tensor products (let me come back to this) and then by a Künneth formula I get exactness. $\square$

So then for the third step, the minimal model for B, we show this to be $(\Gamma(\xi_n^m), \partial_0 + \partial_P)$ and you may also obtain some explicit formulas in small dimensions at least, and the question of how to obtain the general formula is, I dare to call the black hole. I suggest it as an assignment for graduate students if you want to ruin their careers.

Now let me talk about this subtle point, why the functor has all these nice properties. I have the following diagram of forgetful functors.

$$
\begin{tikzcd}
\mathrm{PROPs} \arrow[rr, "\square"] \arrow[dr, "\square_1"] & & \tfrac{1}{2}\text{-PROPs} \arrow[ll, bend right, "F"'] \arrow[dl, bend left, "F_2"] \\
& \mathrm{Diops} \arrow[ur, "\square_2"] \arrow[ul, bend left, "F_1"] &
\end{tikzcd}
$$

So we will also try to see why the argument will fail for F_1, which I claim is not exact.

So I try to do the same thing, $F_1(d) = \Gamma_P(d)/\text{dioperadic composition}$.

[pictures].

So I want to give some idea of why something different will happen for half-props than for dioperads. [pictures]

I still have five minutes, and so I'll give one explicit formula for a differential to see what kind of formulas you would expect. $\partial(\xi_2^3) = \partial_0(\xi_2^3)+$ some correction terms which I explain [pictures].

So what else shall I tell you? This is a black hole, I don't want to go through here, but the conjecture that there is a sequence of polyhedra, the same way that you have one for Stasheff polyhedra, this is not true, but this is probably a good place to stop.

9. Victor Turchin: Embedding calculus and the little disks operads I

Thank you. I'd also like to thank the organizers, Marcy and Philip. This is the study of spaces of embeddings between manifolds. This is a very nice application of the operad theory. The main operad that appears is the little disk operad. The [unintelligible]was invented by Weiss and Goodwillie, the goal was to study embeddings. Let me explain the manifolds functor calculus.

Assume that we have a smooth manifold, we can consider the category of open subsets here, and then we can look at the functors $\mathcal{O}(M) \to \mathrm{Top}$, and look at both the covariant and the contravariant case. We want this to be isotopy invariant, so that if we have $U_1 \subset U_2$ an isotopy equivalence, so that both compositions are isotopic to the identity, then the functor should send isotopy equivalence to homotopy equivalence. The functor calculus provides a sequence of polynomial approximations. In the covariant case, we have a tower $T_0F \to T_1F \to T_2F \to \cdots$, all of which come with a map to F. The T_kF is the kth polynomial approximation. For the contravariant case all the arrows go in the opposite direction, $T_0F \leftarrow T_1F \leftarrow \cdots$

There is a version of this calculus which is so-called "context-free." Consider the category of all smooth manifolds of dimension n. The morphisms are codimension 0 embeddings. If you have a functor $\mathrm{Man}_n \to \mathrm{Top}$, then

Definition 9.1. A functor $F : \mathrm{Man}_n \to \mathrm{Top}$ is polynomial of degree k if for any manifold M and for any collection of closed and pairwise disjoint subsets $A_0, \ldots, A_k$, we get the cube; let me do this in case two to be easier:

$$\begin{tikzcd}
 & F(M\backslash A_1 \cup A_2) \arrow[r] \arrow[d, dotted] & F(M\backslash A_2) \arrow[d] \\
F(M\backslash A_0 \cup A_1 \cup A_2) \arrow[ur] \arrow[r] \arrow[d] & F(M\backslash A_0 \cup A_2) \arrow[r, dotted] \arrow[d] \arrow[ur] & F(M) \\
F(M\backslash A_0 \cup A_1) \arrow[r] \arrow[ur, dotted] & F(M\backslash A_0) \arrow[ur] &
\end{tikzcd}$$

is homotopy Cartesian.

So you can build the value of the functor on M out of its value on smaller pieces.

So say $M \mapsto M \times k$; this is polynomial of degree k. If you take the functor $M \mapsto M^{\times}2$, this is not linear. Let me show this.

$$\begin{tikzcd}
(M\backslash A_0 \cup A_1)^2 \arrow[r] \arrow[d] & (M\backslash A_0)^2 \\
(M\backslash A_1)^2 &
\end{tikzcd}$$

the colimit here will be $(M\backslash A_0)^2 \cup (M\backslash A_1)^2$, but this is not M^2. If you do this in the three dimensional cube, then $M^2 = (M\backslash A_0)^2 \cup (M\backslash A_1)^2 \cup (M\backslash A_2)^2$.

You can take other examples. You can take $M \mapsto M^{\times k}/\Sigma_k$, this is polynomial of degree k. Or $\binom{M}{k}$, the unlabelled configurations of k points, this is also polynomial of degree k. Or you could take the spherical tangent bundle of M, or $M \mapsto M \times A$, these functors are linear.

What about for contravariant functors?

Linear examples would be $M \mapsto \mathrm{Map}(M, A)$ or $M \mapsto \Gamma(p)$ where p is a functorial bundle $E_M \to M$. So the first example is a trivial example of the second.

Another example would be immersions of M in some larger dimension space N, because this is equivalent to sections of a certain fiber bundle, formal immersions, $\Gamma(p, M)$, where you assign a space $E_M \to M$ where E is the space of triples $(m, n, \alpha : T_mM \to T_nN)$ with α a monomorphism. Smale proved his famous immersion theorem, that $\mathrm{Imm}(S^2, \mathbb{R}^3)$ is connected, which follows from seeing the sphere as the union of disks and then seing that this is Cartesian.

Degree k also goes to $\mathrm{Maps}(M^{\times k}, A)$, and we could also ask for Σ_k-equivariance.

The good news is that there is a theorem by Goodwillie–Klein and Weiss, where viewing $\mathrm{Emb}(M, N) \to T_k \mathrm{Emb}(M, N)$, and this is $(1 - m + k(n - m - 2))$-connected, provided $n - m > 2$. This tower becomes closer and closer to the initial space of embeddings. We get closer and closer approximations. So now we want to find k so that this is greater than zero, you can take the fifth polynomial approximation, and then you can understand isotopy classes. To get the fundamental group, you take k higher. But it really becomes a computation similar to calculus.

We can actually describe explicitly the kth polynomial approximation.

I told you that $M \mapsto M^k$ is polynomial of degree k but what is the Taylor tower. In this case $T_iF(M) = \{x_1, \ldots x_k \in M^{\times k} | \#\{x_1, \ldots, x_k\} \leq i\}$. So this functor is not homogeneous.

Okay let's talk about the operadic interpretation. It looks nice in the context free context. Actually before this I should give a formula. Consider the category of disks as a subcategory of the manifolds. I want $\mathrm{Disc}_{\le k}$ to be the subcategory with objects disjoint unions of up to k disks.

Then

$$T_k F(M) = \operatorname*{holim}_{\mathrm{Disc}_{\le k}\downarrow M} F,$$

the homotopy right Kan extension

$$\begin{tikzcd} \mathrm{Disc}_{\le k} \arrow[r, "F\circ i"] \arrow[d, "i"'] & \mathrm{Top} \\ \mathrm{Man}_m \arrow[ur, dotted, "\mathrm{hRan}"] & \end{tikzcd}$$

So the things I'm doing is enriched, and so I can consider now the operad $\mathrm{End}(\mathcal{D}^m)$, the operad of endomorphisms of $\mathcal{D}^m$.

The kth component of the operad is the embeddings of a disjoint union of k disks into a disk. In little disks, your embeddings should be just translation and scaling. Here you also allow all transformations.

So this is equivalent to $B_m^{fr}(k)$. This space of embeddings is equivalent to that.

Theorem 9.1. *(Boavida de Brito–Weiss, T.)*

$$T_k F(M) = h\,\mathrm{Rmod}_{\le k, \mathrm{End}(D^m)}(\mathrm{Emb}(\quad, M), F(\quad))$$

So if your functor is contravariant, $\{F(\mathbf{1}), F(X), F(X^{\otimes}2), \dots\}$ becomes a right module over $\mathrm{End}(X)$ Then $\mathrm{Emb}(\quad, M)$ is still a right module. We look only up to degree k. For $k = \infty$ you get a formula similar to factorization homology.

Just as a remark, if you look at the initial definition,

$$\operatorname*{holim}_{\mathrm{Disc}_{\le k}\downarrow M} F \cong h\,\mathrm{Rmod}_{\mathrm{End}(\mathcal{D}^m)^\delta}(\mathrm{Emb}(\quad, M)^\delta, F(\quad))$$

where the δ means with the discrete topology. Then this was understanding the continuous version to get the same result. First it was Pedro and Michael who resolved it and I gave a different argument.

Another thing is that you can also consider functors from manifolds to chain complexes. In this case you also get

Theorem 9.2. *(Boavida de Brito–Weiss)* $T_k F(M) = h\,\mathrm{Rmod}_{\le k, C_* B_m^{\mathrm{fr}}}(C_*(\mathrm{Emb}, (\quad, M)), F(\quad))$

Now an interesting space of embeddings is the space of embeddings $\mathrm{Emb}(S^m, S^n)$, and assuming $n - m \ge 2$ this has the same π_0 as $\mathrm{Emb}_\partial(\mathcal{D}^m, \mathcal{D}^n)$. So it would be interesting to study this space of embeddings of disks, so it would be interesting to understand the calculus of the closed disk.

So let me give the idea, we should change the category Disc to $\widetilde{\mathrm{Disc}}$. The objects are disjoint unions of the disc or $S^{m-1} \times [0, 1)$. Now we showed that the Taylor tower can be expressed—and let me mention if M is parallelizable, we can reduce this to framed disks, we can reduce to functors which respect the framing, and reduce to the case with unframed disks.

Theorem 9.3. *(Arone–T. (2011))*

$$T_k F(\mathcal{D}^m) \cong h\,\mathrm{Inf}_{B_m}\,\mathrm{Bim}_{\le k}(B_m, F(\quad)).$$

So what are infinitesimal bimodules over an operad? We have so-called infinitesimal left action. The structure is Abelian, you can only insert in one input. The right action is just usual. Since the right action is also unital, we can insert only in one of the inputs. Infinitesimal right action is just like the usual one but the left action is different. We need a compatibility [picutures].

Now the functor F on the category of disks, this left action comes from the boundary conditions. Now the embeddings to a disk is equivalent to the operad being itself, an infinitesimal bimodule over itself.

As a corollary, we get the following.

Corollary 9.1. *(Arone–T.)* $T_k \operatorname{Emb}_\partial(\mathcal{D}^m, \mathcal{D}^n) \cong h \operatorname{Inf}_{B_m} \operatorname{Bim}_{\leq k}(B_m, B_n)$.

The only thing I'm lying about, I should say $\overline{\operatorname{Emb}}_\partial$, where this is a homotopy fiber of $\operatorname{Emb}(\mathcal{D}^m, \mathcal{D}^n) \to \operatorname{Imm}(\mathcal{D}^m, \mathcal{D}^n) \cong \Omega^m V_m(\mathbb{R}^n)$. So this is a linear functor. So we have a little correction to the space.

Now the question is, what about homotopy maps of operads between B_m and B_n. I prefer the truncated case, where you have no more than k inputs. You can study this algebraic structure. If we compare this to the space of embeddings, you get

Theorem 9.4. *(Dwyer–Hess, Boavida de Brito–Weiss, Ducoulombier–T.)*

$$T_k \overline{\operatorname{Emb}_\partial}(\mathcal{D}^m, \mathcal{D}^n) \cong \Omega^{m+1} h \operatorname{Oper}_{\leq k}(B_m, B_n)$$

This is the purpose of my second talk, this theorem.

So Dwyer–Hess did something for $m = 1$, they don't like to work with truncations, Pedro and Michael they understand the truncation case, and we also do the truncation case, but the approaches are very different. They don't use the theorem, but we (and Dwyer–Hess) use the theorem. This really becomes a theory of operads, not calculus.

I'll spend ten or fifteen minutes talking about their approach, although I haven't understood it very well. I'm planning to go over time and spend thirty more minutes. But maybe let me take a little break.

The rational homotopy groups can be computed for the spaces I was discussing and in fact they will be graph complexes. The main reason that things work nicely is relative formality of the little disks operad.

I should have explained, you can always consider a configuration of m-dimensional disks as n-dimensional disks.

Theorem 9.5. *(Tamarkin, Kontsevich, Lambrecht–[unintelligible], T–Willwacher, Fresse–Willwacher) The map of operads $C_* B_m \to C_* B_n$ of singular chains is rationally formal if and only if $n - m \neq 1$.*

So what does the statement mean? The claim is that you can find a zigzag of maps of operads to the induced map $H_* B_m \to H_* B_n$. An equivalence is a map which in every degree induces an isomorphism on homology. So then the two maps of operads are equivalent. What is the homology of the little disks operad? This is a theorem of Fred Cohen, it's either Ass when $m = 1$ or it's the Poisson operad (with bracket of degree $n - 1$) for $m \geq 2$.

What is $B_m(2)$? It's a configuration of two disks. The degree 0 class gives the product and the degree $m - 1$ class gives the bracket, which disappears when you map to S^{n-1}. This is actually crucial for the computation.

It's easy to compute. This is a crucial theorem that, well, you can easily compute, assume $n \geq 2m+3$. Then $H_*(\overline{\mathrm{Emb}}(M,\mathbb{R}^n),\mathbb{Q}) \cong HH^M(H_*B_n)$. Here the higher Hochschild homology, the homology of B_n is a right module for the commutative operad, which is just H_0B_n. So what we get is that $C_*\overline{\mathrm{Emb}}(M,\mathbb{R}^n) \cong h\,\mathrm{Rmod}_{C_*B_m}(C_*(\mathrm{Emb}(\quad,M)),C_*B_n)$, this is almost what we had before, but I can see this as a right module on the right with the [unintelligible], and this is equivalent then to $h\,\mathrm{Rmod}_{C_*B_m}(C_*(Emb(\quad,M)),H_*B_n)$, and now the map factors through the commutative operad. So this is obtained by restriction, and we can use the Quillen adjunction which interchanges restriction and induction. So this becomes $h\,\mathrm{Rmod}_{\mathrm{Com}}(C_*(M^\bullet),H_*B_n)$, and a right module over the commutative operad is the same as natural transformations over finite sets $\mathrm{Nat}_{\mathrm{Fin}}(C_*M^\bullet,H_*B_n)$.

There's a similar theorem that takes place for the rational homology

Theorem 9.6. *(Arone, T.) Let $n \geq 2m+2$. Then*

$$H_*(\overline{\mathrm{Emb}}_\partial(\mathcal{D}^m,\mathcal{D}^n),\mathbb{Q}) = HH^{S^m}(H_*B_n)$$

(this is the pointed version, replacing Fin *with* Γ*) and*

$$\pi_*(\overline{\mathrm{Emb}}_\partial(\mathcal{D}^m,\mathcal{D}^n),\mathbb{Q}) = HH^{S^m}(\pi_*B_n\otimes\mathbb{Q})$$

So this is related to the modular completion of the L_∞ operad, and you should take coinvariants or anticoinvariants, depending on the dimension of M, probably with some sign. There are four different graph complexes, and you take coinvariants or anticoinvariants depending on the dimension and codimension. If you take the suspension of an operad it becomes anticyclic, et cetera. These four graph complexes describe this stuff. If you study, there is a similar— [some discussion]

This discussion in terms of graph complexes, this is recent work of Fresse–T.–Willwacher, but this works for $n-m>2$, the whole range in which you expect it to work.

In particular, there is a graph that looks like this [picture] which corresponds to the Haefliger trefoil.

Let me just give you the picture of this trefoil [pictures].

So it's known that $\pi_0(\mathrm{Emb}(S^m,S^n))$ is an Abelian group for $n-m>2$ of rank at most one. This is a generator which is not torsion.

Let me explain in a few words what we did with Thomas and Benoit. The crucial point is that the map is formal as a map of Hopf operads (operads in coalgebras). We used this strong formality first. We showed the theorem

Theorem 9.7. *(Fresse–T.–Willwacher) For $n-m\geq 2$*

$$h\,\mathrm{Oper}_{\leq k}(B_m,B_n)$$

is $n-m-1$-connected and its rational homotopy type is described by the L_∞ algebra of homotopy biderivations of the map $H_*(B_m)\to H_*(B_n)$, truncated to $\leq k$.

Essentially all the rational information is encoded by this homology map. These are maps of truncated Hopf operads, so you need cofibrant replacements for these guys (cofibrant in chain complexes); in the domain you want componentwise a fibrant coalgebra. Then you look at maps which are derivations of both structures, levelwise for the coalgebra structure. At the limit when $k\to\infty$, but then you need codimension three. The maps between stages in the tower don't become higher and higher and the projective limit of groups doesn't commute with tensoring with rational numbers.

$h\,\mathrm{Oper}(B_m, B_n)$ for $n - m > 2$ is encoded by biderivations of $H_*B_m \to H_*B_n$. When the codimension is 2, things don't become higher and higher connected and tensor product doesn't commute. Maybe it's enough. If you have questions, we can discuss it.

10. June 9: Ralph Kaufmann: Feynman categories II

I do not take notes at slide talks.

11. Philip Hackney/Marcy Robertson: ∞-properads II

Thank you. Last time Marcy gave a quick introduction to the dendroidal category, dendroidal sets, which is some sort of model for higher operads. Today I want to do a similar kind of thing for properads. A reminder of background/notation: if I give you a set of colors, $\mathfrak{C}$, then I want to talk about not profiles in this but *biprofiles*, these are going to be pairs of lists $(\underline{c}, \underline{d}) = (c_1, \ldots, c_m; d_1, \ldots d_n)$; I'm also going to be considering $\mathfrak{C}$-colored graphs and as in Marcy's talk, $\mathfrak{C}$-colored graphs will be colored directed graphs with inputs and outputs, with specified orderings at the inputs and outputs of vertices and also of the graph itself. We'll again write ξ for the coloring function: $\mathrm{Edges}(G) \to \mathfrak{C}$. [picture]

If I give you a biprofile, I'll write $\mathrm{Graph}(\underline{c}, \underline{d})$ which will consist of all $\mathfrak{C}$-colored graphs with $\xi\,\mathrm{in}(G) = \underline{c}$ and $\xi\,\mathrm{out}(G) = \underline{d}$.

Marcy also talked about graph substitution. If I have a graph $H \in \mathrm{Graph}(\underline{c}; \underline{d})$, and if I have a collection $K_v \in \mathrm{Graph}(\xi(\mathrm{in}(v)); \xi(\mathrm{out}(v)))$ for v in the vertices of H [pictures], then I can get a new graph, we've seen plenty of pictures already, $H\{K_v\}_{v \in \mathrm{Vt}(H)} \in \mathrm{Graph}(\underline{c}; \underline{d})$.

This is the kind of setup that lets us do $\mathfrak{C}$-colored properads. This is a pasting scheme in the setting that Martin talked about. $\mathfrak{C}$-colored properads are the thing you get out of this pasting scheme. If you want, a $\mathfrak{C}$-colored properad consists of some spaces $P(\underline{c}; \underline{d})$ (or sets) for each biprofile, these are the operations for this guy. At each vertex I decorate with the appropriate thing [pictures] and this should give me a function that goes from

$$\gamma_P^G : P[G] = \bigotimes_{\mathrm{Vt}(G)} P(\xi\,\mathrm{in}(v); \xi\,\mathrm{out}(v)) \to P(\xi\,\mathrm{in}\,G; \xi\,\mathrm{out}\,G).$$

You should be doing associativity, identity, and so on for these guys.

Maybe more important for us is, what is a map of these things? This came up after Marcy's talk yesterday. There was some confusion; maps of colored properads, a map $f : P \to Q$, this consists of

- $f_0 : \mathrm{Col}(P) \to \mathrm{Col}(Q)$; this doesn't have to be the identity.
- $f_1 : P(\underline{c}; \underline{d}) \to Q(f_0\underline{c}; f_0\underline{d})$.

So this is what we mean by map. One thing to notice is whenever we have a graph like this, I can do the operations one at a time, like first contract one pair of vertices, I can always contract two vertices at a time. So this is determined by γ_P^G on partially grafted corollas and the graph with just an edge (for identities) and the one vertex graphs (for symmetric group actions).

So this gives us a nice category of properads. Then we can get $\Gamma(G)$, the free properad on an uncolored graph. We take as our colors the edges of the graph. The morphism spaces, the operations, will be generated by the vertices.

What are all the operations? The operations are $\hat{G}$-decorated graphs. This means that something in $\Gamma(G)(\underline{c};\underline{d})$, the first piece of data is a graph H in Graph$(\underline{c};\underline{d})$ so the edges are colored by edges of G, and a function from the vertices of H to the vertices of G which is compatible with the coloring of H.

[pictures]

If I look at this, I can build it iteratively using partially grafted corollas [pictures]

Now given this collection of properads, one for every graph, what do maps look like?

The maps $f : \Gamma(G) \to \Gamma(H)$, the color sets are just the edges, so I have a function f_0 from the edges of G to the edges of H and then I have a map f_1, I can specify it just on generators, from the vertices of G to $\{\mathrm{Vt}(H)\text{-decorated graphs}\}$. These should satisfy some conditions. So $f_1(v) \in \Gamma(H)(f_0 \operatorname{in} v; f_0 \operatorname{out} v)$. That's the H-decorated graph I want to get out of this.

Now we know the maps, let's write down, suppose that we have $f : \Gamma(G) \to \Gamma(K)$, then I can define the image of f as follows. I take G, and then f_0G is an Edges(K)-colored graph, and so I take $[f_0G]\{f_1(u)\}_{u\in\mathrm{Vt}(G)}$ in Graph$(f_0 \operatorname{in} G; f_0 \operatorname{out} G)$, and is $\mathrm{Vt}(K)$-decorated.

A detailed example is coming, but before that, I'd like to say, when you have a $\mathrm{Vt}(K)$-decorated graph, whether it's a subgraph of K or not. This is taking a maximal composition of G stuff, mapping it over, and maybe it's a subgraph, maybe not. Let's look at an example related to this.

[picture]

So here's a great example of what an image of a map looks like. Here I'm just gluing these two things.

Here's a fact. If the image of f is a subgraph of K, then f is uniquely determined by what it does on edges. This particularly holds if the target K is a simply connected graph. WE can see that this is not the case in the example I did.

It turns out that it's hard to do things if you don't have a property like this, so from now on we'll assume it.

So the *graphical category* Γ has objects uncolored graphs and morphisms $\Gamma(G,K)$ is some subset of the properad maps $\Gamma(G) \to \Gamma(K)$, consisting of those f such that $\mathrm{im} f$ is a subgraph of K. So just as a note, in this example that we did, it was enough to check that images of one of the vertices was a subgraph, but it's not so easy in general.

[pictures]

You have degeneracy maps from H to $H\{|\}$, where I remove a $(1,1)$-vertex and replace it by a straight edge. The other ones are face maps, which are a lot like what Marcy was talking about, except we have to be a little bit careful if we're going to talk about, inner cofaces I'll draw a schematic picture [picture]. If I have something that looks like a partially grafted corolla with nothing between them, then I can map in. So if you like, with the vertex G, I insert a partially grafted corolla at v. From the domain there's no problem; for the codomain you have to think harder. Again, for outer coface maps, it's easier to start with the domain and graft a corolla on an exterior edge.

Let's just wrap up, I'm going to talk next time about how these maps generate the whole category Γ, some decomposition properties and such, but now we have Γ and some feel for what it's like.

So *graphical sets* are presheaves on Γ, that is $\mathrm{Set}^{\Gamma^{\mathrm{op}}}$, and now one can do all the usual things, looking at representables $\Gamma[H] = \Gamma(\quad, H)$, faces of representables $\partial_\alpha[H] \subset \Gamma[H]$ where α is a coface, and as soon as you start to do horns you can define $\Lambda^\alpha[H] \subset \Gamma[H]$ where $\Lambda^\alpha[H] = \cup_{\beta\neq\alpha}\partial_\beta[H]$.

Then *quasiproperads* have the lifting condition

$$\begin{tikzcd} \Lambda^\alpha[H] \arrow[r] \arrow[d] & X \\ \Gamma[H] \arrow[ur, dotted] & \end{tikzcd}$$

where α is an inner coface. For a properad then I get a nerve thing in gSet, that is, $P \mapsto \mathrm{Properad}(H, P)$, and this is fully faithful and lands in quasiproperads (that's not immediate because I've taken this subcategory).

12. Clark Barwick: Parametrised higher category theory and parametrised higher algebra I

Different parts of this are joint with Dotto, Glasman, Nardin, Shah, and Schlank.

Thanks for the invitation to come speak. I'm going to take a minute here and write down a list of papers. I have to confess that after a couple of late night conversations, I decided to do something completely different, and I thought maybe it would be more productive if I said something about what higher categories are for and what you do with them in practice.

I just want to emphasize that everything is joint work with these guys, the bourbon seminar.

B	Spectral Mackey functors I
BGN	Dualizing (co)cartesian fibrations
BGS	Spectral Mackey functors II
BG1	Cyclonic and cyclotomic spectra
BG2	The noncommutative syntomic realization.
BDGNS	Parameterized higher category theory and parameterized higher algebra
G	Calculus = equivariance (this is not the right title)
DS	G-equivariant calculus= even more equivariance.

It'll be a better use of our time if I start with a general discussion of higher category theory. I probably can't reproduce every proof at the board, but I can probably reproduce every definition at the board. Please speak up if I say something that's imperfectly understood.

Let me start with an example, it shows a lot about what the story is about parameterized higher category theory.

Let $F \subset E$ be a finite Galois extension with group G, obviously a finite group.

You can build a category out of this, I'll call it $\mathrm{Subext}(E/F)$, this is an ordinary category whose objects are separable subextensions $F \subset K \subset E$ and whose morphisms are field homomorphisms over F but not under E.

There's a theorem that you probably know, it's the main theorem of Galois theory, which is the following, this category here has a much simpler identification that you already know, this is equivalent as a category to the opposite of the orbit category of G. What's the equivalence? Morally, what extension do I get for $[G/H]$? I get E^H. I want to see conjugation actions on my fields.

I could do something fancier, look at finite separable extensions of F under E, and again I have this category where I don't respect the embedding, but any of these happens to be a product of these field extensions in a unique fashion. Here you're also allowing, for your orbits, to take disjoint unions of orbits, and so this is equivalent to the opposite of finite G-sets.

To any of these guys, there's a way to associate a category. If I have a subextension, I can create a functor

$$\mathrm{Subext}(E/F) \to \mathrm{Cat}$$

where I send K to finite dimensional K-vector spaces, $\mathrm{Vect}^{\mathrm{fd}}(K)$.

What do I do to maps? If I take $\varphi : K \to K'$ over F, then I take the tensor product $\otimes_K K'$, it's a functor from O_G^{op} (opposite of the order category) to Cat.

But there's an easy objection. The corresponding diagram commutes but only up to a natural isomorphism

$$\begin{tikzcd} \mathrm{Vect}^{\mathrm{fd}}(K) \arrow[d] \arrow[dr] & \\ \mathrm{Vect}^{\mathrm{fd}}(K') \arrow[r] & \mathrm{Vect}^{\mathrm{fd}}(K'') \end{tikzcd}$$

You could fix this with a rectification, but I don't want to take that point of view. Let me instead take the following category, where the objects are pairs (K, V) where $F \subset K \subset E$ and V is a finite dimensional vector space over K. The maps will be $(K, V) \to (K', V')$ will be nothing more than a homomorphism of subextensions, a map $\varphi : K \to K'$ over F, but also the information of a map $V \to V'$ as K-vector spaces. I guess K-linear map is the preferred term.

What's the use? Here I had to wave my hands about this not being a functor. The standard way to fix that problem is to look instead at some big category built with overcategories of the source. Instead here I've constructed something $\mathrm{Vect}(E/F)$, which has pairs: a subextension and a vector space over that subextension. I've obviously got a forgetful functor down to $\mathrm{Subext}(E/F)$, which is another name for the orbit category O_G^{op} and now this is a Grothendieck opfibration. I really am getting something of exactly the size that I want to contemplate. The fiber over K is the K-vector spaces, but I haven't had to cheat, I haven't passed outside of categories and functors.

Do people like Grothendieck opfibrations? Do you swing with that?

This is a nice example of a parameterized ∞-category (or a parameterized 1-category, in this case), a G-category (I'll explain this in a bit).

As you know, the tensor product is a left adjoint, left adjoint to the forgetful functor. Because I'm talking about finite separable extensions is that $\otimes_K K'$ has both adjoints. A strange fact of Galois theory is that there is an equivalence between the left and right adjoint.

You've seen something like this before. You can look at this larger category over products of fields, $\mathrm{Proj}^{\mathrm{fg}}(\prod K_i)$, and again I have a tensor product functor and can tensor up from $\mathrm{Vect}\, F$, and again I have a left and right adjoint which are equivalent.

The $\mathrm{Proj}^{\mathrm{fg}}(\prod K_i)$ is $\coprod \mathrm{Vect}^{\mathrm{fd}}(K_i)$, and these are $\oplus$, both product and coproduct. We like this.

Here's the first big idea of prameterized higher category theory, that you want to think of these adjoints $\mathrm{Vect}^{\mathrm{fd}}(K') \to \mathrm{Vect}^{\mathrm{fd}}(K)$ as a kind of generalized direct sum, an "indexed direct sum."

What do I mean, "indexed direct sum?" Let's think about this in the particular case, think of this as going $K \to K'$, that this is $E^H \to E^{H'}$, and let's say $H' < H$ to give ourselves an idea of what's going on here. Why do I call this the indexed direct sum? I want to think of this as $V \mapsto \bigoplus H/H'V$, this is an H-orbit of V. When I say indexed, I mean indexed by these orbits.

Why is this a reasonable thing to do? There's a lovely theorem in Galois theory, the normal basis theorem. Let me say it in the case of the full extension from F to E. So E admits an F-basis of the form $\{g\theta\}$ for g in your Galois group for some θ in E. So there is an element so if you take this element's conjugates under the Galois group action, you get a basis for E as a vector space over F. Now if V is a vector space over E with basis $\{v_1, \ldots, v_n\}$, then it has an F-basis of the form $\{g\theta v_i\}$ over all g and all $1 \le i \le n$, and now you see why I want to regard V over F, as, well,

$$\bigoplus G/eV = V_F$$

This is a formal generalization of the thing you know, which is that the direct sum is the product and coproduct in vector spaces.

The motivating question for all of parameterized higher category theory is "how do you make that precise?" There's something funny because saying V is a direct sum of copies of V seems like an unlikely assertion, but in what sense is this true? This example (this is about just 1-categories), this kind of example motivated us to think about parameterized ∞-category theory.

So now I need to give some idea of what G-direct sums can be, what a G-product or G-coproduct should be where it's indexed not by a set but by an orbit, in a way that makes precise what happens in category theory. I also want exotic examples from homotopy theory. You could think about chain complexes over these different Ks, and think of the homotopy theory of chain complexes, and try to think of how to write down this functor, with a derived tensor product. The version of the Grothendieck opfibration, though, is trivial to write down once you have the machinery in place which is why I like that point of view.

This leads me to ∞-categories. I was looking last night at various introductions to higher categories. I realized that there's a grave sin committed by almost all introductions, which is to say that most examples come from model categories. This is not the way people think about them and this is not true. I'm going to give, you can recognize homotopy universal properties of the world's favorite homotopy theories.

When people say that the way you construct the ∞-category of spaces as taking the model category and constructing an ∞-category out of this, they're not lying to you, but this is not the best way, there's a universal property. The idea (and this at least goes back to Dugger, I apologize if I'm snubbing anyone), is that you can write down generators and relations for a homotopy theory.

For now think of relative categories, categories equipped with some form of weak equivalence. Then you can write down what you mean by a homotopy limit or colimit. You can think that way and I'll make things precise in a moment.

Let me give some examples.

Example 12.1.

- *Top* is freely generated under homotopy colimits by $*$. What does that mean? That means, first of all, this is already familiar to you, because any simplicial set can be written as the colimit of its simplices. If you left-Kan extend [unintelligible]along itself you get [unintelligible], that's the fancy way.

  So everything is a complicated homotopy colimit of points. You know this, if you take the pushout of two points into a point and into a point, you get the circle.

  So homotopy colimit preserving functors from Top into D is the same as D.

  Then this is like the free thing on one generator. How do I get a bigger thing?
- Like $\mathcal{P}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Top})$, and I'll say this is the free thing generated under homotopy colimits by C. I have a Yoneda embedding, super not obvious, from C into $\mathcal{P}(C)$, and if you restrict along that Yoneda embedding, j^*, then I get

$$\mathrm{Fun}^L(\mathcal{P}(C), D) \xrightarrow{j^*} \mathrm{Fun}(C, D)$$

  and I'm going to tell you that's an equivalence.

  Those were generators, what about relations. That's governed by left Bousfield localization. Let's see how that works for this kind example.
- If S is a set (small set) of maps I want to be weak equivalences, then I can do that. So $L_S\mathcal{P}(C)$ is the full subcategory spanned by the X such that for $B \to A$ in S, I have $\mathrm{Map}(A, X) \to \mathrm{Map}(B, X)$ are equivalences. I really want you to think of this as generated by C with relations given by S. Why do I know that you get to think that way. This functor here has a left adjoint, and if you restrict along it, you have the fact that any morphism in S will be taken to an equivalence in D. When you write an Abelian group, the relations you write down give a universal property. In order to specify a map, you tell me what to do with the generators, and the relations should be taken to equivalences in the target. And the same thing is happening here.

Let's do some examples of this. My favorite examples come from loop space theory. Let $C = \Delta$ and let S be Segal maps that also have a grouplike condition. Now what happens is that $L_S\mathcal{P}(C)$ is equivalent to 1-fold loop spaces. What is this doing? It's giving generators and relations for the homotopy theory of loop spaces. It goes further, if you take $C = \Theta_n$ and S is still the Segal maps and the grouplike guys, then $L_S\mathcal{P}(C)$ is equivalent to n-fold loop spaces.

Then I could do $C = \Gamma$ and the same S, and then $L_S\mathcal{P}(C)$ is infinite loop spaces.

This is supposed to be motivation for you to consider writing down homotopy theories with generators and relations, and it's extremely convenient to do this with generators and relations. I haven't been honest about my model, and I'll come to that, but let me give you a different model that I won't use, so complete Segal spaces, $C = \Delta$ and the "Segal maps" take $X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \times \cdots \times_{X_0} X_1$ for $n \geq 2$ along with a "completeness" condition like the grouplike condition that I won't get into because I don't have time.

Then this is again, complete Segal spaces are $L_S\mathcal{P}(C)$

Now what baout $C = O_G$ with $S = \varnothing$. This is sometimes called Elmendorf but I think McClure should get credit as well, which is that $\mathcal{P}(O_G)$ is equivalent to the homotopy theory, the underlying category is that of G-CW complexes, these are spaces, I'll say CW complexes, equipped with an action of G. The weak equivalences, a map from X to Y is a weak equivalence if $X^H \to Y^H$ is a weak equivalence of spaces. This homotopy theory is freely generated by the orbit category.

This is a certain amount of motivation for the following definition.

If D is any homotopy theory which I'm really thinking of as an ∞-category, then the homotopy theory of G-objects in D is $\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, D)$. This is a definition. This is interesting, it tells you there's different kinds of G-objects. There are naive ones, where I just look at maps from BG, and there's this more refined thing that has this as the full subcategory spanned just by the orbit G/e. This gives "genuine fixed points" for any $H \leq G$.

Warning: if you think about finite G-sets, and plug finite sets in for D, you won't get finite G-sets out. That's a little weird, these aren't G-sets. When I think about actual fixed points about finite G-sets, there are no homotopy questions I can ask about that. If I think about a space with a G action, and a homotopy equivalent space, and look at their H-fixed points, I might not get a homotopy equivalence. This is enforcing that I get a homotopy equivalence. I can get F_G, that's the localization at the injections. That's some extra condition in this category.

Okay, there are two things you'd want to do with ∞ categories. You'd like to write down universal properties. For this you don't need to prefer a model, all of them let you do this in an essentially unique (orientation-preserving) way. But you want to actually be able to write things down, and define objects, this is something that I think is not written in the papers I found on the internet last night. You can actually write down a quasicategory by hand. This is not working with something defined up to something defined up to something defined up to something, you're actually working with an object.

Let me give you an example that you've probably not contemplated before. Marcy, I apologize, this will be a repetition. Here's one of my favorite quasicategories.

I'll define a simplicial set $A^{\mathrm{eff}}(F_G)$, the *effective Burnside category* of finite G-sets. I'll now tell you that the n-simplices are the set of diagrams

$$\begin{tikzcd}
 & & & x_{0,n} & & & \\
 & & & \vdots & \ddots & & \\
 & & x_{02} \arrow[dl] \arrow[dr] & \cdots & x_{n-2,n} \arrow[dr] & & \\
 & x_{01} \arrow[dl] \arrow[dr] & & x_{12} \arrow[dl] \arrow[dr] & \cdots & x_{n-1,n} \arrow[dr] & \\
x_{00} & & x_{11} & & x_{22} & \cdots & x_{nn}
\end{tikzcd}$$

and all of the squares are pullbacks.

This is a quasicategory, it's not a category, doesn't come from a model category or relative category. This is equivalent to a 2-category. Let's see why that is, partially at least, let's see how to fill a horn.

So there are not many functors $\Delta \to \Delta$. I'll look at $\mathrm{id} * \mathrm{op}$, this goes from n to $2n+1$. The first n is in the same direction, the second one is in the other direction.

Then I can take a simplicial set and pull back along this, this takes a simplicial set X and gives a new one $\widetilde{\mathcal{O}}(X)_n^{\mathrm{op}} = X_{2n+1}$, this is the twisted arrow category of X, so this is a subset of the simplicial set whose n-simplices are $\mathrm{Hom}(\widetilde{\mathcal{O}}(X)^{\mathrm{op}}, F_G)$. This is a simplicial subset, and that's precisely the one hich requires these squares to be pullbacks.

Let's fill a horn. Take $\Lambda_1^2 \to A^{\mathrm{eff}}(F_G)$, and we want to fill a horn to Δ^2.

$$\begin{tikzcd} & x_{01} \arrow[dl] \arrow[dr] & & x_{12} \arrow[dl] \arrow[dr] & \\ x_{00} & & x_{11} & & x_{22} \end{tikzcd}$$

and then you can just pull back and get $x_{01} \times_{x_{11}} x_{12}$ for x_{02}, so you've fileld the horn.

This is deceptive, it's hard even to fill a 3-horn. You end up doing an induction on walks going down, and there's a bunch of special cases.

Let me tell you another thing, one of my favorite movies is this movie Real Genius. You go into a math lecture and you see that people have recording devices on their desks, and in each iteration there are more and more recording devices on the desks. In the final iteration you see that the teacher is gone and he just has a tape player going on his desk.

Okay this is going to be the main reason that quasicategories work so well. This is the ability to write down a functor to Top without writing down a functor to Top. I gave you a universal property, having to map out of Top. How do I map *into* Top. I want to be able to write down $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to Top for an ∞-category. I have to tell you one more thing, you never want to write functors to Top, you want to write Grothiendieck opfibrations. Writing down the data of a functor to Top is a terrible task. For every commutative triangle I need a homotopy, then I need homotopies of homotopies, and this is an infinite amount of data. But we have the following lovely theorem.

Theorem 12.1. *(Joyal)*

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Top}) \cong \{\textit{left fibrations } X \to \mathcal{C}\}$$

So you know inner fibrations, they have horn filling for inner horns. These are maps of simplicial sets such that for $n \geq 1$ and for any $0 \leq k < n$, the horn inclusion has a filler:

$$\begin{tikzcd} \Lambda_k^n \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathcal{C} \end{tikzcd}$$

The first thing to say is that the fiber of a left fibration is a Kan complex. It's obvious that it satisfies the left lifting condition, and it's a consequence that it satisfies the right one. The things over any point are really spaces.

The other interesting point is $k = 0$, look at Λ_0^1 inside Δ^1, this is $\{0\}$ including into $0 \longrightarrow 1$, so this says that for a morphism in $\mathcal{C}$ with a lift of the source, you automatically get a lift of the target and the map. So call the map $a \to b$, and then you have X_a the fiber over a and X_b the fiber over b, and it's super non-obvious, but you can rectify and get a functor. You have to use a model for Top to show this.

You construct this as a Quillen equivalence of model categories and then see that the underlying [unintelligible]are equivalent. This is like using a ladder to get to the roof and kicking the ladder down. You never need to fret about mapping to Top again. This is not the way it's done in Higher Topos Theory, but this is the good way.

Check it out, I've got this twisted arrow category of X, opposite, here's the excuse for thinking of it as a twisted arrow category, the nerve of a category, you get a quasicategory, and I can do this construction to it, so I get the $2n+1$-simplices of the nerve of $\mathcal{C}$. This is *isomorphic* to the nerve of the twisted arrow category. The objects are arrows, and the maps from f to g is a factorization of g through f; this is a 1-simplex:

$$\begin{tikzcd} \cdot \arrow[r, "f"] & \cdot \arrow[d] \\ \cdot \arrow[u] \arrow[r, "g"] & \cdot \end{tikzcd}$$

and if you put 0, 1, 2, 3 in, you see the 3-simplex.

Okay so for $\widetilde{\mathcal{O}}(X) \to X^{\mathrm{op}} \times X$, which says look at the source and target. The fun theorem is that $\mathrm{Fun}(\widetilde{\mathcal{O}}(X) \to X^{\mathrm{op}} \times X)$ is a left fibration. That means that this thing corresponds to a functor from $X^{\mathrm{op}} \times X$ to Top. It classifies the functor which to two points gives the fiber over those two points, if I look at the fiber over (x, y), I am looking at, the zero simplices are 1-simplices with source x and target y. The one-simplices are 3-simplices that bunch this up. It's harder to explain this, but it's true, the 1-simplices are homotopies, this a degenerate 3-simplex where you've contracted 01 and 23. The trend continues. In other words, $\mathrm{Map}_X(x, y) \coloneqq \widetilde{\mathcal{O}}(X)_{x,y}$. The 0-morphisms are maps, the 1-morphisms are homotopies, and so on.

If X isn't a quasicategory, then $\widetilde{\mathcal{O}}(X)$ isn't a quasicategory either, but this is always a left fibration even when X sucks.

The vertical point of view on $X \to$ Top gave you this way of doing this without writing down the infinite amounts of data. You have explicit control. If I wanted to use Joyal's theorem, this came from a Quillen equivalence of simplicial model categories. [missed]. When you extract the functor from Joyal's method, you have no control over the n-simplices, they're huge. Here now they're totally under control.

Mark brought up the fact that there are more general kinds of fibrations (by the way there is the notion of right fibration, which I'll give you five minutes to write down the definition of) and I'll tell you there's something like this when you replace Cat with Cat_∞, the category of ∞-categories. This has a nice universal property, but you can use the generators and relations I gave for complete Segal spaces. Then

Theorem 12.2. *(Joyal)*

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty) \cong \{\textit{coCartesian fibrations } X \to \mathcal{C}\}$$

This relaxes left fibrations. I had left and inner horn fibrations. Here the 0th horn only needs to be filled for a coCartesian edge.

Definition 12.1. So suppose $p : X \to \mathcal{C}$ is an inner fibration. An edge $\Delta^1 \xrightarrow{f} X$ is *coCartesian* if I have a lift whenever I have a diagram like this:

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] & \mathcal{C} \end{tikzcd}$$

Now an inner fibration $p : X \to \mathcal{C}$ is a coCartesian fibration if for any edge $\eta \in C_1$, I'll think of $\eta : a \to b$ and for any lift of a to x in X, there exsits a coCartesian edge $x \to y$ covering η.

This is saying no matter where you roam in your base, if I have a map out of that simplex to somewhere else, then I can take the fiber over the first vertex to the fiber over the second. The fibers all have to be quasicategories, and I can go from one fiber to the next, which is the idea behind the theorem.

The proof, again, is a pain. But my point is to *use* these theorems, not *prove* them.

There's a dual story for Cartesian fibrations and edges, for contravariant functors to Cat.

This is a theorem from Higher Topos Theory, actually Corollary 3.2.2.13, but the correct proof is in Appendix B.4 of Higher Algebra.

So okay, suppose X is covariant and Y is contravariant as functors from $\mathcal{C} \to \mathrm{Cat}_\infty$. Then I want a construction (twisted fun) $\widetilde{\mathrm{Fun}}(X,Y)(\mathcal{C}) = \mathrm{Fun}(X(\mathcal{C}), Y(\mathcal{C}))$. So I can use covariant funtoriality in the first variable and contravariant in the second variable to get from $c \to d$ to $\mathrm{Fun}(X(d), Y(d)) \to \mathrm{Fun}(X(c), Y(c))$. No one wants to write down functors to Cat. So let's write down a fibration instead. We have a coCartesian fibration $X \to \mathcal{C}$ and a Cartesian fibration $Y \to \mathcal{C}$. I want a Cartesian fibration to $\mathcal{C}$ to represent $\widetilde{\mathrm{Fun}}$.

Definition 12.2. Given $X \to \mathcal{C}$ a coCartesian fibration and $Y \to \mathcal{C}$ a Cartesian fibration, define a Cartesian fibration $\widetilde{\mathrm{Fun}}(X,Y) \xrightarrow{p} \mathcal{C}$ as follows. For any $K \xrightarrow{\pi} \mathcal{C}$, I'll tell you how to take $\mathrm{Mor}_{\mathcal{C}}(X, \widetilde{\mathrm{Fun}}(X,Y))$, and this is in bijection with the set of maps $\mathrm{Mor}_{\mathcal{C}}(K \times_{\mathcal{C}} X, Y)$

Theorem 12.3. *(The Cartesian workhorse, 3.2.2.13 HTT) p is a Cartesian fibration.*

It's hard to overstate how important this way of building things is. I've probably used it seventy times.

This is a fun theorem to use. So let's write down, finally, the definition of parameterised higher category. I apologize, I thought this was a better use of our time.

Definition 12.3. A $G-\infty$-category is a coCartesian fibration $X \to O_G^{\mathrm{op}}$.

Why didn't I write this down as a Cartesian fibration to O_G? If you pass to ∞-operads, those are closer to coCartesian fibrations, so unless you want to spend your life solving duality problems, you wnat to build things on the coCartesian side.

A G-functor from X to Y over O_G^{op} which carries coCartesian edges of X to coCartesian edges of Y. This is precisely the same thing, if you think of Joyal's theorem, a map of these things is a natural transformation. But I'm not going to say that this is a functor from O_G^{op} to Cat. I don't like this. It was easier to write this down as an opfibration rather than a pseudofunctor. This becomes exponentially worse passing to the ∞-world. This is better because it doesn't require the infinite process.

Let's do some. Let me add something, you could ask about a G-∞ groupoid or space, then you are talking about a left fibration. They don't need to carry coCartesian edges to edges because all edges are coCartesian.

Examples. Well, there are always easy examples of everything so let's do some of those first. If V is a finite G-set, then maybe we want to build a corresponding G-space in this sense. Then we get a G-space $\underline{V}$, a functor down to O_G^{op}, and I want the fibers, how will I do this? The fiber over G/H should be the H fixed points of my G-space V. I don't want to say that this fiber is the fixed points, this isn't nice because there's an ambiguity due to conjugation, this is just an orbit category. So I want my fibers to be discrete simplicial sets, the H-fixed points. The finiteness is irrelevant. What is V^H? Let's think it through. If I look in finite G-sets and want to extract fixed points, then I map $[G/H]$ to V (as G-sets). Now I want fibers that look like maps that vary in the first parameter. That's exactly what you can do with the Cartesian workhorse.

Everything in sight is an ordinary category. So let's actually just do this. The objects will be G/H and x in V^H. The maps $([G/H],x) \to ([G/K],y)$, I want a left fibration $[G/K] \to [G/H] \in O_G$, such that the map $V^H \to V^k$ carries x to y. [missed]

In a minute, I want to say that restricting to O_G isn't important. This will still work for any ∞-category. What if B is a random ∞-category? It's a coproduct [missed]. Just to give you a little thrill, the twisted arrow category $\mathcal{O}(F_G)$, this maps down to $F_G^{\mathrm{op}} \times F_G$, and the fiber over this thing is the set of maps of finite G-sets. I wnat to embed the orbit category crossed with V, and embed it in $F_G^{\mathrm{op}} \times F_G$ and form the pullback and this is what I'll call $\underline{V}$. Thisis precisely the story I told up here. But now I put almost no effort into writing this down.

Let me give an idea of *how* to start building this kind of fibration if you want something like a functor. Emily said during the break: there is a *universal* left fibration, well, there's a generators and relations picture for Top_*, so this is $\mathrm{Fun}(\Delta^1, \mathrm{Top}) \times_{\mathrm{Fun}(\{0\},\mathrm{Top})} \{*\}$, this is a perfectly good category, and there's a universal left fibration down to Top, that's just the forgetful functor, this actually really is (by the way, you can take composites along homotopy equivalences and break left fibrations. I could have screwed this up. This really is a left fibration) a left fibration. The fiber over $\{X\}$ is X, the space of points in X. This is one example where the functor to Top is easy. Why is this universal. Suppose you're a quasicategory and you want to write a functor to Top, you want to see the left fibration, that's

$$\begin{array}{ccc} \mathcal{C} \times_{\mathrm{Top}} \mathrm{Top}_* & \longrightarrow & \mathrm{Top}_* \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathrm{Top} \end{array}$$

This doesn't answer our question, given a functor how do you guess the fibration. The story is the basic procedure is guess and check. You start by trying to write down a fibration with the fibers you're after. If I want to write down something

$X \to \mathcal{C}$, I want X_s over s, and over $s \to t$ I know I want $X_s \to X_t$ because I know what I want the Cartesian edges to be. This you can usually actually get away with, even though it's a shit algorithm. Sometimes you get in trouble.

Working vertically like this replaces constructions with theorems. Writing down a functor to Top is a painful construction. A good example of this, when you think about Waldhausen K-theory, you want to say that the n-simplices are filtered objects, but then you need chosen collections of subquotients all the way down. That kind of process, I've completely lost my train of thought. Why'd I bring that up?

Let me restart. Ah, that definition is a big old definition, writing it down, that's the hard part. Once you've got it, you can dance the night away. Writing down, here, is not the hard part, but *checking* that it's a coCartesian fibration can be a nightmare. It took no energy to write down the definition of $\underline{V}$. To show that it's a fibration requires a horn-filling argument. If you're me, a babe in the woods, then [missed], and if you don't want to bother Emily, you can look in DAG X. He does it directly, doesn't finesse it. The complication moves from writing definitions to proving complicated theorems, that something is a fibration.

Let's look at the example of the G-∞-category of G-spaces. What do we want out of this? It's supposed to be a functor from O_G^{op} to Cat_∞. For $[G/H]$ I'll choose the ∞-category of H-spaces. This is a perfectly good functor, and this is the kind of thing I want to write down, but as a fibration to put it in my vertical world. Now we might think the Cartesian workhorse will help us here. Write down, I want a coCartesian fibration, so the dual of the Cartesian workhorse. We need a coCartesian fibration, I'll call it $\mathrm{Top}^G \to O_G^{\mathrm{op}}$, such that, I want $\mathrm{Hom}_{O_G^{\mathrm{op}}}(K, \mathrm{Top}^G) \cong \mathrm{Hom}_{O_G^{\mathrm{op}}}(K \times_{O_G^{\mathrm{op}}} ?, \mathrm{Top} \times O_G^{\mathrm{op}})$. This represents the constant functor at Top. Now how do I fill in the blank, say with X? Well $X \to O_G^{\mathrm{op}}$ should be a Cartesian fibration and it should have as its fiber over $[G/H]$ is supposed to be $\mathcal{O}_H^{\mathrm{op}}$. So what is O_H^{op}, this, well, $O_G/[G/H] \cong O_H$, this is not canonical because of the conjugation action, but this is an okay thing to calculate. So now I want the fiber to be $O_G^{\mathrm{op}}/[G/H]$.

I know a form for the fiber, so what kind of fibrations do I see where the fibers are overcategories? One really good time when I see this is the arrow category. So this will be $(O_{G/[G/H]})^{\mathrm{op}}$; then we have this pullback.

$$\begin{tikzcd}
(O_{G,[G/H]})^{\mathrm{op}} & \mathrm{Fun}(\Delta^1, O_G)^{\mathrm{op}} \arrow[d, "s"] \\
\{[G/H]\} \arrow[r] & O_G^{\mathrm{op}}
\end{tikzcd}$$

The source functor into any category D is always a Cartesian fibration. Target is coCartesian, because Hom is contravariant in the first variable and covariant in the second. This is a classic situation which happens all the time. This is also not in higher topos theory. This is a standard issue that shows up all the time. If I have a coCartesian fibration oriented one way, the opposite is a Cartesian fibration but it doesn't represent the same functor. Suppose p is a Cartesian fibration $X \to \mathcal{C}$, then $p^{\mathrm{op}} : X^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a coCartesian fibration. Then the first one looks like a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ and so does the second, but they're not equal. In the first case I map a to the fiber X_a. In the second case I get X_a^{op}, these two are not the same functor.

So what do you do? You fix it. If you spend much time in this space you encounter this problem a lot. So we wrote down a fix. Here's a *small* thing you can do. I told you that the effective Burnside category was a quasicategory, and now I'll tell you that you can be general with it. If you have $\mathcal{C}$ along with subcategories $\mathcal{C}_\dagger$ and $\mathcal{C}^\dagger$, wide subcategories of $\mathcal{C}$ so they contain all objects (and indeed for me, also all equivalences), and you have a further fact, a subcategory, by the way, of an infinity category, you want two maps, if they're in the same homotopy class, you don't want one but not the other in your subcategory.

So I call these guys ingressive and egressive. If you have this and if, a pullback of an ingressive along an egressive is ingressive and vice versa, then you get an ∞-category $A^{\mathrm{eff}}(C, C_\dagger, C^\dagger)$ and the n-simplices will be these big triangular diagrams, and the backward guys are all egressive while the forward arrows are all ingressive.

This is a typical three-simplex in the effective Burnside category.

Let me finish the story and tell you how to change a Cartesian fibration into a coCartesian fibration which represents the same functor. The effective Burnside category uses both the category and its opposite. We'll use that a lot next time. So let's do it. So we have the problem of this nice Cartesian fibration and we want a coCartesian fibration to represent the same functor. Now I'll build an effective Burnside category of X, and what do I want to have happen? My base is $\mathbb{C}^{\mathrm{op}}$. I really want this to contain all equivalences. This is the same as the Burnside category $A^{\mathrm{eff}}(\mathcal{C}, \iota\mathcal{C}, \mathcal{C})$, where $\iota\mathcal{C}$ is the equivalences of $\mathcal{C}$, so the backward maps are arbitrary and the forward maps are equivalences. I have an embedding of $\mathcal{C}^{\mathrm{op}}$ where I use id for the forward maps.

Now in the effective Burnside category, I want, so I have a map in the opposite category, so I want to go $X_a \to X_b$. I need a functor $X_a \to X_b$. It's supposed to be pulling back and then the identity on the right side. So I want my backward maps to Cartesian edges and the forward maps to lie over equivalences. I'll take, then, $A^{\mathrm{eff}}(X, X \times_{\mathcal{C}} i\mathcal{C}, X^{\mathrm{Cart}})$, and I'll call the pullback $X^\vee$. I'll call my pullback $p^\vee$ and then the theorem is that $p^\vee$ is a coCartesian fibration and the functor it corresponds to from $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ (only defined up to a contractible choice) is equivalent to the corresponding functor for p.

$$\begin{tikzcd}
X^\vee \arrow[r] \arrow[d, "p^\vee"] & A^{\mathrm{eff}}(X, X \times_{\mathcal{C}} \iota\mathcal{C}, X^{\mathrm{Cart}}) \arrow[d] \\
\mathcal{C}^{\mathrm{op}} \arrow[r] & A^{\mathrm{eff}}(\mathcal{C}, \iota\mathcal{C}, \mathcal{C})
\end{tikzcd}$$

13. June 10: Philip Hackney/Marcy Robertson: ∞-properads III

So last time we talked about this category Γ that is supposed to encode higher versions of properads as Ω encodes higher operads and Δ encodes higher categories. You all have a handout. Everything I said last time is joint with Marcy and Donald Yau, it's probably in our book. If I do what I want today then we might get to things that aren't in the book, but that's okay.

So here's a notion, a generalization of the normal notion of Reedy category due to Berger and Moerdijk.

Definition 13.1 (Berger–Moerdijk–Reedy category)**.** A *BMR* structure on a small category $\mathbb{R}$ consists of wide subcategories $\mathbb{R}^+$ and $\mathbb{R}^-$ and a degree function $\mathrm{Ob}(\mathbb{R}) \to \mathbb{N}$ satisfying:

(1) non-invertible morphisms in $\mathbb{R}^+$ (respectively $\mathbb{R}^-$) raise (respectively lower degree; isomorphisms preserve degree).
(2) $\mathbb{R}^+ \cap \mathbb{R}^- = \mathrm{Iso}(\mathbb{R})$
(3) Every morphism f factors as $f = gh$ such that $g \in \mathbb{R}^+$ and $h \in \mathbb{R}^-$ and this factorization is unique up to isomorphism.
(4) If $\theta f = f$ for $\theta \in \mathrm{Iso}(\mathbb{R})$ and $f \in \mathbb{R}^-$ then θ is an identity.
(5) If $f\theta = f$ for $\theta \in \mathrm{Iso}(\mathbb{R})$ and $f \in \mathbb{R}^+$ then θ is an identity.

If you're familiar with normal Reedy categories, and I won't write the definition, but these are $\mathbb{R}$ where $\mathrm{Iso}(\mathbb{R})$ is just the identities. You don't have any nontrivial isomorphisms. That includes Δ, Δ^{op}, others. Other examples, the dendroidal category that Marcy talked about, Ω, this is a, I'll call these Reedy categories, finite sets, pointed finite sets, Λ, their opposites as well.

If you have a model category on $\mathcal{M}$, under reasonable assumptions, functors $\mathbb{R} \to \mathcal{M}$ will have a model category structure. Let me tell you what that's about. For that I need matching and latching objects. I have $\mathrm{Ob}(\mathbb{R}^+(r))$, which is a full subcategory of $\mathbb{R}^+ \downarrow r$, which consists of those maps with target r which are not invertible. Similarly, the objects of $\mathbb{R}^-(r)$ are the maps $\alpha : r \to s$ which are noninvertible. These are the categories which give us $L_r(X) = \mathrm{colim}_{\alpha \in \mathbb{R}^+(r)} X_s$, which maps to X_r via the colimit of α, which in turn maps to $\lim_{\alpha \in \mathbb{R}^-(r)} X_s = M_r(X)$. For any object r in $\mathbb{R}$ I have this. For us $\mathcal{M}$ is a (let's say cofibrantly generated) model category and X is a functor in $\mathcal{M}^{\mathbb{R}}$. I'm just using bicompleteness so far, but now I want the model category structure.

If you have this, then you say a map $f : X \to Y$ in $\mathcal{M}^{\mathbb{R}}$ is

- a Reedy cofibration if $X_r \cup_{L_r X} L_r Y \to Y_r$ is a cofibration in $\mathcal{M}^{\mathrm{Aut}\, r}$ for all r.
- a Reedy weak equivalence if $X_r \to Y_r$ is a weak equivalence in $\mathcal{M}^{\mathrm{Aut}(r)}$ for all r (I can check this in $\mathcal{M}$).
- a Reedy fibration if $X_r \to M_r X \times_{M_r Y} Y_r$ is a fibration in $\mathcal{M}^{\mathrm{Aut}\, r}$ for all r (I can check this in $\mathcal{M}$).

Theorem 13.1. *(Reedy, Kan, Berger–Moerdijk) $\mathcal{M}^{\mathbb{R}}$ is a model category with these classes.*

As a note, we'd like this guy to inherit some properties from $\mathcal{M}$. If $\mathcal{M}$ is left proper, cellular, combinatorial, then so is $\mathcal{M}^{\mathcal{R}}$. There is a caveat that for left properness this isn't written down anywhere. Left proper I need an injective model structure on $\mathcal{M}^{\mathbb{R}}$ to know how to prove that, so for me if $\mathcal{M}$ is combinatorial and left proper then so is $\mathcal{M}^{\mathbb{R}}$. I had a lot of trouble with left proper.

Okay. Here's the theorem on Γ.

Theorem 13.2. Γ *is Reedy.*

I'll give you some of the proof. The degree function is the number of vertices. The plus maps are injective on edges, f_0 is injective from $\mathrm{Edges}(H) \to \mathrm{Edges}(G)$. The minus maps are surjective on edges, f_0 is surjective from $\mathrm{Edges}(H) \to \mathrm{Edges}(G)$ but you need something else. Let's see why you need more. So we could look at this outer coface map [pictures]. So you should also require that $d(H) > d(G)$ (possibly not good enough) or for every vertex in $\mathrm{Vt}(G)$ there is a vertex in $\mathrm{Vt}(H)$ with $v \in f_1(v')$.

Here's another characterization of both of these categories. $f : H \to G$ is in Γ^+ if we can write it as a composition of isomorphisms and coface maps. It is in Γ^- if we can write it as a composition of isomorphisms and codegeneracy maps.

So this requires a proof if you want to take that definition. Note that codegeneracies decrease degree and satisfy the definition I gave; cofaces increase degree and are injective on edges.

So maybe this could have been the definition.

Let's look at the decomposition, for a map $f : G \to K$ in Γ; for each vertex v of G, I get $f_1(v)$ a subgraph of K. I want to look at the collection of all of these so that the subgraph is a single edge, $T \subset \mathrm{Vt}(G)$ such that $f_1(v) =|$ for $v \in T$. Then first we plug in an edge to all v in T; call this graph G_1. This is an iteration of codegeneracy maps. I also have the image of f, which is a subgraph of K. Then the inclusion into K is a composition of outer coface maps. If I'm ignoring the coloring, I can factor:

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[d] \arrow[drr] & & K \\ G_1 \arrow[r] & G_2 \arrow[r] & \mathrm{im}(f) \arrow[u] \end{tikzcd}$$

what else can I do, I can go from G_1 to G_2, an isomorphism where I relabel edges, this is $f_0(G_1)$, and for the next guy I have an inner coface map from G_2 to $\mathrm{im}(f)$.

This is the existence part of the third axiom. Let's move on to something else.

Let's write down a model structure, and I'm losing all of my erasers. We've now established, partly established that Γ is Reedy, so also, since I included the (unnecessary) last axiom, then Γ^{op} is also Reedy. Then I can look at $\mathrm{sSet}_0^{\Gamma^{\mathrm{op}}}$ Here my 0 means that $X_| = X_0$ is discrete. This, everything we had about simplicial sets to Δ^{op} being a model category, this is also true; this thing carries a Reedy model structure. I want to write down a model structure for ∞-properads here, and I'll do it by localization, which Clark taught us about yesterday. So I'll localize with respect to Segal core inclusions; I'll tell you what these are and what you should be getting out of this. For each G you have $\mathrm{Sc}[G]$, for every vertix in G, I can look at the representable on a corolla at that vertex

$$\mathrm{Sc}[G] = \cup_{v \in \mathrm{Vt}(G)} i_*^v \Gamma[C_v]$$

where i_*^v from $C_v \to G$ is an iterated outer coface.

Localization is why I was saying things about cellular and left proper. If C is the set of Segal core inclusions, then I can get $\mathcal{L}_C\,\mathrm{sSet}_0^{\Gamma^{op}}$ and this should be a model, it should have Segal properads as its fibrant objects. How do these look, what defines them and why is this a reasonable thing to call this? If I take this, what are the fibrant objects, first it should be fibrant in the underlying thing, levelwise fibrant, but it also has to be local. The map

$$\mathrm{map}^h(\mathrm{Sc}[G], X) \leftarrow \mathrm{map}^h(\Gamma[G], X)$$

should be an equivalence for all G. So we can omit the h which are not important. The right hand side is X_G; the left hand side is the limit over the vertices of X_{C_v}, and I'm just asserting that this is an equivalence, $X_G \to \lim X_{C_v}$, so if I tell you something at X at each vertex, then I can get a composition along the graph G, the map $X_G \to X_C$, the maximal thing I can do, and this $\lim X_{C_v} \to X_C$ is what my composition would be.

This stuff is what's not in the book, or I probably have a better handle on these issues.

Here's a question, if I take a simplicially enriched properad, I can take the nerve and I end up in the category $\mathcal{L}\,\mathrm{sSet}_0^{\Gamma^{op}}$, the question is, is this a Quillen equivalence. We haven't talked about the right hand side at all, but it has a model structure written down by Marcy, Donald, and I, so it's a question, is this a Quillen equivalence.

If you do this and show this, you can have a copy of our book. There's a prize involved. I went over by four minutes, I'm sorry.

14. Victor Turchin: Embedding calculus and the little disks operads II

So let me put on the board two [unintelligible]

Theorem 14.1. *(Arone–T. 2011) Consider embeddings modulo immersions*

$$T_\infty \overline{\mathrm{Emb}}_\partial(\mathbb{D}^m, \mathbb{D}^n) \cong h\,\mathrm{Inf}\,\mathrm{Bim}_{B_m}(B_m, B_n)$$

embeddings fixed in a neighborhood of the boundary, or rather the limit of the Goodwillie tower of this extension of them, is equivalent to these bimodules, and for $n - m > 2$, the tower converges so I have the same theorem without T_∞.

Let me remind you of $\overline{\mathrm{Emb}}$. So it's the homotopy fiber of $\mathrm{Emb}(\mathcal{D}^n, \mathcal{D}^m) \to \mathrm{Imm}_\partial(\mathbb{D}^m, \mathbb{D}^n)$, and this latter is just $\Omega V_m(\mathbb{R}^n)$, this Stiefel things. I'll give some hint of these bimodules.

Let me write down this other theorem.

Theorem 14.2. *(Dwyer–Hess, Boavida de Brito–Weiss (this is the only one that appeared), Ducoulombier–T.)*

$$T_\infty \overline{\mathrm{Emb}}_\partial(\mathbb{D}^m, \mathbb{D}^n) \cong \Omega^{m+1} h\,\mathrm{Oper}(B_m, B_n).$$

It's natural, instead of looking at weird bimodules, it's natural to look at maps of operads; then you get a delooping. Let me give the special case $m = 1$, which is really due to Dev Sinha. Then B_1 is naturally equivalent to the associative operad. In fact, B_n is equivalent to a certain operad, the Kontsevich operad, and you have something like

$$\begin{tikzcd} B_1 \arrow[d] & W_1 \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & \mathrm{Ass} \arrow[d] \\ B_n & W_n \arrow[l, "\cong"'] \arrow[r, "\cong"] & K_n \end{tikzcd}$$

now an infinitesimal bimodule over Ass is a cosimplicial object, and $h\,\mathrm{Inf}\,\mathrm{Bim}_{\mathrm{Ass}}(\mathrm{Ass}, K_n) = h\,\mathrm{Tot}\,K_n(\quad)$. For $m = 1$ this was proved by Dwyer and Hess and then I gave a different proof, if you have a map of operads $\mathrm{Ass} \to \mathcal{O}$, then $\mathrm{Tot}\,\mathcal{O}(\quad) = \Omega^2 h\,\mathrm{Oper}(\mathrm{Ass}, \mathcal{O})$. I should have said, $\mathcal{O}$ needs to be double reduced, $\mathcal{O}(0) \cong \mathcal{O}(1) \cong *$. This is also true in the truncated case for $\mathrm{Tot}_{\le k}$ and $\mathrm{Oper}_{\le k}$. We are using a very explicit cofibrant replacement. Now there's all this business about the Deligne cohomology conjecture, that on the Hochschild complex of an associative algebra you get an action of the operad of chains on little squares. Afterward, McClure and Smith [unintelligible], and this is an explicit delooping, this gives a high dimension generalization.

So the proof of Dwyer and Hess and my proof with Ducoulombier rely on my theorem with Arone.

Let me give you a brief sketch of ideas of Dwyer and Hess' proof. They prove a theorem

Theorem 14.3. *(Dwyer–Hess) In a monoidal model category, for a map of monoids $M_1 \to M_2$, we have, well, M_2 becomes a bimodule over M_1, and we can look at $h\operatorname{Bim}_{M_1}(M_1, M_2)$, and we can compare with $\operatorname{Mon}(M_1, M_2)$, and this space will naturally be a loop space of the second one provided the mapping space from $\mathbf{1}$ to M_2 is contractible. So a map of monoids will be a delooping of the maps of monoids.*

So how does this work? We can consider a map of operads $P \to Q$, you know operads are monoids with respect to the $\circ$ product, we have a monoidal structure, well, Dwyer and Hess consider non-symmetric operads, then you have just sequences of spaces, and the space of maps of bimodules, so Q becomes a bimodule over P, and then $h\operatorname{Bim}_P(P, Q) = \Omega h\operatorname{Oper}(P, Q)$, and here we need the condition that $Q(1) = *$. Probably you also maybe need a technical condition on P. Marcy proved this theorem in the colored Σ case, so she should know, maybe $P(0) \cong P(1) \cong *$. So we have this delooping, and we take $P = \operatorname{Ass}$, and we obtain that $h\operatorname{Bim}_{\operatorname{Ass}}(\operatorname{Ass}, \mathcal{O}) \cong \Omega h\operatorname{Oper}(\operatorname{Ass}, \mathcal{O})$. Now we need a second delooping, the second delooping is obtained by, well, $h\operatorname{Tot}(\mathcal{O}(\quad)) \cong \Omega h\operatorname{Bim}_{\operatorname{Ass}}(\operatorname{Ass}, \mathcal{O})$ provided that $\mathcal{O}(0) \cong *$. Here you need the right model, take the following monoidal model category: right modules over Ass with tensor product $(P \boxtimes Q)(n) = \bigsqcup_{i+j=n} P(i) \times Q(j)$. Then monoids with respect to this structure are bimodules, and bimodules over Ass are cosimplicial objects. That's how this works.

Now for high dimensions.

Theorem 14.4. *(Dwyer–Hess; Ducoulombier–T.) If you have $B_m \to \mathcal{O}$ is an operad map and $\mathcal{O}(0) \cong \mathcal{O}(1) \cong *$, then*

(1)
$$h\operatorname{Inf}\operatorname{Bim}_{B_m}(B_m, \mathcal{O}) \cong \Omega^{m+1} h\operatorname{Oper}(B_m, \mathcal{O}).$$

(2) *If $B_m \to M$ is a B_m-bimodule map. When you have a bimodule, it's not automatically an infinitesimal bimodule, but with this kind of map you can mimic empty insertions. Then we get*
$$h\operatorname{Inf}\operatorname{Bim}_{B_m}(B_m, M) \cong \Omega^m h\operatorname{Bim}_{B_m}(B_m, \mathcal{O}).$$

So the second one implies the first one by Marcy's theorem. This has more implications than to the study of embeddings. Let me give more motivation and then some idea of the proof.

We can consider any space of maps $\operatorname{Maps}^{\mathbb{S}}_{\partial}(\mathcal{D}^m, \mathcal{D}^n)$, where these maps avoid certain multisingularities, triple intersections or something like that, and for these guys, it's a difficult question whether the Goodwillie tower converges. Still we can apply the theorem, and then the infinitesimal bimodule which controls this tower comes from [unintelligible].

Consider the sequence $\{\operatorname{Maps}^{\mathbb{S}}(\sqcup_k \mathcal{D}^m, \mathcal{D}^n), m\}$, this is a B_m-bimodule. Therefor T_∞ for this tower can also be delooped in this way. You could look at $\operatorname{Imm}^{(\ell)}_{\partial}(\mathcal{D}^m, \mathcal{D}^m) \hookrightarrow \operatorname{Imm}_{\partial}(\mathcal{D}^m, \mathcal{D}^n)$, which avoid ℓ-self intersections, and the fiber is $\overline{\operatorname{Imm}}^{(\ell)}_{\partial}(\mathcal{D}^m, \mathcal{D}^n)$, and then $B^{(\ell)}_n$ is the space of collections of disks which can overlap but no ℓ of them have a common point. This is a bimodule over B_n. Then T_∞ of the space of immersions is described

$$T_\infty \operatorname{Imm}_\partial^{(\ell)}(\mathcal{D}^m, \mathcal{D}^n) \cong h \operatorname{Inf}\operatorname{Bim}_{B_m(B_n, B_n^{(\ell)})}.$$

So I should also mention, for embeddings we have not just an action of B_m but also of B_{m+1}. Where does this come from? Morally speaking, it comes from, well, for one dimensional knots [picture] we have not just little intervals but little squares, this is homotopy commutative, we can shrink a knot and pull it through another. You need framing, I should say, to pull it through. On this space of embeddings you don't have this, but on embeddings modulo immersions you have this. You can't do this on more general immersions. These embeddings, they're really operads and can be delooped one more time.

The approach of Dwyer–Hess to this theorem, they're using the fact that $B_m \cong \underbrace{\mathrm{Ass} \otimes \cdots \otimes \mathrm{Ass}}_{n}$, the Boardman–Vogt tensor product, they use this decomposition and apply the philosophical decomposition that I erased, how exactly it works I don't know, I think it's probably technical, that's why they're slow in writing it down.

Our approach is more direct, and the proof is very similar to my proof of the second delooping, with an explicit cofibrant replacement. For any operad $\mathcal{P}$ (doubly reduced), and any $\mathcal{P}$-bimodule maps $P \to M$, we construct a map, a natural map

$$\operatorname{Maps}_*(\Sigma\mathcal{P}(2), h \operatorname{Bim}_{\mathcal{P}}(\mathcal{P}, M)) \to \operatorname{Inf}\operatorname{Bim}(\mathcal{P}, M)$$

and we write down when this is an equivalence, and for the little disks this is satisfied.

Our approach, I should say, works for the truncated case as well. In Dwyer–Hess, it's more difficult. You have to look at the tensor product of truncated operads and then it's not clear how well it works.

Now I want to talk about the approach of Boavida de Brito and Weiss. How do they prove that $\overline{\operatorname{Emb}}(\mathcal{D}^m, \mathcal{D}^n) \cong \Omega^{m+1} h \operatorname{Oper}(B_m, B_n)$.

Their result is weaker and stronger. They can't do immersions or anything, but it's stronger because their deloopings respect the action of the little disks. We have $\operatorname{Emb}_\partial(\mathcal{D}^m, \mathcal{D}^n)$, which is mapped to $\Omega^m V_m(\mathbb{R}^n)$, the Stiefel manifold, and ther's a natural map $V_m(\mathbb{R}^n) \to h \operatorname{Oper}(B_m, B_n)$, so you have a map to $\Omega^m h \operatorname{Oper}(B_m, B_n)$, and the theorem of Boavida de Brito and Weiss is:

Theorem 14.5. *This sequence is a fiber sequence.*

So they also give

$$\operatorname{Emb}_\partial(\mathcal{D}^m, \mathcal{D}^n) \cong \Omega^m \operatorname{hofib}(V_m(\mathbb{R}^n) \to h \operatorname{Oper}(B_m, B_n)).$$

We just proved equivalence on the level of spaces without any action of the disks. As a consequence, when you take the homotopy fiber, you get Ω^{m+1}, as I said before.

To give an idea of the techniques that they're using, the crucial things are configuration categories. You don't need M to be smooth, and they define $\operatorname{Con}(M)$, it's a topological category. The objects of the category are the disjoint union of embeddings of k labelled points to M, $\operatorname{Emb}(\underline{k}, M)$. If $x \in \operatorname{Emb}(\underline{k}, M)$ and $y \in \operatorname{Emb}(\underline{\ell}, M)$, then $\operatorname{Mor}(x, y) = \{(j, \alpha)\}$ where $j : \underline{k} \to \underline{\ell}$ and α is a *reverse exit path* from x to $y \circ j$.

So x is a bunch of labelled points in your manifold M and you have a bunch of other points y. Then a reverse exit path you need to say which points go to which points. [picture]

Then it's a path where once points collide, they should stay collided. It's kind of sticking configurations. We mean a map from $[0,t]$ (with $t \geq 0$). Then we have a natural functor from $\mathrm{Con}(M) \to \mathrm{Fin}$ by forgetting the paths. Now the theorem is the following.

Theorem 14.6. *If $n - m \geq 3$, there is a homotopy Cartesian square*

$$\begin{tikzcd} \mathrm{Emb}(M,N) \arrow[r] \arrow[d] & h\,\mathrm{Map}_{\mathrm{Fin}}(\mathrm{Con}(M),\mathrm{Con}(N)) \arrow[d] \\ \mathrm{Imm}(M,N) \arrow[r] & \Gamma \end{tikzcd}$$

where Γ is the space of sections of $E \to M$ where $E = \{(m,n,\alpha)\}$ where m is in M, n is in N, and α is in $h\,\mathrm{Map}_{Fin}(\mathrm{Con}(T_mM),\mathrm{Con}(T_nN))$, which you'll see in a second is equivalent to $h\,\mathrm{Oper}(B_m,B_n)$.

So what do they consider? They take the nerve of the category $\mathrm{Con}(M)$, this is a simplicial space, and the nerve of Fin. Then you need to consider the Rezk model category structure on simplicial sets. This is in homotopy theory of homotopy theories. I'm not familiar with this work, but the fibrant objects are complete Segal spaces. They work in the overcategory, the space of maps in this model category of objects over $N\,\mathrm{Fin}$. They define the model structure on this overcategory, this is not a fibrant object, $N\,\mathrm{Fin}$, so you need something technical. You take the space of maps in this model category of simplicial spaces over $N\,\mathrm{Fin}$. The claim is, there are two important statements.

Proposition 14.1. *If you apply this to a map of disks, $h\,\mathrm{Map}_{\mathrm{Fin}}(\mathrm{Con}^{\partial}(\mathcal{D}^m),\mathrm{Con}^{\partial}(\mathcal{D}^n))$ (I should say the space changes when you let points go to the boundary), this space is contractible.*

It actually factors through topological embeddings, which is contractible by the [unintelligible]trick.

Proposition 14.2. *The space $h\,\mathrm{Map}_{\mathrm{Fin}}(\mathrm{Con}(\mathbb{R}^m),\mathrm{Con}(\mathbb{R}^n)) \cong h\,\mathrm{Oper}(B_m,B_n)$.*

These configuration categories over Fin, it's equivalent to a certain construction over C_{B_m}. If we have a sequence of maps of sets, you can assign to this a level tree. So once you have an operad, you can construct a simplicial space $C_{\mathcal{O}}$. Then they show that the nerve of the configuration category is equivalent to the simplicial space over $N\,\mathrm{Fin}$ you obtain in this way. First they replace and then they show that for any $\mathcal{O}_1,\mathcal{O}_2$ with $\mathcal{O}_1(0) \cong \mathcal{O}_2(0) \cong *$, then $h\,\mathrm{Map}_{\mathrm{Fin}}(C_{O_1},C_{O_2}) \cong h\,\mathrm{Oper}(O_1,O_2)$. To do this they make contact with dendroidal spaces, they move through infinity operads or permutads, they use this technique that I don't understand very well, but I wanted to give you an idea of the construction.

15. Clark Barwick: Parametrised higher category theory and parametrised higher algebra II

So, last time I tried to give a convincing argument for why you should care about higher categories, and within that why you should use quasicategories, and I tried to convince you that these could be explicit, so you're never saying things about equivalence classes. Interspersed with that I introduced the concept of a G-∞-category, which is more than an ∞-category, and more than an ∞-category with a

G action. It's a coCartesian fibration $X \to O_G^{\mathrm{op}}$, a G-functor is a map over O_G^{op} that preserves coCartesian edges. We didn't talk about motivation, but the examples we saw that were interesting, when we considered E/F, this Galois extension with group G, we began by writing down a coCartesian fibration $\mathrm{Vect}_{E/F} \to O_G^{\mathrm{op}}$. There are other examples that are worth noticing. Here's an example I did last time, there's G-spaces, and the Elmendorf–McClure theorem that says usual G-spaces is like functors from O_G^{op} into [missed].

So we learned about the homotopy theory from presheaves on O_G, and this is the category of G-spaces as usually thought of. I suggested but didn't quite prove that G-spaces should form a G-category. That's what I said, I didn't say how, so let's do that but honestly. I ran into a wall about opposites and got distracted, but let's think about this. This is morally the assignment that to every $[G/H]$ assigns H-spaces. I want a coCartesian fibration that represents the functor. I need to write down the coCartesian fibration. We want a coCartesian fibration, maybe I should give this a name, $\mathrm{Top}^G \to O_G^{\mathrm{op}}$ such that the fiber over $[G/H]$ is $\mathrm{Fun}(O_H^{\mathrm{op}}, \mathrm{Top})$. Remember, we decided that O_H for $H \leq G$, is the same as a slice category of the orbit category of G (not naturally), $O_H \cong (O_G)/[G/H]$. Then the fiber I want over $[G/H]$ is really functors $\mathrm{Fun}((O_G)^{\mathrm{op}}_{[G/H]}, \mathrm{Top})$. So in the domain we need something contravariant and I should have the boring fibration on the right. So we'll define Top^G as, I should use my Cartesian workhorse that says that when I have a Cartesian and coCartesian fibration, I can pair them and get a coCartesian fibration. So I'll say this is

$$\mathrm{Top}^G \coloneqq \mathrm{Fun}_{O_G^{\mathrm{op}}}(X, \mathrm{Top} \times O_G^{\mathrm{op}})$$

where X is a Cartesian fibration to O_G^{op} with fibers the slice categories.

We got stuck with this, we look from $\mathrm{Fun}(\Delta^1, O_G^{\mathrm{op}})$, maybe the smart money would have been to stick with this, to O_G^{op}. Then the source functor is always a Cartesian fibration. The fibers of this thing, over $[G/H]$, it's the category of arrows whose source is $[G/H]$, and so I get $(O_G^{\mathrm{op}})_{[G/H]/}$.

Now I'm happy, this has the functoriality I want, life is good. So X is this.

If D is an ∞-category, I can define an ∞-category of G-objects in D in the same way. Top was just along for the ride here. So we get $D^G \to O_G^{\mathrm{op}}$. This is not just an object with a G-action, it has all the information about H-fixed points.

Here's a fun theorem, if you like the first theorem of our book. The theorem is that D^G, the G-∞-category, is the cofree G-∞-category generated by D. This means there is natural equivalence betwen $\mathrm{Fun}_{O_G^{\mathrm{op}}}(C, D^G) \cong \mathrm{Fun}(C, D)$, so $D \rightsquigarrow D^G$ is right adjoint to taking the "total category" of a G-∞ category. A real category theorist probably already knew this, but this is the first hint that there will be access to G-homotopy theories in this subject. In this story, so far, we haven't used anything particularly about the orbit category of G. The only thing we used was slice categories. We haven't used any structure. So we can generalize this as follows.

Suppose B is any ∞-category. Then a B-∞ category is a coCartesian fibration $X \to B^{\mathrm{op}}$. I'm simply saying, you can work with everything you like. If you don't want genuine G-objects but rather functors from BG to your thing, then just replace the orbit category with BG. There are really great examples where you want to replace this category, most of these are not my invention. The first example is

$O_{G,\mathcal{F}}$, orbits of the form $[G/H]$ where H is in a family $\mathcal{F}$ closed under conjugation. This is an important tool especially if you want to do certain kinds of induction.

A more exotic example is $O_{©}$; this is $(2,1)$-category. The objects are "orbits" of the form $\mathbb{Q}/\frac{1}{m}\mathbb{Z}$, and the maps are $\mathbb{Q}/\mathbb{Z}$-equivariant maps. You'd be unimpressed so far, but you can put in intertwiners to make this a 2-category.

Another example that is one of my favorites is finite sets of size at most n and surjective maps. That's a super interesting category, it turns out.

Once you have all this in place, what do we want to do? We have the theory of G-spaces. My job is to look at cohomology theories on G-spaces? Well, what are cohomology theories on spaces?

Definition 15.1. A *reduced excisive* functor $H : \mathrm{Top} \to \mathrm{Top}$ is one with the following properties.

(1) $H(\varnothing) = *$
(2) H preserves filtered homotopy colimits (this is a technical condition), filtered just means hocolims indexed on filtered posets.
(3) If you have a homotopy pushout square

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ U' \arrow[r] & V' \end{tikzcd}$$

then H takes this to a homotopy pullback.

You probably think of a homology theory as being a graded Abelian group and the way you get the Abelian group out is by applying π_*, and everything is canonically pointed.

This is a reduced excisive functor, so this is the same thing as [missed].

Now I'll define Sp, the category of spectra, as the full subcategory of functors from Top to Top spanned by the reduced excisive functors. This is a definition of spectra. This has a nice clean description. This is the free stable ∞-category with colimits on one object. Stability means you can't tell the difference between pushouts and pullbacks.

This is the ∞-category of homology theories for spaces, and I want to do the same thing for G-spaces and that should give me G-spectra.

Let me comment on the category of spectra. Observe, and this is something we talked about in Vect, that Sp acts a lot like Abelian groups, observe that Sp has direct sums. It has a 0 object, the 0 homology theory, and if you form finite products and if you form finite coproducts, there's a map from one to the other. Ther's a map $X \vee Y \to X \times Y$, and the functor you're writing down is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and that's an equivalence. The maps from one object to another is a grouplike E_∞ space so that's also really good, but I'm going to focus on direct sums.

Now what about G-spectra, and more generally B-spectra for some crazy ∞-category B. I need now to pause and point out an issue that we already have. What was I going to say? I'm completely blanking. Well okay, let me make a proto-definition.

Definition 15.2. A *reduced G-excisive* G-functor $E : \mathrm{Top}^G \to \mathrm{Top}^G$ is one with the following properties

(1) for all $H \leq G$, we have $E_H : \mathrm{Top}_H \to \mathrm{Top}_H$ (here Top_H is the ∞-category $\mathrm{Fun}(O_H^{\mathrm{op}}, \mathrm{Top})$) on the fibers over $[G/H]$ is reduced excisive.

If you want something for excision, we want something more than just fiberwise, we need G-∞-category limits and colimits. I want to say that a G-pushout should go to a G-pullout. But you might think this is weird because there's no canonical way that you can act on a square. So we'll formulate the next condition, and to do that I need to say something complicated about a cube. Now maybe you'll think that this is very technical and tedious, and you're right, but if you have this first condition you can cut down to fewer cases.

One thing you learn in category theory is that if you have coequalizers and coproducts you have all colimits. If you're working in homotopy theory, and you have G-homotopy coproducts and G-diagrams indexed on Δ, then you have everything you want. So the good news is that being a sifted colimit is the same in the ordinary world and the G-world. So it turns out it's enough to understand coproducts. This nice feature of having direct sums is what I'll ask for here as well.

(2) E carries finite G-coproducts to finite G-products. This is really a definition.

I only need to define G-products and G-coproducts. This will exactly agree, define the full subcategory Sp^G in $\mathrm{Fun}(\mathrm{Top}^G, \mathrm{Top}^G)$ spanned by the reduced G-excisive guys. This is exactly the same as G-spectra in [unintelligible]from the 70s, but defined in a homotopy invariant way without any model.

This whole story here is that I don't need G, it's done nothing for me, so why don't I replace it with B, to talk about finite G-(co)-products, I need to understand finite G-sets, and there's something critical (and unusual) about finite G-sets. So let's talk about the passage from $O_G \to F_G$, this is a (finite) coproduct completion. That means every object in F_G can be written in a unique manner as the disjoint union of its orbits. If you demanded that I look at coproduct preserving functors, then (∞-categorically)

$$\mathrm{Fun}^{\sqcup}(F_G, D) \xrightarrow{\cong} \mathrm{Fun}(O_G, D).$$

So we can take $B \to F_B$, but there's something really nice about F_G which is that it has pullbacks. If you want a concrete model for F_B; take functors from B^{op} into sets, take B there by Yoneda, and look at the smallest subcategory that contains coproducts. This is something you may have seen in a bizarre form.

$$\begin{tikzcd}
& \bigsqcup_{x \in H\backslash K/L} [G/H^x \cap L] \arrow[dl, dotted] \arrow[dr, dotted] & \\
{[G/H]} \arrow[dr] & & {[G/L]} \arrow[dl] \\
& {[G/K]} &
\end{tikzcd}$$

This is (or some version is) called the Mackey decomposition formula. It's something you like. This is a key property, this having pullbacks.

Definition 15.3. B is *orbital* if F_B has pullbacks

There are tons of these things and they tell you all sort of wonderful things.

If you have this condition, then this story works perfectly. Okay, I want to give you a sense that there are real theorems in this field, so let me do that.

The first theorem is that you might be a little unhappy, I haven't told you G-products and coproducts very directly. You don't have to be unsatisfied because of the following theorem.

Theorem 15.1. *(Gwilliam–May) If B is orbital, then Sp^B is equivalent to $\mathrm{Fun}^{\oplus}(A^{\mathrm{eff}(F_B)}, \mathrm{Sp})$.*

What is this effective Burnside category? You have morphisms as spans with pullback as composition. If you're unhappy about not telling the truth about what B-spectra are, then you can be happy because now they're Mackey functors on finite B-sets.

What else is true for ordinary ∞-categories? [missed]. The same thing is true here. The B-∞-category Top^B is freely generated by a single object under B-colimits, just as you'd expect.

This is kind of amusing. This gave you a cofree characterization, now we have a sort of free characterization. If I want to tell you what a colimit-preserving functor in $\mathrm{Fun}^L_B(\mathrm{Top}^B, D)$ is, that's just D.

[some discussion]

Okay, there's more. There's a lot more but I'm going to have to stop. Maybe I'll emphasize, this theorem provides you with a universal property for Top^B, it's like the role the integers play in Abelian groups. So Top^B is the unit for a symmetric monoidal structure on B-∞-categories.

What will I tell you now? Once you have B-stable B-∞-categories, then Sp^B is freely generated under colimits by a single object. Once again this gives this thing the status of the unit in B-stable B-∞-categories and then [unintelligible]and that's Hill–Hopkins–Ravenel's norm.

I want to give you some examples to bring things back to earth a little bit. I said that what G-spectra are. What about these other things? If you take $\mathrm{Sp}^{©}$, these are what we call cyclonic spectra, and for those of you who like equivariant things, these are equivarinat spectra relative to the family of finite subgroups. You can think about topological Hochschild homology as having a property of [unintelligible]and cyclotomic spectra even, but I'll address this in questions.

Finally, take this truncated category, this is Glasman's theorem. What are spectra with respect to that? They're Mackey functors from the category $F_{\mathrm{Fin}\leq n}$, but that turns out to be n-excisive functors from Sp to Sp and everything you can see going from different n is visible here, this is a huge elaboration of Goodwillie's version that says something about working with just Σ_n.

16. June 15: Dimitri Zaganidis: The quasi-category of homotopy coherent monads in an $(\infty, 2)$-category

I'm going to start by unpacking what might be a scary title. When I talk about an $(\infty, 2)$-category, what I have in mind is mainly a category enriched in quasicategories. The letter will be often K, you can pick Cat or you can pick $q\,\mathrm{Cat}_\infty$ or other things. Mainly if you take M a model category enirched over sSet_J, then K could be the cofibrant fibrant subcategory.

There's a whole model structure of simplicial categories where the fibrant objects are these guys, and this is a model for $(\infty,2)$-categories. This is supposed to generalize what we know about monads and adjunctions in Cat to these guys.

Now I want to talk about homotopy coherence. So the idea is to encode homotopy coherence as simplicial functors $\mathcal{C} \to K$ where $\mathcal{C}$ is maybe actually a 2-category and so by $\mathcal{C}$ I mean $N_*\mathcal{C}$. I will not write the nerve again. It's not a problem because it's a full and faithful embedding from 2-categories into simplicial categories.

These kinds of ideas were first introduced by Vogt (around 1973) and Cordier–Porter (around 1986). You can see this in the definition of the homotopy coherent nerve of a simplicial category $\mathcal{C}$, which is $\mathcal{N}(\mathcal{C})_m = \mathrm{sCat}(\mathbb{C}\Delta[m], \mathbb{C})$.

The second instance is more recent, by Riehl and Verity (2013) and in that case it's about coherent monads and adjunctions, that's the second paper of a series in order to try to make the theory of $(\infty,1)$-categories more accessible, and I really like that, I have to say.

Maybe I should say right now what they mean by coherent monads and adjunctions. So homotopy coherent adjunctions and monads. The first thing, I should first introduce two categories, Mnd and Adj. These are universal categories in some sense, they are the universal, the free 2-category containing a monad, and the free 2-category containing an adjunction. So a 2-functor from Mnd to $\mathcal{C}$ are in bijection with monads in $\mathcal{C}$, natural in $\mathcal{C}$, and the same, $2-\mathrm{Cat}(\mathrm{Adj}, \mathcal{C}) \cong \{\text{adjunctions in } \mathcal{C}\}$. The maps will just pick the image of a monad in Mnd.

Definition 16.1. A *homotopy coherent monad* is a simplicial functor $\mathrm{Mnd} \to K$. A *homotopy coherent adjunction* is a simplicial functor $\mathrm{Adj} \to K$.

Then if you choose K to be Cat, you get monads and adjunctions in the plain old sense.

Now I will go on to the second idea in this talk, which comes from Street, "The formal theory of monads" (1972). Let $\mathcal{C}$ be a 2-category. There is a 2-category $\mathrm{Mnd}(\mathcal{C})$ of monads in $\mathcal{C}$ whose objects are monads in $\mathcal{C}$, whose 1-cells are monad morphisms, and whose 2-cells are modifications of these things. If you want to think about what should be the morphisms between monads or adjunctions, the first thing that comes to mind is to look at simplicial natural transformations, but if you pick K to be Cat you only get strict things. So that doesn't work very well. Maybe I could also take [unintelligible]and do some stuff, but then I would not get the full morphisms, but only those where the 1-cell is an identity.

Definition 16.2. Let's say I have two monads, (B,t,μ,η) and (B',t',μ',η'). Then a *monad morphism* is a pair $B \xrightarrow{f} B'$ and a two-cell

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\swarrow"] & B' \arrow[d, "t'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

Now a *homotopy-coherent monad morphism* is a simplicial functor $\mathrm{Mnd}[1] \to K$. Here $2-\mathrm{Cat}(\mathrm{Mnd}[1], \mathcal{C}) \cong \{\text{monad morphisms in } \mathcal{C}\}$. So this is a relaxation.

Maybe now you have the objection, how do I compose such things? The second question is whether there are such things, do they exist? Are there many of them? The third question is whether these things induce morphisms on the level of the algebras? The monad morphisms are made to induce morphisms, one-cells, between the categories or objects of algebras. The sanity check would be to check whether this induces morphisms on the level of algebras.

Now the rest of the talk will be devoted to answer these questions "yes" (not the first one).

Now the plan will be to give a quick review of the 2-categories Mnd and Adj and later $\mathrm{Mnd}[n]$, the 2-category with n composable monad morphisms, and $\mathrm{Adj}[n]$.

[slides] The picture you should have in mind is

$$\begin{tikzcd} \Delta_+ \arrow[r, phantom, "\circlearrowright"] & B \arrow[r, bend left, "\Delta_{-\infty}"] & A \arrow[l, bend left, "\Delta_{+\infty}"] \arrow[r, phantom, "\circlearrowleft"] & \Delta_{-\infty,+\infty} \end{tikzcd}$$

Here + means the empty ordinal is allowed, $-\infty$ means that morphisms preserve the minimal element and $+\infty$ means that morphism preserve the maximal element.

So Riehl and Verity described a simplicial category ADJ which is isomorphic to $N_* \mathrm{Adj}$. A n-morphism is a *strictly undulating squiggle* on $n+1$ lines. [pictures]

Definition 16.3. An *adjunction morphism* is a square

$$\begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[d, "b"'] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left, "u"] \arrow[d, "a"] \\ B' \arrow[r, shift left, "f'"] \arrow[r, phantom, "\perp"] & A' \arrow[l, shift left, "u'"] \end{tikzcd}$$

so that $bu = u'a$

Proposition 16.1. $\mathrm{Mnd}[n] \xrightarrow{j} \mathrm{Adj}[n]$ *is fully faithful and injective on objects.*

I want this to be true which is why I pick this maybe slightly strange definition. [slides]

That was really 2-categorical up to now, and now maybe I can apply these ideas. As you have already noted, $\mathrm{Mnd}[\quad] : \Delta \to \mathrm{sCat}$ and that gives me a nerve functor

$$\begin{tikzcd} \mathrm{sCat} \arrow[r, phantom, "\perp"] & \mathrm{sSet} \arrow[l, shift right, "\mathbb{C}_{\mathrm{Mnd}}"'] \arrow[l, leftarrow, shift left, "\mathcal{N}_{\mathrm{Mnd}}"'] \end{tikzcd}$$

and now the monadic nerve of $\mathcal{K}$ is given by $\mathcal{N}_{\mathrm{Mnd}}(K) = \mathrm{sCat}(\mathrm{Mnd}[n], K)$ so for $\mathcal{C}$ a 2-category, $\mathcal{N}_{\mathrm{Mnd}}(\mathcal{C}) = N(\mathrm{Mnd}(\mathcal{C})|n)$ and that is [missed].

In the last ten minutes I'll try to say that this is a quasicategory under the condition that [unintelligible]is closed under weighted [missed] limits. All the examples we care about are closed under those limits because they come from model categories. So probably it's not worth working too hard about that.

So first of all, to prove the theorem I should start with a lifting theorem. If you,

Theorem 16.1. *If I am given a subcategory $\mathcal{A}$ of $\mathrm{Adj}[n]$ satisfying conditions and F is also subject to some conditions (a bit unpleasant to write) and $K \twoheadrightarrow \mathcal{L}$ is a local isofibration between $(\infty, 2)$-categories, then there exists a lift $\bar{F}$*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, hook] & K \arrow[d] \\ \mathrm{Adj}[n] \arrow[r] \arrow[ur, dotted, "\bar{F}"] & \mathcal{L} \end{tikzcd}$$

So that's not easy. Fortunately, Riehl and Verity have a theorem of the same form where $n = 0$, and looking at that you can produce this case, they did all the hard work.

The sketch of the proof is that the conditions allow you to decompose the map $\mathcal{A} \to \mathrm{Adj}[n]$ as a transfinite composite of pushouts of maps of the form

$$\begin{tikzcd} \mathbf{2}[\Lambda^k[n]] \arrow[d] & \mathbf{3}[\partial\Delta[n-1]] \arrow[d] \\ \mathbf{2}[\Lambda^k[n]] & \mathbf{3}[\Delta[n-1]] \end{tikzcd}$$

where these $\mathbf{2}[X]$ has two objects 0 and 1 with maps from 0 to 1 given by X. and $\mathbf{3}[X]$ is like

$$\begin{tikzcd} & \bullet \arrow[dr, "X"] & \\ \bullet \arrow[ur, "{\Delta[0]}"] \arrow[rr, "{\Delta[0] \bigstar X}"'] & & \bullet \end{tikzcd}$$

and now the lifting for $\mathbf{2}$ gives you lifting for free and for $\mathbf{3}$ the lifting comes from the universal property of the colimit.

Then using

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & N_{\mathrm{Mnd}}(K) \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

we get something like

$$\begin{tikzcd} \mathrm{Mnd}^k[n] \coloneqq \mathbb{C}_{\mathrm{Mnd}}(\Lambda^k[n]) \arrow[r] \arrow[d] & K \\ \mathrm{Mnd}[n] = \mathbb{C}_{\mathrm{Mnd}}(\Delta[n]) \arrow[ur] & \end{tikzcd}$$

and so to get

$$\begin{tikzcd} \mathrm{Mnd}^k[n] \arrow[r] \arrow[d] & K \\ \mathrm{Adj}^k[n] \arrow[ur, dotted] & \end{tikzcd}$$

I put this in the following:

$$\begin{tikzcd} & \mathrm{Mnd}^k[n] \arrow[r] \arrow[dl] \arrow[d] & K \\ \mathrm{Mnd}[n] \arrow[dr] & \mathrm{Adj}^k[n] \arrow[ur, dotted] \arrow[d, dotted] & \\ & \mathrm{Adj}[n] \arrow[uur, dotted] & \end{tikzcd}$$

And let's see, given a morphism

$$\begin{tikzcd} \mathrm{Mnd}[1] \arrow[r, "M"] \arrow[d, hook] & K \\ \mathrm{Adj}[1] \arrow[ur, dotted, "A"'] & \end{tikzcd}$$

I can get

$$\begin{tikzcd} \mathbb{A}(B,1) \arrow[r, shift left] \arrow[d] & \mathbb{A}(A,1) \cong \mathrm{Alg}(\mathbb{T}) \arrow[l, shift left] \arrow[d] \\ \mathbb{A}(B,0) \arrow[r, shift left] & \mathbb{A}(A,0) \cong \mathrm{Alg}(\mathbb{S}) \arrow[l, shift left] \end{tikzcd}$$

and given

$$\begin{tikzcd} B \arrow[d] & \mathrm{Alg}(\mathbb{T}) \arrow[l] \arrow[d, "f"] \arrow[dl, phantom, "\cong"] \\ B' & \mathrm{Alg}(\mathbb{T}') \arrow[l] \end{tikzcd}$$

there exists a morphism $\mathrm{Mnd}[1] \to K$ so that the induced morphism $\mathrm{Alg}(\mathbb{T}) \to \mathrm{Alg}(\mathbb{T}')$ is equivalent to f.

17. Nora Ganter: Not even wrong!

Let me start by thanking the organizers, this is promising to be a great meeting. This is a report on joint work with Matthew Ando. I've not read the book with this title, but this is a criticism of string theory. I'll try to offer my five cents about the discussion about this. The first thing to start with is this.

17.1. What makes a model of reality right or wrong? If the criticism of string theory is not even wrong, we should have an idea of what it means to be right or wrong. Let's start with an example, namely planetary movements. We could think about the earth [picture]. We have here the sun and the moon, I want to draw the arrow of the direction of the sun, and there are stars, and to be fair, the model is a little more sophisticated, Anaximander and Anaximenes and Ptolemy became famous for these. These were predictive models. The sun and the moon do not go around, they go around on the side, the stars are on concentric spheres.

Of course we had the heliocentric model, with the stars including the earth going around in some sort of orbits. This is Copernicus, Kepler, Galileo, and so on. We could take this discussion further, I don't want to dive too deeply in this. We could say we only really care about relative movement, both of these are predictive. So why do people prefer the heliocentric? It's not about right or wrong.

The guiding principles are elegance and simplicity. Especially in the second point, the heliocentric model won over the geocentric model.

Now here's where string theory in its current state has to hanswer to some critical question. We can talk about a grand elegant unifying theory that will make everything so simple. Then we go homo and parameterize our boundaries. Maybe here in elegance, we should not parameterize our boundaries. A point can be made that that is not an elegant thing. When my daughter asks what's the smallest thing in the world, I have to start explaining about (∞, n)-groupoids and so, maybe also simplicity could be improved.

I'd like to emphasize that this work is in progress. This is very much still in progress. Maybe I'll add a question mark: "Not even wrong?"

So what's the scenario we're trying to model.

17.2. Extended field theories with target space. Let's start by drawing the target space [picture], M a smooth manifold. Inside M we might have a bunch of particles, a finite set of charged particles, x_1^+, x_2^-, x_3^+, and x_4^-, and now these particles evolve. So for instance x_1 and x_2 might start moving like this. We orient our strings and then they cancel out when they meet. So x_3 could go from here to some other point y_1 and because it didn't cancel it stays positively charged, and the same here, the x_4^- can transform into a negatively charged y_2. We also myight have a scenario where something materializes out of nowhere and vanishes again.

Now imagine the second way to go from (x_1, x_2, x_3, x_4) to (y_1, y_2). We could instead have x_1 evolve to y_1 and x_2 into y_2, and [pictures].

You can guess what is at the next level. You could now go from blue to green and have an oriented membrane. The orientation should be mis-matched with blue, which is the input, and matched with the output green.

Let me first of all tell you what I mean about all of these things. Let's start by saying what is a d-dimensional world-sheet in M. This would be a map σ from a compact oriented smooth d-dimensional manifold Σ (possibly ith boundary) to M. I want to consider this up to an equivalence relation, up to thin bordism with corners. What does that mean? A bordism between Σ_1 and Σ_2 and I want something 2-dimensional with boundary this stuff but also maybe some other boundary [pictures].

The differential has nowhere full rank; nor should β restricted to its free boundary. An example is this. The point x_1^+ and x_2^+ are thinly bordant if and only if these two particles are in the same location. Or x_1^+ and x_2^- are together thinly bordant to zero if they coincide. This cancellation is very important. What else can you do with thin bordism? Reparameterization of paths. You think of the image. It's closer to our imagination to reparameterize. Let's go back to this and do more examples in a second. I want to get to our category.

17.3. The thin bordism chain complex. We have cancellation, a strict monoidal structure with disjoint union and inverses. So we get a chain complex. So $\mathrm{Bor}_\bullet M$ is a chain complex in topological Abelian groups; $\mathrm{Bor}_d M$ is the set of d-dimensional worldsheets in M in the sense that Dave was saying earlier, up to thin bordism. Her + is the disjoint union and 0 is the empty set.

Then the boundary is boundary ∂. Now I can [unintelligible]in degree d and use my globular Dold–Kan correspondence and $\mathrm{Bor}_\bullet^{(d)}(M) \leftrightarrow \mathscr{B}\mathrm{or}_d(M)$, a strict topological Picard d-groupoid.

Let's unravel this. The objects on the right side are elements of $\mathrm{Bor}_0 M$, black or purple things, and then the blue and green are 1-morphisms, so $1\,\mathrm{Hom}(x, y) = \{\gamma \in \mathrm{Bor}_1 M | \partial_\gamma = y - x\}$ where $-$ is disjoint union with the particle of the opposite charge.

Then $2\,\mathrm{Hom}(\gamma, \beta) = \{\sigma \in \mathrm{Bor}_2 M | \partial\sigma = \beta - \gamma\}$. Et cetera. The monoidal product is disjoint union and composition, if I have a path from x to y and a second path to z, now we can do the same thing, taking the path and forgetting y in the middle, up to thin bordism anyway. Under normal circumstances you'd be screaming that this is not smooth, but because of reparameterization this is smooth.

I should say again what I mean about an extended field theory with target space. The particle theory wants to forget that the 2-morphisms ever existed, just have particles and the strings connecting them, go from $\mathscr{B}\mathrm{or}_1(M)$ with $\sqcup$ and $\varnothing$ to $(\mathrm{Vect}_{\mathbb{C}}, \otimes, \mathbb{C})$, and because this is a Picard groupoid, you might as well think of this as lines in $\mathbb{C}$. So a line bundle with connection is an example of what people have considerid. You want to say that this is a symmetric monoidal functor, continuous in an appropriate sense.

The next level up is *open-closed string theory*. Here you want to say, start with $\mathrm{Bor}_2(M)$ with disjoint union and $\varnothing$ and assign to it a 2-vector space, with $\boxtimes$, and $\mathrm{Vect}_{\mathbb{C}}$. So a bundle gerbe with connection will give you that formalism. The source is Picard so everything should be invertible so you should really land in 2-lines. There is a categorical parallel transport and higher dimensional parallel transport.

Okay. We want now to get into the part that makes people nervous that we're actually paying attention, what do we mean by "continuous in the appropriate sense" and even topological? So we replace our lines with $\mathbb{B}^d\mathbb{C}^\times$, so now we have no trouble to say what our topology is. We want to throw this into our globular Dold–Kan correspondence, this is a strict topological Picard d-groupoid, which correpsonds to $\mathbb{C}^\times[d]$. So rather than talking about anafunctors, I'll work in chain complexes. I was happy to see Simona and possibly David on the list of participants because, let's put the definition in writing,

Definition 17.1. An *extended d-dimensional field theory with target space M* is a span

$$\begin{tikzcd} & P_\bullet \arrow[dl, "\sim"'] \arrow[dr] & \\ \mathrm{Bor}_\bullet^{(d)} & & \mathbb{C}^\times[d] \end{tikzcd}$$

I haven't talked about where this lives, so this isn't a complete definition, but let's steer clear of this.

[some discussion]

I do want to talk about equivalence, so I should talk about the cone of that arrow. In homological algebra, it has been studied, for the first time in my career I, when I was a student we were banned from reading this counterexample book and now for the first time in my career I'm thinking about what it means to be T1 but not Lindelöf or whatever. Actually this has been studied by Morris who is here in Ballarat. So in order to make sense of this, I should tell you about exact sequences.

I want to look at a sequence

Definition 17.2.

$$0 \to A \to B \to C \to 0$$

is a *short exact sequence* in $\mathrm{Top\,Ab}$ if (Morris and collaborators asked for a global section, that's too far) if there exists a (continuous) local section $C \to B$ around 0. In orther words, if B is a principal A-bundle over C.

The digression I did was motivation for doing this exact structure.

So now what do you want to put in $P_\bullet$? I'm thinking about projective resolutions but this is familiar to many and too complicated for those who haven't seen it.

What do projectives look like? Well under what circumstances is $\mathbb{Z}U$ "projective," meaning with respect to epimorphisms of this form:

$$\begin{tikzcd}
& & A \arrow[d, tail] \\
& & B \arrow[d, two heads, "\text{principal } A\text{-bundle}"] \\
U \arrow[r, "\eta"] \arrow[urr, dotted, "\exists?"] & \mathbb{Z}U \arrow[r] \arrow[ur, "\exists?"] & C
\end{tikzcd}$$

So we want U to be homotopically discrete, paracompact, Hausdorff.

So [unintelligible]$A \twoheadleftarrow U$ and $A \twoheadleftarrow \mathbb{Z}U \leftarrowtail K$

and then [missed]

If you look at just, if you fix diffeomorphism class of a world sheet Σ, then you can look at the function space M^{Σ} and cover this with open sets written by [unintelligible], and this will map $\mathbb{Z}M^{\Sigma} \to \mathrm{Bor}_d M$. So we want

$$\begin{tikzcd}
\text{Čech} \arrow[r] \arrow[d] & \text{Cartan–Eilenberg} \arrow[d] \\
\mathbb{Z}M^{\Sigma} \arrow[r] & \mathrm{Bor}_d M
\end{tikzcd}$$

So I want the right hand vertical arrow to be my $P_\bullet \to \mathrm{Bor}_\bullet^{(d)}$. [unintelligible]not sure if this is projective in my structure. I think I will stop here.

18. Daniela Egas Santander: Sullivan diagrams and homological stability

This is joint work with Felix Boes.

I want to describe some concrete computations on a graph complex, and let me start by putting this into context, why I would be talking about this at this conference.

Context. S is an oriented surface, with boundary, with genus, and parts of the boundary are parameterized, there's a marked point, and we say $\partial_P S$ is the parameterized boundary of S, and I'll always ask that this is nonempty. If you're an algebraic topologist, you might be interested in $\mathrm{Diff}^+(S, \partial_p S)$, maybe you want to study surface bundles with fiber S, and you want the homotopy type of its classifying space. Then if we have this one parameterized boundary component, then this is equivalent to $M(S)$, the unmarked boundary should be thought of as punctures. We strangely know a lot and very little about the homology of these groups. These things have homological stability; when the genus is big enough, the homology groups are independent of the surface. The stable homology, which is the homology of the infinite genus surface, we know well, this is Madsen–Tillman–Weiss and later Galatius. In the unstable range, these homology groups still appear to be rather mysterious. One strategy is called combinatorial reduction. This means I take this complicated geometric object and model it by a combinatorial object that I can handle concretely. This is not really a "reduction" because there's a combinatorial explosion. Some computations have been done for small genus and small number of boundary components, but there are some computations of Harer and Zagier, they compute $\chi(\mathcal{M}(S^1_g))$ and I tell you this because there are a bunch more classes, this tells you, that we don't have access to.

Why would we want to look at these combinatorial models? They appear in string topology. So $LM = \mathrm{Map}(S^1, M)$, then string topology studies $H_*(LM)$ or maybe $H^*(LM)$, some maps $H_*(LM)^{\otimes p} \to H_*(LM)^{\otimes q}$, this was explained by Nora with different targets, we want this to assemble to a topological conformal field theory. This goes from $D\,\mathrm{Cob}$, with disjoint union to $(\mathrm{Ch}, \otimes)$. So I'll have objects natural numbers and think of addition as disjoint union of circles, and the maps $\mathrm{Hom}(p, q)$, are all topological types of cobordisms from p circles to q circles, and this is $\bigoplus C_*(\mathcal{M}(S_{p,q}))$ with monoidal structure by disjoint union and with composition by gluing cobordisms at the boundary. In order for this to make sense you have to choose the right chain model for moduli space but it can be done. Then you see if you look at functors like this that send the circle to $H_*(LM)$, you get things that look like these string topology operations. This is actually a PROP, sometimes called the Segal PROP. The work of certain people say that whenever you take such a TCFT which hits the homology of the free loop space of the manifold, this factors through $\overline{D\,\mathrm{Cob}}, \sqcup$, where the objects are the same, but the morphisms are $\bigoplus_S$ Graph complexes. These are born combinatorially. The question is, what are the underlying spaces of these graph complexes. We should know what we're parameterizing this over. We should know something about the homotopy type of the underlying spaces.

These things also form a PROP, and you're asking about the homotopy type of the bicollections of this PROP.

I want to tell you about one such complex, which is the complex of Sullivan diagrams. I'll describe exactly what I mean by them, this is the first approach we have, and so I hope to say that we can say a lot of things and use them to evaluate what people have done about string topology.

(1) What I'll describe today is what Tradler–Zenailian, Wahl–Westerland, Kaufmann (in the closed part), what Ben Ward will use ($g = 0$, + conditions). There is something done by Cohen–Godin which is definitely different and something else done by Drummond-Cole–Poirier–Rounds which looks suspiciously the same but we don't know.
(2) Joint with Kuper, we showed that the chain complex of Sullivan diagrams is the cellular complex of the harmonic compactification of moduli space (this compactification is due to Bödigheimer). So I'll say something combinatorial but this has geometric meaning. This is why some people sometimes say that string topology "compactifies."
(3) There's a third disclaimer: if one chooses the right chains for moduli space, a specific combinatorial reduction, then one can write a projection onto Sullivan diagrams. The hope is that we can use this to detect unstable classes. Even though there's a combinatorial explosion in both, actually SD is much smaller. If we have some guesses of unstable classes, we might be able to find things. The hope is to detect stable classes.

I hope that now I convinced you that it's worth looking at some combinatorial object that we haven't looked at before. Now let me define things properly.

Definition 18.1. A *fat graph* or *ribbon graph* is a graph together with a cyclic ordering of the edges (or flags) incident at each vertex.

Let's make some examples [pictures].

They are called fat or ribbon because from them you can construct a surface by thickening the graph. You thicken the edges to strips and the vertices to disks and then glue together according to the combinatorial structure given by the cyclic ordering. [picture]

The topological type, let me tell you in case you haven't seen it before, is completely determined by the graph, because you can retract to the graph. The only thing you have to know is how many boundary components your surface has, which you can read off from the graph [pictures].

The last thing you need to know, if Γ has a leaf, something like this or that [pictures], we can define a subgraph of Γ given by tracing the boundary cycle. [pictures].

Now this is a definition due to Godin, quite insightful

Definition 18.2. (Godin) A *p-admissible* fatgraph is a fatgraph with leaves (which may or may not have labels), at least p of which are labeled, a bijection to $\{\ell_1, \dots, \ell_p, \dots\}$ so that $\Gamma_{\ell_1}, \dots \Gamma_{\ell_p}$ are disjoint embedded circles in Γ.

[pictures]. In general an admissible thing looks like this: [pictures]. These are enough to model moduli space.

Definition 18.3. (Sullivan) We say that Γ_1 and Γ_2, p-admissible, $\Gamma_1 \sim \Gamma_2$, if Γ_1 can be obtained from Γ_2 by slides away from ground circles. [pictures].

A *p-Sullivan diagram* Σ is an equivalence class.

Now you can imagine how to build a space of these guys.

The space $1-\mathrm{SD}$ is a (semi)-simplicial set. The K-simplices are Sullivan diagrams Σ with $k+1$ edges on the ground circle. The faces are obtained by contracting edges.

Anyhow this is the space, and well, you can convince yourself that even though we are taking equivalence classes, when you have a Sullivan diagram, you have a well-defined topological surface, and $1 - \mathrm{SD}$ splits into connected components of topological type S. The surface S might have decorations in the boundary or not. Let me describe certain components:

- SD_g^m, which have genus g, $m+1$ boundary components, and only one marked and labelled.
- $\mathrm{SD}_{g,m}$, which has one labelled boundary and m unlabeled but parameterized boundary, and
- $\widetilde{SD}_{g,m}$, which has $m+1$ marked and labelled boundary.

This looks combinatorial, but it keeps coming up. One can do this for $p \geq 2$, and then you'll have a multisimplicial set indexed by the number of circles you have.

Computations. The homology we get by plugging this into a computer is really suspicious.

Theorem 18.1.

$$\pi_*(\mathrm{SD}_{g,m}) = \pi_*(\mathrm{SD}_g^m) = \pi_*(\widetilde{\mathrm{SD}}_{g,m}) = 0$$

for $* < m - 2$ The range is sharp for $g = 0$. We have a similar thing for genus. You have the one special boundary component, then $S_{g,1}^m$, you can look at the diffeomorphism group and look at what happens when you glue the torus minus two disks, and can extend by the identity on the glued torus because my diffeomorphism is the identity on the boundary. This gives me

$$B\operatorname{Diff}^+(S^m_{g,1}) \to B\operatorname{Diff}^+(S^m_{g+1,1})$$

and I have something similar between $\mathrm{SD}(g,m)$ and $\mathrm{SD}(g+1,m)$, and the stabilization maps behave well.

What we can also say

Theorem 18.2. *When $m > 2$, then the stabliization map is k-connected, where $k = g + m - 2$, which is stronger than homological stability.*

When $m = 2$ these are not simply connected, just connected, but $\pi_1(\mathrm{SD}(g,m)) = \begin{cases} \mathbb{Z} & g = 0 \\ \mathbb{Z}/2 & g \neq 0 \end{cases}$ and the stabilization map sends the generator to the generator.

Why do I want to mention this? These actually come from $H_1(B\operatorname{Diff}(S))$. We have these quotient maps and [unintelligible], and in the diffeomorphism group this is exchanging the punctures by a half-twist.

Showing these are nontrivial and generators in Sullivan diagrams is easy. So finding something above this would be pretty easy, and you'd know it was nontrivial.

We could use the PROP structure of this, the circles are to define composition. Ben will tell us about composition, insertion one on top of the other, these compositions, can we build infinite families coming from below like this?

I think that's all I wanted to tell you. I was very quick.

Let me tell you one last comment. Gabriel asked if you showed this on homology, we show that the matching is cyclic so everything should die, you should be able to write a flow to see that things are contractible, using Ralph's model and Hatcher type surgery arguments. We're looking at a subcomplex, we should be able to do this on the space level, and use that to say the homotopy type for higher p. The bigger p is, the closer we get to moduli space. If we look at the surface from p boundaries to one boundary, and genus zero, then $\mathrm{SD}_{0,p-1}$, this is quasi-isomorphic to $\mathcal{M}_{0,p+1}$, and these things, the moduli space can't tell the difference between incoming and outgoing boundary components, but these spaces change drastically, saying what's in and what's out. It's interesting to understand this filtration of the moduli space. Okay, now I'm done.

19. Sinan Yalin: Moduli spaces of bialgebras, higher Hochschild cohomology and formality

Thanks to the organizers and to the kangaroos.

My talk will be in three parts.

(1) Algebraic structures
(2) moduli spaces of such structures
(3) Applications to deformation theory problems related to bialgebras and E_n algebras.

19.1. Algebraic structures. I'll start by briefly recalling the definition of a PROP. I'll work in $\mathbb{Z}$-graded chain complexes over a field which I will suppose for simplicity to be characteristic zero.

Definition 19.1. A *dg PROP* is a collection $P = \{P(m,n) \in \mathrm{Ch}\}$ with

- actions of Σ_m on the left and Σ_n on the right,
- $\circ_V : P(k,n) \otimes P(m,k) \to P(m,n)$ which is supposed to be composition, and
- $\circ_h : P(m_1,n_1) \otimes P(m_2,n_2) \to P(m_1+m_2, n_1+n_2)$, like concatenation.
- There is an exchange law relating $\circ_H$ and $\circ_V$.
- There are also maps corresponding to the identity operation $\mathbf{k} \to P(n,n)$ neutral for $\circ_V$.

An example is the endomorphism PROP $\mathrm{End}_X(m,n) = \mathrm{Hom}(X^{\otimes m}, X^{\otimes n})$ for $X \in \mathrm{Ch}$.

Definition 19.2. A *P-algebra structure* on X is a PROP morphism $P \to \mathrm{End}_X$. That is, $P(m,n) \to \mathrm{Hom}(X^{\otimes m}, X^{\otimes n})$. These are equivariant and compatible with $\circ_H$ and $\circ_V$

Some examples are associative and coassociative bialgebras, which is free on a two to one product and one to two coproduct subject to associativity, coassociativity, and the bialgebra relation which says that bialgebras are algebras in coalgebras or vice versa.

These kinds of structure occur very often in representation theory or quantum groups for instance.

Another interesting example is Lie bialgebras, this is a Lie algebra and Lie coalgebra structure which are compatible in a certain way. This is freely generated by a bracket and cobracket, and mod out by some relations, Jacobi and coJacobi along with the compatibility or Drinfeld 5-term cocycle relation. I won't write it.

These structures appeared in differential geometry and mathematical physics. When people are interested in tangent spaces of Poisson Lie groups, the tangent space of a Lie group is a Lie algebra, and the tangent space of a Poisson–Lie group is a Lie bialgebra. These also came up in Drinfeld's work on quantization. About ten years later these showed up in algebraic topology, where the composition of the bracket and cobracket is trivial. This appeared in the word of Goldman and Turaev, studying free loops on surfaces and this has been generalized to equivariant string topology by Chas and Sullivan, namely, the S^1-equivariant homology of LM, this has this kind of structure.

Theorem 19.1. *(Fresse 2010) Differential grade PROPs form a model category with fibrations and weak equivalences aritywise.*

This is analagous to operads. When you have this, a natural way to define homotopy versions is as follows:

Definition 19.3. A *homotopy P-algebra* is a P_∞-algebra where $P_\infty \xrightarrow{\sim} P$ is a cofibrant resolution of P

There are several motivations and examples where such things appear, such as

- the A_∞ structure on the cohomology of a space, given by higher Massey products or
- the E_∞ structures which appear in different places in homology, such as in classifying homotopy types in the word of Sullivan in rational homotopy theory or Mandell over finite fields.
- The L_∞ structures in deformation quantization of Poisson manifolds à la Kontsevich.

A priori the notion is dependent on the choice of resolution. We'd like the associated categories of algebras to be equivalent.

Theorem 19.2. *(Y., 2013) If* $\varphi : P_\infty \xrightarrow{Q}{}_\infty$*, then there is an equivalence* φ^* *of* $(\infty, 1)$*-categories between* $(Q_\infty\text{–alg}, \textit{quasi-isomorphisms}) \to (P_\infty\text{–alg}, \textit{quasi-isomoprhism})$*.*

There is no model category here on algebras over a PROP so this requires much more work than for algebras over operads.

19.2. Moduli spaces. The geometric idea is that if you have a collection of objects or structures and a relation between them, you want to associate to this a space $\mathcal{M}$ whose points are your objects. You want the connected components to give you the equivalence classes of such objects. But you want something more, you want infinitesimal deformations of your moduli space. You want the deformation theory of the points and these are, in a sense I'll be precise about after this, you can call this a moduli problem. These problems turn out to be controlled by algebraic data, by a differential graded Lie algebra, or by their homotopy version, an L_∞ algebra.

This story goes back to Deligne, Drinfeld, Feigin, Kontsevich, Hinich, Manetti, and the final form of such a principle was given by Pridham and Lurie.

To define this you need some tangent properties, some geometric structure. So geometric structure is needed to get a tangent space.

Now the question is how to formalize properly these ideas and construct a moduli space encoding the structures we are interested in.

So a possible formalization is via derived algebraic geometry, which is a "homotopical perturbation or thickening of algebraic geometry."

I'll only say a few words. In classical algebraic geometry, affine schemes are the opposite category of commutative algebras. Then schemes are sheaves, functors from commutative algebras to sets. Then if you want to study problems where you have automorphisms, you work with stacks, working with sheaves valued in groupoids.

Now in the derived setting what people do is homotopically thicken these definitions.

Your derived affine schemes is the opposite category of commutative differential graded (nonpositively graded) algebras. Then your stacks are simplicial sheaves up to homotopy from nonpositively graded commutative differential graded algebras to simplicial sets.

So here in sSet you can study moduli problems for which you have weak equivalences between your objects.

The fact of adding the differential at the inputs allows us to encode derived data, a conceptual way of discussing nontransversal intersection, [unintelligible], lots of things, and this setting is very interesting to me. This is where we want to live to find our moduli spaces.

Definition 19.4. $P_\infty\{X\}$ is $\mathrm{Map}(P_\infty, \mathrm{End}_X)$. You can take a model that looks like $\mathrm{Mor}_{PROP}(P_\infty, \mathrm{End}_X \otimes \Omega_\bullet)$ where $(\mathrm{End}_X \otimes \Omega_\bullet)(n,m) = \mathrm{End}_X(n,m) \otimes \Omega_\bullet$ and $\Omega_\bullet$ is the Sullivan algebra of the standard simplex (this is a simplicial dga).

This has many properties, like

- $P_\infty\{X\}$ is a Kan complex.
- $P_\infty\{X\} = [P_\infty, \mathrm{End}_X]$

- $P_\infty \xrightarrow{\sim} Q_\infty$ implies that $Q_\infty\{X\} \xrightarrow{\sim} P_\infty\{X\}$.

Theorem 19.3. *(Y., 2015)*

(1) *The functor $P_\infty\{X\}$ from nonpositively graded commutative differential graded algebras to simplicial sets which takes R to* $\mathrm{Map}(P_\infty, \mathrm{End}_X \otimes R)$ *is a stack.*

(2) *For a given $\varphi : P_\infty \to \mathrm{End}_X$, the tangent complex of $P_\infty\{X\}$ over φ satisfies the following:*

$$H^*(\mathbb{T}_varphi[-1]) \cong H^* \underbrace{\mathrm{Def}(\varphi)}_{\text{explicit } L_\infty\text{-algebra}}$$

where this thing is given in terms of deformations of PROPs.

Remark 1. (1) So not only can we explicitly compute this, but the formal moduli problem of deformations of φ is given by the following

$$\mathrm{hofib}_\varphi(\underline{P_\infty\{X\}}(R) \to \underline{P_\infty\{X\}}(\mathbf{k}))$$

for the augmentation $R \to \mathbf{k}$, the "space of R-deformations" of φ.

So I wanted to say that $\mathrm{Def}(\varphi)$ is the tangent L_∞ algebra of this moduli problem in the sense of Lurie–Pridham, Deligne, and so on.

(2) There is an associated obstruction theory, like obstructions to infinitesimal deformations, and actually if you try to work in an underived setting, you don't see these obstruction groups, which really only live in this derived world.

Up to a degree shift and so on, you recover the theories you know, Hochschild, Chevalley–Eilenberg, Gerstenhaber–Schack, et cetera. So it's really the right kind of thing you want to consider.

19.3. Bialgebras vs E_n-algebras.

I'll start with some recollections and properties.

19.3.1. *E_n-algebras.* . There is a sequence of topological operads

$$E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_n \hookrightarrow \cdots \hookrightarrow E_\infty$$

which are "more and more homotopy commutative." So topological E_n-algebras in spaces model n-iterated loop spaces. We'll be interested in differential graded E_n algebras, which are algebras over C_*E_n. These structures have many interesting properties.

Theorem 19.4. (1) $H_*E_1 \cong \mathrm{Ass}$

(2) *$H_*E_n \cong P_n$, the operad of n-Poisson algebras, meaning you have a commutative and associative algebra sturcture and a Poisson bracket in shifted degree $1-n$.*

(3) *For $n \geq 2$, $C_*E_n \xrightarrow{\sim} P_n$ (as an operad) (Tamarkin $n = 2, \mathbb{Q}$, Kontsevich, Lambrechts, [unintelligible], Fresse–Willwacher, others)*

I'm interested in the cohomology theory of such structures. So what's the cohomology theory of E_n algebras?

Definition 19.5. Let A be an E_n-algebra. The *higher Hochschild or E_n-Hochschild complex* of A is given by

$$CH^*_{E_n}(A, A) = R\operatorname{Hom}_{\mathrm{Mod}_A^{E_n}}(A, A),$$

the E_n-A-module morphisms.

Remark 19.1. For $n = 1$ and A associative, you see the usual Hochschild complex. It turns out that these higher Hochschild complexes have a nice sturcture on them controlling this deformation theory.

Theorem 19.5. *(Francis, Lurie, Ginot–Tradler–Zeinalian) $CH^*_{E_n}(A, A)$ is an E^*_{n+1}-algebra.*

This is like taking the derived center; this can be made precise but I don't have time. This is also known as the *higher Deligne conjecture*.

The deformation complex has all this nice structure on it. Let me explain the link to bialgebras and why we care.

19.3.2. *Link to bialgebras.* There have been several motivations to study these relations.

- attempts to relate categories of bialgebras and E_2-algebras and their respective deformation theories; if you take the cobar on a bialgebra you get an E_2-algebra. People wondered for a long time what the relationship was.
- Motivation from Gerstenhaber–Schack, who introduced a cohomology theory for bialgebras (in quantum group theory), $C^*_{GS}(B; B)$, which has a Hochschild and a coHochschild part. They remarked a few things, like
  - the fact that this complex has a cup product like the Hochschild complex for an associative algebra.
  - They expected a (shifted) homotopy Lie algebra that was compatible with this cup product. These should gather into an E_3-algebra
- Kontsevich (around 2000) conjectured (in working on deformations of Poisson manifolds) that if you look at deformations of the symmetric bialgebra, which is an algebra but also a coalgebra, and he conjectured that this has an E_3-algebra structure which is formal. This $H^*_{GS}(\mathrm{Sym}(V))$ which should imply Drinfeld and Etingof–Kazhdan for [unintelligible]quantum groups.

With work in collaboration with Gregory Ginot we solved all these questions.

Theorem 19.6. *(Ginot–Y.)*

- *There is a fully faithful conservative ∞-functor from E_1-algebras augmented as algebras and conilpotent as a coalgebra to augmented E_2-algebras. The categories are not equivalent but rather we embed one into the other, some kind of modified cobar construction.*
- *This embedding lets you study the cotangent deformations; if B is a biagebra, then the moduli problem $\mathrm{Bialg}_\infty\{B\} \cong E_2\{\widetilde{\Omega}B\}$ so the homotopy theory is entirely controlled by E_2 algebras.*
- *$C^*_{GS}(B; B) \xrightarrow{\sim}_{E_3} CH_{E_2}(\widetilde{\Omega}B)$. So you can study deformation theory of bialgebras by way of higher Hochschild homology.*
- *We solved Kontsevich's problem in a more general way, this gives a generalization of Drinfeld and Etingof–Kazhdan quantization theory.*

20. June 16: Chris Rogers: What do homotopy algebras form?

Thanks to Marcy and Phil for letting me participate in this nice workshop in conference. The goal of this talk is to summarize the results of three papers. Two are joint work with V. Dolgushev and the third with V. Dolgushev and A. Hoffnung. This will be a story about an enrichment of a category of homotopy algebras of a certain type, which should tell you something about the homotopy theroy of these homotopy algebras. This is reminiscient of the situation with chain complexes, because you can enrich them over themselves and then with Dold–Kan you move to a simplicial version. This is like a non-Abelian version of this story.

The things that go into these papers are very well-known; our contribution is taking things that are well-known and putting them together in the right way. This will be very explicit, too.

We'll work with with homotopy algebra structures on differential graded vector spaces, $\mathbb{Z}$-graded, characteristic zero (I don't know the analog, if one exists, in positive characteristic; I'll always be passing between invariants and coinvariants $V_G \xrightarrow{\cong} V^G$, $v \mapsto \sum_{g\in G} gv$). This works for $\mathrm{Cobar}(\mathcal{C})$ algebras, an algebra over the cobar of a coaugmented dg cooperad satisfying mild technical conditions, e.g., if we have the situation where $\mathcal{C}(0) = 0$ and $\mathcal{C}(1) = \mathbf{k}$, this will work. So A_∞, L_∞, C_∞, Ger_∞.

Let me remind people how A_∞ algebras work, with special emphasis on A_∞ morphisms. An A_∞ structure on (A, ∂) will be a codifferential $(\mathit{boundary} + Q)$ of degree 1 on $\mathcal{C}(A) \coloneqq (\bar{T}(s^{-1}A), \Delta)$ and $Q|_{s^{-1}A} = 0$.

There's a well-known fact that Q is determined by its restriction, the projection of this codifferential to the cogenerators $pr_{s^{-1}A} \circ Q$. What about A_∞ morphisms? I'll write these as $A_1 \xrightarrow{F} A_2$, this is a degree 0 differential graded coalgebra morphism $F : \mathcal{C}(A_1) \to \mathcal{C}(A_2)$. These are *not* strict maps. So many people know, these maps are uniquely determined by the projection to the cogenerators $F' \coloneqq \mathrm{pr}_{A_2} \circ F : \mathcal{C}(A_1) \to s^{-1}A_2$.

Let me remind you of the formula to recover the full coalgebra morphism from the projection:

$$F(x) = \sum_n \underbrace{F' \otimes \cdots \otimes F'} \circ \Delta_{n-1}$$

where $\Delta_n : \mathcal{C}(A) \to \mathcal{C}(A)^{\otimes n+1}$.

What's the right notion of weak equivalence? It's the A_∞ quasi-isomorphisms. Say that F is an A_∞ quasi-isomorphism if $F'|_{s^{-1}A_1} : (A_1, \partial) \to (A_2, \partial)$ is a quasi-isomorphism in differential graded vector spaces.

Let's say something about convolution algebras. This is something that's been around for a while, various people cite various sources for its origins. This was in Fregier, Markl, Yau, and at the same time maybe Merkulov–Vallette, around 2009.

Let A_1 and A_2 be A_∞ algebras. I'll look at a cochain complex $\mathrm{Conv}(A_1, A_2)$. As a graded vector space it will be $L = \mathrm{Hom}_{\mathbf{k}}(\mathcal{C}(A_1), s^{-1}A_2)$, and this has a differential given by the differential on A_2 along with the full codifferential $(\partial_1 + Q_1)$ on $\mathcal{C}(A_1)$.

This also has more structure, degree 1 "brackets" $\{\ldots\}_k$, which are k-ary maps that go from $\bar{S}^k(L) \to L$ defined using the A_∞ structure of A_2 by

$$\{f_1, \ldots, f_k\}(x) = \sum_\sigma \mathrm{pr}_{A_2} Q_2(f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k)}) \circ \Delta_{k-1}$$

These brackets satisfy L_∞-like relations. They give this convolution a stucture of a (shifted) L_∞ structure. There is not just this L_∞ structure but also a very nice filtration on this complex, $\mathcal{F}_n L = \{f : \mathcal{C}(A_1) \to s^{-1}A_2 | f(a_1, \ldots, a_k) = 0 \forall k < n\}$. This is a decreasing filtration, where the first piece is the whole complex, $L = \mathcal{F}_1 L \supset \mathcal{F}_2 \supset \cdots$ so that $L = \lim_{\leftarrow} L/\mathcal{F}_i L$. The last cool thing about this is that the brackets are compatible with the filtration in a nice way, $\{\mathcal{F}_{i_1} L, \ldots \mathcal{F}_{i_k} L\} \subset \mathcal{F}_{i_1+\cdots+i_k} L$.

What does this do for us? The punchline, which is well-known to experts, is that this $\mathrm{Conv}(A_1, A_2)$ is a filtered (complete) shifted L_∞ algebra.

If L is a shifted L_∞ algebra, I can look at a degree zero element $\alpha \in L^0$ and define the curvature of α to be

$$\partial\alpha + \sum_{k\geq 1} \frac{1}{k!}\{\alpha, \ldots, \alpha\}_k$$

This is called the Maurer–Cartan equation, I can defined $MC(L)$ to be the degree zero elements of zero curvature. Usually you're used to thinking of this as degree 1 but, you'll see why I put them in degree zero and why I'm working in the shifted case.

The cool fact is that when I look at the convolution elements of this L_∞ algebra, if I look at $Hom_{A_\infty}(A_1, A_2)$, this is bijection with $MC(\mathrm{Conv}(A_1, A_2))$, sending a coalgebra morphism to its projection $F \mapsto \mathrm{pr}_{A_2} \circ F$.

I think it's fair to say that people have asked, seeing this, whether the category of A_∞ algebras is enriched over things like this, and then MC turns out to be a functor and by base change or enrichment you can recover A_∞ algebras.

First we'd like to talk about a symmetric monoidal structure on shifted L_∞ algebras. So let's talk about a monoidal category of these algebras. The objects of the category will be shifted L_∞ algebras and the morphisms will be the usual thing you think of when you think of L_∞ algebras, but continuous with respect to the filtration.

This means that $F : \bar{S}(L_1) \to L_2$; I get a filtration on $\bar{S}(L_1)$, the obvious thing, and for all n there exists an m such that $F(\mathcal{F}_m \bar{S}(L_1)) \subset \mathcal{F}_n L_2)$.

So what was I saying? Those are our morphisms, and the monoidal structure is just direct sum. If I have two such shifted L_∞ algebras takes direct sums of brackets and filtrations. The unit is then the 0 shifted L_∞ algebra.

What's an example of a morphism in this category. Going back to these convolution L_∞ algebras, well I forgot one thing I should say. Why do I care about the continuity condition? If I have a continuous morphism $F : L_1 \to L_2$, it will give me a map from $MC(L_1)$ to $MC(L_2)$. If I look at the Maurer–Cartan elements of $L_1 \oplus L_2$, then this is $MC(L_1) \times MC(L_2)$. So this gives me a strong monoidal functor.

An example of a morphism in this category that's interesting is one that gives us a notion of composition with respect to the convolution L_∞ algebras. If I have A_i are all A_∞ algebras (or homotopy algebras satisfying my conditions), then there's a nice candidat $\mathrm{Conv}(A_2, A_3) \oplus \mathrm{Conv}(A_1, A_2) \xrightarrow{c} \mathrm{Conv}(A_1, A_3)$, this should be a map satisfying compatibilitiy conditions, this will by the symmetrization

$$c(g_1 + f_1, \ldots g_n + f_n) \coloneqq \mathrm{sym}(\sum_i g_i \circ (f_1 \otimes \cdot \otimes \hat{f}_i \otimes \cdots \otimes f_n) \circ \Delta_{n-2})$$

and the cool thing is that passing to Maurer–Cartan sets this gives composition of A_∞ morphisms. We're on our way now to enriching over the category of L_∞ algebras.

There's actually something even better. We can upgrade MC to a monoidal functor to simplicial sets. This will be a non-Abelian version of chain complexes and Dold–Kan. So Deligne–Getzler–Hinich (or DGH) ∞-groupoids, suppose I have an L_∞ algebra with a nice filtration, complete and so on, L, then well, let me tensor it in the right way with $\Omega(\Delta^n)$, the polynomial de Rham forms on the n-simplex, I can take the completed tensor product with this differential graded commutative algebra

$$L\hat{\otimes}\Omega(\Delta^m) = \varprojlim (L/\mathcal{F}_k L) \otimes \Omega(\Delta^n)$$

and I'll define $\mathcal{MC}_n(L)$ to be $MC(L\hat{\otimes}\Omega(\Delta^n))$.

This gives me a simplicial set, but it's a bit better:

Theorem 20.1. *$\mathcal{MC}(L)$ is a Kan complex.*

This uses the fact that this filtered condition makes it pronilpotent, and you use a famous result of Getzler.

What you're really getting is a monoidal functor $\mathcal{MC}_\bullet : (sL_\infty, \oplus) \to (\mathrm{Kan}, \times)$. This is something we use to say that enriching over this thing is the right thing to do.

At this point you'd expect me to write down a theorem, let's enrich A_∞ algebras over shifted L_∞ algebras. We have almost everything we need. We have a nice notion of a mapping object, we have a composition, the only thing I didn't say yet is a unit and identity laws. You can prove that composition is associative. This means for any A_∞ algebra, I need a map $j_A : 0 \to \mathrm{Conv}(A, A)$. This is a bummer. This is bad. At this point you get sad.

I want to get the identity that takes A to A, I should take this under MC to $*$ mapping to the identity A_∞ morphism but any L_∞ morphism will take 0 to 0.

We can fix this. So let's twist again (there's been a lot of dancing). I'll tell you two facts about Maurer–Cartan elements and twisting by them. This will fix the problem for the last bit of the enrichment. If L is a shifted L_∞ algebra and α is a Maurer–Cartan element of L, then from this data I can form a new L_∞ algebra L^α, which has the same underlying complex but a new differential and new brackets.

The differential

$$\partial^\alpha(x) \coloneqq \partial x + \sum_{k\geq 1} \frac{1}{k!}\{\alpha^k x\}_{k+1}$$

and you do the same thing for the brackets

$$\{x_1, \dots, x_m\}^\alpha = \sum_{k=0}^{\infty} \frac{1}{k!}\{\alpha^k x_1 \cdots x_m\}_{k+m}$$

and I can twist morphisms too. If I have $L_1 \xrightarrow{F} L_2$, and $\alpha \in MC(L_1)$ then $L_1^\alpha \xrightarrow{F^\alpha} L_2^{F_*(\alpha)}$ where

$$F^\alpha(x) = \sum_{k\geq 0} \frac{1}{k!} F(\alpha^k x)$$

These let me define a symmetric monoidal category. The objects are shifted L_∞ algebras. The morphisms are pairs (α, F) where F is an L_∞ morphism and α is a Maurer–Cartan element in L_2, so $L_1 \mapsto L_2^\alpha$. You may wonder how you compose, and it all follows from rules of twisting. At the level of spaces, this is basically letting me change basepoints, these things, I want to be able to get to the end in a leisurely way, these things compose just fine. Let me write down the point.

You still have $\mathcal{MC}_\bullet : (sL_\infty^{MC}, \oplus) \to (\mathrm{Kan}, \times)$, but the point is that $\mathrm{Hom}_{sL_\infty^{MC}}(0, L)$, and this is $MC(L)$. This is the same thing as the zero simplices of the corresponding Kan complex $\mathrm{Hom}_{\mathrm{sSet}}(*, \mathcal{MC}_\bullet(L))$, and this will probably convince you that this will solve the problem of having the identity. Now the identity will be 0 with the Maurer–Cartan element corresponding to the identity.

Let me put in a theorem that kind of summarizes everything and that will be good.

Theorem 20.2. *(Dolgushev–R., Dolgushev–Hoffnung–R.)*

1. *There is an sL_∞^{MC}-enriched category of A_∞ algebras.*
2. *"Base change" via $\mathcal{MC}_\bullet$ gives the category of A_∞-algebras enriched in ∞-groupoids*
3. $\pi_0(A_\infty - \mathrm{alg}_\Delta) \cong A_\infty - \mathrm{alg}(\{\textit{quasi-isos}^{-1}\})$

21. Gabriel C. Drummond-Cole: Operadic convolution in probability theory

I do not take notes at my own talks.

22. Ben Ward: Chain models for moduli space operads

Thank you so much and thanks to Marcy and Phil for gathering us all together here. So let me start by defining an operad, and by that I mean a specific one, not the concept, it will be called Tree and Tree(n) will be the $\mathbf{k}$ span of trees with n labelled vertices modulo some identification, the edges are directed but I'll mod out by that (with signs; switching directions of edges gives a -1). There's an S_n action on the vertices. This is the warm-up example. So Tree(2) is the span of

$$\bigcirc_1 \longrightarrow \bigcirc_2$$

Then Tree(3) is the span of

$$\bigcirc_1 \longrightarrow \bigcirc_2 \longrightarrow \bigcirc_3$$

$$\bigcirc_1 \longrightarrow \bigcirc_3 \longrightarrow \bigcirc_2$$

$$\bigcirc_2 \longrightarrow \bigcirc_1 \longrightarrow \bigcirc_3$$

This has an operad structure which is given by insertion and then sum over connections. So if I insert the 2 to 1 operation $\bigcirc_1 \longrightarrow \bigcirc_2$ into $\bigcirc_1 \longleftarrow \bigcirc_2$ by $\circ_1$ then I get

$$\bigcirc_1 \longrightarrow \bigcirc_2 \longrightarrow \bigcirc_3$$

which is

$$\bigcirc_1 \longleftarrow \bigcirc_2 \longrightarrow \bigcirc_3 \qquad +$$

$$\bigcirc_3 \longleftarrow \bigcirc_1 \longrightarrow \bigcirc_2$$

This is similar to the pasting schemes we had last week.

Lemma 22.1. *There is a morphism from* Lie *(the operad whose algebras are Lie algebras) to* Tree *via the map which takes ℓ_2 to the* 2 *to* 1 *operation.*

To prove this, all I need to do is check that this satisfies skew symmetry (it does) and that the image of the Jacobi identity is zero on this side.

I'm trying to go at a measured pace; this is obvious to half the audience and the other half doesn't think about this.

I need to check that

$$\ell_2 \circ_1 \ell_2 + (123)\ell_2 \circ_1 \ell_2 + (132)\ell_2 \circ_1 \ell_2 = 0$$

and this is

$$\bigcirc_1 \longleftarrow \bigcirc_2 \longleftarrow \bigcirc_3 \qquad +$$

$$\bigcirc_3 \longrightarrow \bigcirc_1 \longleftarrow \bigcirc_2 \qquad +$$

$$\bigcirc_2 \longleftarrow \bigcirc_3 \longleftarrow \bigcirc_1 \qquad +$$

$$\bigcirc_1 \longrightarrow \bigcirc_2 \longleftarrow \bigcirc_3 \qquad +$$

$$\bigcirc_3 \longleftarrow \bigcirc_1 \longleftarrow \bigcirc_2 \qquad +$$

$$\bigcirc_2 \longrightarrow \bigcirc_3 \longleftarrow \bigcirc_1$$

which is zero.

If $\mathcal{O}$ is a cyclic operad, then $\bigoplus \mathcal{O}(n)^{S_{n+1}}$ is a Tree-algebra.

Later I'll give a variant with a more concrete presentation. I've been to several conferences in a row that have the name "higher structures in geometry and physics." This is a phrase that means different things to different people. There's a very different feel depending on who is in the audience. I think it's incumbent on me to say what this phrase means to me. Let me give some examples of objects I want to study and then talk about things. I'll amke a table

object	"lower structure"	algebraic invariant	higher structure
X a space	commutative algebra	$H^*(X)$	E_∞ (perhaps in the sequence model) algebra on cochains
A_∞ algebras	Gerstenhaber algebra governing deformations	$HH^*(A, A())$	a G_∞ algebra, but I want a particular model, an M-algebra, where M is the minimal operad of Kontsevich–Soibelman.
N a (closed oriented) manifold	"string topology" (it's not really fair to call this a lower structure); BV-algebra and gravity algebra	$H_*(LN)$ and $H^{S^1}(LN)$	FM and $M_{\circlearrowleft}$
N compact symplectic manifold	"Gromov–Witten invariants" or hypercommutative	$QH^*(N)$	$(FM)_{hS^1}$

Maybe I'm spending a lot of time making this table but what higher structures in geometry and physics means to me is: "higher" means "chain level" and "structures" means "operations," well, "in" means "in" and "geometry and physics" means "algebra and topology."

First I want to give references for relationships and then I want to discuss the combinatorial chain models, trying to give a feel for how they work.

I'll do this here. Let me first discuss how these operads relate to each other. As a way to understand some of these by analogy, let's consider what's hopefully a more familiar sequence, the Lie, associative, and commutative operads. So the operad As that encodes associative algebras, because it has one for each order of arguments, I could call this As = $H_*(D_1)$, where this is the space of configurations of little intervals inside a big interval, labelled, in any order, this only has H_0, and the component is determined by the order. That's how we see this correspondence. This sits inside $H_*(fD_1)$, and a framing is a point on the boundary, this is a marking on each component. Then I can map to the commutative operad, and As receives a map from the Lie operad. This is hopefully a more familiar version. This is the E_1 version of this story. Then in the E_2 line, I can start with $H(D_2)$, drawing the same [picture], one dimension up, and if you know this, it's a theorem, and if you don't know this it's a definition, this is Gerst, and this sits inside the version where I mark points on the boundary, and that's the operad for BV algebras, the identification with the S^1 action gives me HyComm. It's appropriate to mention Gabriel's work here, $BV//S^1$ = HyComm (Drummond-Cole), and the Koszul dual of the commutative is Lie, and the Koszul dual of HyComm is Grav. So I get

$$\text{Lie} \longrightarrow \text{As} = H_*(D_1) \longrightarrow H_*(fD_1) \longrightarrow \text{Com}$$

$$\begin{tikzcd}
\Sigma H(\mathcal{M}_{*+1}) = \text{Grav} \arrow[r] \arrow[d, "\sim"] & \text{Gerst} = H_*(D^2) \arrow[r] \arrow[d, "\sim"] & H_*(fD_2) = BV \arrow[r, "//S^1"] \arrow[d, "\sim"] & \text{HyComm} = H_*(\overline{\mathcal{M}}_{*+1}) \arrow[d, "\sim"] \\
M_{\circlearrowleft} & M & fM & (fM)_{hS^1}
\end{tikzcd}$$

Chronologically, M came first, building on work of Gerstenhaber and Voronov. Then fM came next, there ar models of this, include Westerland–Wahl, Tradler–Zeinalian, and then I have papers for $M_{\circlearrowleft}$ and $(fM)_{hS^1}$ were third and fourth. But I'll go in order left to right instead.

I haven't given full definitions, let me define

Definition 22.1. $M_{\circlearrowleft}(n)$ is spanned by planar, *stable*, black and white trees with n white vertices.

Here stable means no monovalent or bivalent black vertices or edges between black vertices. If you allow the last thing, the computations will be longer, this is just for exposition. So for arity three, I get [pictures].

The operad structure is insert and sum. I want to make this differential graded so I'll define a differential colloquially as "collapse white angles," by which I mean sum over all ways of doing this. [pictures].

I can quickly calculate the homology of this guy, the first Betti number is 1 and the second Betti number is 2. This does calculate the homology of the moduli space with four punctures, this is the sphere minus three points, and I can retract this, Daniela talked about this yesterday.

In arity four, we don't get a combinatorial *explosion*, but we do get a sort of controlled demolition, we get a twenty dimensional space, and go down one degree, you get no symmetries, this is 24 dimensional, and here you get something 6 dimensional. It's a little hard to calculate the homology, it's not impossible. You can see the bottom homology is one dimensional. The top homology, all you have is the Lie thing, and when I start composing that, I should get cycles. The dimension 2 is the dimension of Lie words on three letters, since Jacobi says I go three dimensional to two dimensional. The same thing happens here. The homology will be identifiable with Lie words on four letters

Lemma 22.2. $H^{\top}(M_{\circlearrowleft}) = \text{Lie}$

Nwo you can calculate the middle dimension using Euler characteristic, this middle one has dimension 5. You can look at the Poincaré polynomial of these spaces, due to Getzler, this is $(1-2t)(1-3t) = (1-5t+6t^2)$. Now my combinatorial sklls are reaching their limit so I'll just tell you the theorem at this point. Hopefully my reason for, well,

Theorem 22.1. (1) *$g_n \in H_0(\mathcal{M}_{n+1})$ maps to the class of the graph with one black central vertex and n white vertices, this induces an isomorphism of operads. For this to make sense, you have to know that with this suspension, you can generate all the homology from suspension and by operadic composition.*

I would say that, this is an original result, I don't know where to point you to find the chain model, maybe you can find this as a suboperad of things that people wrote down in string topology. You can also give an explicit presentation for the relations here, you can give the gravity relations.

(2) *(this uses the first statement) There exists a zigzag of weak equivalences of operads between* $\Sigma H(\mathcal{M}_{*+1})$ *and* $M_{\circlearrowleft}$.
(3) *If* $A_\infty \xrightarrow{\mu} \mathcal{O}$ *is a map of cyclic operads, then* $(\prod s\mathcal{O}(n^{\mathbb{Z}_{n+1}}), \{\mu, \quad\})$ *is an algebra over* $M_{\circlearrowleft}$.

The point is, at the level of homology and cohomology, you know that the cyclic cohomology of the Frobenius algebra, [unintelligible], then this is Connes' complex computing cyclic cohomology. So we get actions on the complexes computing cyclic cohomology or $H_\bullet^{S^1}$.

It's better to go a little slow than to race through all of this, I only have a few minutes left. How do you combinatorially go from step one to step two to step three to step four?

What you can do is go from $M_{\circlearrowleft}$ to M by adding roots to the trees. The only thing I can do is sum over all roots. Once you have the minimal operad of Kontsevich–Soibelman, you can do a similar thing, pick out an angle to mark white vertices. You include this with the trivial marking, you can always mark the trivial flag. We're increasing combinatorial complexity, the middle two are finitely generated, and we go back to being infinitely generated, adding cells that correspond to the cohomology of BS^1, we want something, fM acts on homotopy BV algebras, and we'd like to take the additional operations killing the BV operator into account. This includes in by adding cells, and you can ask what new comes about? The new thing that comes in is the fundamental classes of these manifolds. I'm happy to discuss any of this but for now my time is up.

23. Sophia Raynor: Hierarchical networks and coloured modular operads

Thank you, thank you Marcy and Phil, it's been a great conference, I don't know about you, but I'm kind of tired, I'm not going to do much here, certainly not maths. I want to make a suggestion, and then I want your input. I've already had some great feedback from Mark, I think maybe Ben and Martin can help me, maybe others.

I want to suggest that coloured modular operads (an appropriate notion thereof) provide a suitable framework for considering questions which arise in the study of complex networks.

Today:

(1) I'll provide some support for this suggestion,
(2) I'll outline a formalism which lends itself to this project,
(3) and describe some applied collaborations

Let's talk a little bit about complex networks, hierarchical complex networks, complex problems. We have huge amounts of data on everything and much of it is arranged in networks. This sometimes is related to problems of scale. You can't understand all the data, but only emergent properties that come from zooming out and looking at the system as a whole. I worked with people at the blue brain project who want to reverse engineer a brain, and what they're doing should tally with fMRI scans and what happens in them. We have lots of problems of extrapolating from local fine scale information to global structure.

There's always a question when you take scientific measurements if your measurements are scale-specific, there's problems with changes of scale. There's not really a framework right now across applied sciences for talking about this kind of problem. You have lots of computing power in biology, so people run lots of simualtions at different scales.

The project as a whole is:

(1) to find a suitable (categorical) formalism for hierarchical networks and investigate its structure and properties.
(2) Then we'd like to identify properties (respectively invariants) that could potentially be relevant (repsectively computable) in "real life" applications (these may be application specific), then
(3) Adjust and apply the model appropriately to real data.

I'll do a little of step one, maybe a sentence of step two, and none of step three today.

Let me give an informal definition of hierarchical networks

Definition 23.1. A *network* $\mathcal{N} = (\mathcal{G}, \alpha)$ is

- a finite graph $\mathcal{G}$ whose edges and nodes are decorated "via some map α" by some elements of a given set.

You could imagine a set of cities or a set of dynamical systems or something to decorate a network with. You want some kind of stability under automorphisms of the underlying graph $\mathcal{G}$ and it will be connected if $\mathcal{G}$ is connected.

I'll tell you a hierarchical network on a slide.

Definition 23.2. A *(k-level) hierarchical network* is an underlying network $\mathcal{N}$ together with a (k-times) iterated partitioning of $\mathcal{N}$ into connected subnetworks.

You're all thinking "Aha! graph substitution." We could zoom out, crushing nested edges. The hierarchy is defined on increasing subsets of internal edges of the underlying graph.

Another point of view is to zoom in to reveal fine local structure, zozming in and exploding vertices, after seeing hereditary pasting schemes with graphs, you're probably thinking, oh, modular operads are just colored operads so we could think about taking the small components and building up our network. We can collapse or build up and so on, all according to the coloring.

I suppose the point [pictures], we have a network of networks of networks of networks and we have a canonical simplicial structure here.

There seems to be an iteration of a monadic thing, so some kind of simplicial structure. One thing is that this tree structure is an invariant of the hierarchical network. You could think of invariants at each level or invariants of the network tree itself. This is related to work with David Spivak and I think we'll be interested in Martin's operads, because we care about the levelwise structure.

I want to make some assumptions and put some conditions. I want to assume, first of all, I'm not interested in *clustering algorithms*. There are lots of people working on efficient robust clustering algorithms. What you have to do is going to depend on the data on the network. I can't say anything about that.

(1) I'd actually like a framework in which any connected subgraph can form a cluster. The network could be any shape of connected graph.
(2) It shouldn't matter if I miss or add extra clustering layers. In my picture if I cluster first blue and then green, I could have just taking only green.
(3) I want the rule for clustering to depend only on the underlying subgraph.
(4) I want collapse of structures to give a network of the same type. Okay, fMRI scans are not the same as cells or cell connections. I want to be able to compare what's going on at one level and at the other level. I want some sort of maps from my hierarchical data into a hierarchy where everything is of the same "type."

    So the first condition makes me think about hereditary pasting schemes. Condition 2 gives me [unintelligible], and I'm looking at some algebra for 3 and 4.
(5) Finally any decoration, including orientation (or not!) should be part of the data of specific problems and not of the underlying formalism. Networks can be undirected (facebook) or directed (disease spread) or some kind of mixture, where for some maybe we don't care or something like that.

That's what I'm actually going to talk about in the formalism today.

I'm going to give, jumping ahead a few steps from that but not quite, I'll give a partial definition on this board

Definition 23.3. (partial) A k-level hierarchical network of type A are elements of a $k+1$-times iterated free coloured modular operad, they are $(k+1)$-simplices in some simplicially enriched coloured modular operad $T^\bullet A$ where, what do we need to define in this definition? A *coloured modular operad* is an algebra over some free monad $T : \mathcal{GS} \to \mathcal{GS}$, so $\mathcal{GS}$ is the presheaf category on el and finite sets is a full subcategory of el, and T is given by a pasting scheme of undirected graphs with leaves.

What haven't I defined? I haven't defined *graphs*, *the category* el*,* type*, or* the monad T*. That's what we're going to do.*

I want to talk about categories of graphs. I'll revive the formalism of Joyal–Kock. There's a problem with their note (of 2011) which is why the full paper wasn't written. I'm not thinking about genus or units, so I avoid their problems, a lot of work comes from trying to solve the problem when I have units, but that's a different story.

Their definition of a graph is similar to the standard definition of flags with involution, but the involution has no fixed points.

Definition 23.4. I graph $\mathcal{G} = (E, H, V, s, t, \tau)$ is a diagram of finite sets

$$\tau \circlearrowright E \qquad\qquad H s \xrightarrow{t} V$$

where s is injective and τ is an involution with no fixed points. So E_o is $E - \mathrm{im}(s)$ and $E_I = \{e \in E | e, \tau(e) \in \mathrm{im}(s)\}$.

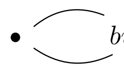
The edge with no vertices looks like

$$\circlearrowright \underline{2} \longleftarrow \varnothing \longrightarrow \varnothing$$

or the corolla, there are n loose ends and n attached ends to a single vertex

$$n \sqcup n \longleftarrow n \longrightarrow *$$

Or something else,

 *bullet* $\circlearrowright 2 \sqcup 2 \longleftarrow 2 \sqcup 2 \longrightarrow 2$

or

$$\subset \bullet \qquad \circlearrowright 2 \longleftarrow 2 \longrightarrow *$$

Then a map of graphs is a triple f_E, f_H, and f_V which makes the diagrams commute:

$$\begin{tikzcd}
E \arrow[d, "f_E"] & E \arrow[l, "\tau"'] \arrow[d, "f_E"] & H \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "f_H"] & V \arrow[d, "f_V"] \\
E' & E' \arrow[l, "\tau"] & H' \arrow[l, "s"] \arrow[r, "t"] & V'
\end{tikzcd}$$

We also want the map t and f_H to be the pullback of t' and f_V

So for instance we have a map of graphs

$$\begin{tikzcd}
\circlearrowright 2 \sqcup 2 \arrow[d] & 2 \sqcup 2 \arrow[l] \arrow[r] \arrow[d] & 2 \arrow[d] \\
\circlearrowright 2 & 2 \arrow[l] \arrow[r] & *
\end{tikzcd}$$

So within this graph category I can pick out Gr*, the full subcategory of connected graphs, and inside that is* el*, the set of connected graphs with no internal edges.*

Observations, $\mathrm{Gr}(1, \mathcal{G}) \cong E(\mathcal{G})$ *and there is a canonical fully faithful embedding of finite sets with isomorphisms to* el *by taking x to the corolla c_x, the graph $x \sqcup x \leftarrow x \to *$*

So the category $\mathrm{el}(1,1)$ *has* $\{\mathrm{id}_\epsilon, \tau_\epsilon\}$*, and* $\mathrm{el}(c_x, c_y) \cong$ *bijections*(x, y)*.*

We had the observation before and so $\mathrm{el}(1, c) = c$ *and* $\mathrm{el}(c_x, 1) = \varnothing$*.*

Why do we like this category? What can we do with it? The key thing is that for any graph, let's just stick to the connected graphs, $\mathcal{G} \in \mathrm{Gr}$*, we have the category of* elements *of* $\mathcal{G}$*,* $\mathrm{el}(\mathcal{G})$*, given by* $\mathrm{el}(\mathcal{G}) = \mathrm{el} \downarrow \mathcal{G}$*, the restriction of the slice category* $\mathrm{Gr} \downarrow \mathcal{G}$*.*

Most importantly, it holds that $\mathcal{G}$ is canonically isomorphic to the colimit

$$\operatorname*{colim}_{(c,f)} c$$

So you build a graph by sticking it together what is happening over the edges. So it's pretty trivial.

Definition 23.5. The category $\mathcal{GS}$ of graphical species is presheaves on el. So for $F \in \mathcal{GS}$ we have a set of colours togehter with an involution, τ_C, for every finite set X you have $F(X)$ with an action of Aut X and then you have X projections compatible with the involution.

The point here is that we can extend a graphical species to a presheaf over the whole category of graphs. This defines sieves over each graph, which give us a canonical Grothendieck topology. The [missed] with respect to this topology.

For $F \in \mathcal{GS}$ and $\mathcal{G}$ in Gr*, we define $F(\mathcal{G})$ to be the limet*

$$F(\mathcal{G}) = \lim_{(c,f) \in \mathrm{el}(\mathcal{G})} f(c).$$

Very quickly I'll give two examples and then hurry up.

(1) $F = \mathrm{Comm}$, *the terminal graphical species, then* $F(\mathcal{G})$ *is* $\mathcal{G}$. *You can see that* $F(\mathcal{G})$ *is the graph* $\mathcal{G}$ *decorated compatibly by the elements of* F. *This is a single term, so this looks like* $\mathcal{G}$.
(2) *I could also take* F *where* $F(|)$ *is either "in" or "out" and do* $*$ *for other things in* el. *so then we get orientations on* $\mathcal{G}$. *You can add colours or extra information as you wish.*

I have to be quick, so we want to think about this monad T *now, and for that I want*

Definition 23.6. The groupoid $X - \mathrm{Gr}_{\mathrm{iso}}$, where X is a finite set, has objects $\mathcal{X} = (\mathcal{G}, \rho)$ where $\rho : E_0(\mathcal{G}) \to \mathcal{X}$ and with morphisms isomorphisms of $\mathcal{G}$ compatible with the labelings. Then define the free monad $T : \mathcal{GS} \to \mathcal{GS}$ by $TF(|) = F(|)$ and $TF(c_X)$ as the colimit

$$\operatorname*{colim}_{(\mathcal{G},\rho)\in X-\mathrm{Gr}_{\mathrm{iso}}} F(\mathcal{G})$$

Normally when you see this kind of free monads, you get a tensor product of evaluation at vertices. If F *of the stick graph is a singleton, this reduces in this way. By* $F(\mathcal{G})$ *beoing defined as the limit or colimit over* $\mathcal{G}$, *you can add information on the edges within the formalism. So elements of this guy are isomorphism classes decorated by* F. *This is the sense, I think, in which I have all these things.*

I'll take two minutes to mention some applications. One is the brain things, we've been talking about that, I mentioned that, In general, there are a lot of problems in computer science that come under the umbrella of constrained satisfaction problems, this is to do with getting complex systems of multiple variables, finding solutions when you have constraints, and an issue in computer science, apparently, is a way to organize constraints at different scales.

An example with extremely real world applications, in the UK, something that people were worried about a few years back was badgers, they've been implicated in the spread of TB in cows, cows spread to badgers and badgers to cows. They spent millions on a badger cull, killed thousands of badgers but it was very expensive, thousands of pounds per badger. The cows are mapped, carefully tracked, but we only have coarse information on badgers. I've started talking with someone in the government who works with badgers about applying operadic methods to this problem.

24. June 17: Simona Paoli: Weakly globular structures in homotopy theory and higher category theory

[I do not take lectures at slide talks].

25. David Cardechi: Étale homotopy theory for higher stacks

Thanks, and thanks to Phil and Marcy for being awesome and stuff. I couldn't remember my title so I put both of them up here.

I'll start with some things that aren't higher at all. Before getting to étale homotopy theory I'll say some things about étale cohomology. This was an important ingredient in the proof of the Weil conjecture, introduced by Grothendieck and Artin, the first step in producing a Weil cohomology theory.

Here are two important properties you should know about it.

First of all, it's a way to attach cohomology theories to a scheme.

(1) *Étale cohomology is defined for a scheme X over* any *base, or Deligne–Mumford stack, in any characteristic. Something defined over $\mathbb{F}_p$, you don't know what a cohomology theory should be.*
(2) *If X is a complex variety and A a finite Abelian group, on the one hand you can look at $H^n_{ét}(X, A)$; on the other hand you could give X the usual topology (this is the analyticization X_{an}) and this étale cohomology is isomorphic to $H^n_{\mathrm{sing}}(X_{\mathrm{an}}, A)$.*

If you're a topologist you might care about other invariants, so, say,

(1) *what about homotopy groups?*
(2) *Maybe I can't get a topological space but maybe I can get a homotopy type whose cohomology is the étale cohomology?*

The answer is almost, but not quite. You can't get a homotopy type, but you can get a pro-homotopy type. I'll explain what that means in a second.

I'll be a bit agnostic about what it means, I'll start with a naive description and move to something a little more sophisticated later.

So let me give a reminder or an introduction about pro-objects.

The idea is that I have $\mathcal{C}$ a category, and then I produce a new category $\mathrm{Pro}(\mathcal{C})$, *which are formal cofiltered limits of objects of $\mathcal{C}$. If you don't know what that means, direct or inverse limits, that's enough for a definition. So this has a universal property, there's an embedding of $\mathcal{C} \overset{j}{\hookrightarrow} \mathrm{Pro}(\mathcal{C})$. If $\mathcal{D}$ has cofiltered limits, then j induces an equivalence of categories between*

$$\mathrm{Fun}_{\mathrm{cofilt}}(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

I'll give you two different concrete descriptions. Concretely, I'll put $\mathrm{Pro}(\mathcal{C})$ *into* $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$. *Suppose C is an object of $\mathcal{C}$, then I'll define a functor $\mathcal{C} \to \mathrm{Set}$ which takes $D \mapsto \mathrm{Hom}(C, D)$. We'll say F is in $\mathrm{Pro}(\mathcal{C})$ if and only if it can be written as*

$$F = \lim_{\leftarrow c_\alpha} j(c_\alpha)$$

in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$.

This is still not very concrete. When $\mathcal{C}$ has nice properties, you can describe these functors; when $\mathcal{C}$ has finite limits, then $\mathbb{F}$ is in $\mathrm{Pro}(\mathcal{C})$ if and only if $\mathbb{F}$ preserves finite limits. This gives a concrete description, this is finite limit-preserving functors to Set, *or well the opposite category.*

Let me give the example that you might have seen before. Take $\mathbb{C}$ to be the category of finite groups. Then $\mathrm{Pro}(\mathcal{C})$ is the category of profinite groups. There's a functor, given any group I can construct any group called its profinite completion $\mathrm{Gp} \to \mathrm{Pro}(\mathrm{Fin\,Gp})$ *where*

$$G \mapsto \lim_{\leftarrow N \trianglelefteq G} G/N$$

where this limit should be in quotes because it's formal.

If G is finite then $\hat{G} = G$ and $\mathrm{Pro}(\mathrm{Fin\,Gp}) \subset \mathrm{Top\,Gp}$, and this will be those which are actual cofiltered limits of finite discrete groups.

Let's do another example, taking $\mathcal{C} = \mathrm{Ho}(\mathrm{Top})$. Then $\mathrm{Pro}(\mathcal{C})$ might be something you'd call prohomotopy types, although later we'll do something better.

What we can try to do is something like what we did for groups, but for homotopy groups.

Definition 25.1. A space (I'll be agnostic about what kind) X is *π-finite* if it has only finitely many non-trivial homotopy groups and π_0 is finite, all of which are themselves finite.

So now you can try to play this game, can I do a profinite completion of a topological space to a a profinite object?

If I want to get pro-objects here I can't give a good description but it exists. Artin–Mazur show that there exists a "profinite completion functor"

$$\mathrm{Pro}(\mathrm{Ho}(\mathrm{Top})) \to \mathrm{Pro}(\mathrm{Ho}(\mathrm{Top}^\pi))$$

where Top^π *is π-finite spaces. This is very hard to describe. It's impressive that they did this, but because you did this for the homotopy category, it's hard to get your hands on what properties it has. Call this functor* $\hat{}$. *We had this nice comparison theorem between étale cohomology of a complex variety with finite coefficients and its singular cohomology. Now we can improve that a lot. Before I say it, let me tell you something. I'll black box the étale homotopy type for now. They also show that if X is a locally Noetherian scheme, then they can produce a pro-object $[X]_{ét}$ in* $\mathrm{Pro}(\mathrm{Ho}(\mathrm{Top}))$, *called the* étale homotopy type of X. *I won't describe this now and will give a better definition using more recent technology.*

Theorem 25.1. *(Profinite comparison theorem) If X is a pointed connected scheme over $\mathbb{C}$ of finite type, then there is*

$$[X]_{ét} \longleftarrow [X_{\mathrm{an}}]$$

such that the induced map on profinite completion is an isomorphism.

The homotopy groups, all toplogical invariants are the same. This is a slight lie, no one catch me.

I want to formulate all of this using infinity categories.

First of all I want to say that doing profunctors does not commute with taking homotopy categories. The theorem will be more general, get rid of many of the conditions, and I'll do this before the homotopy category. This is somewhere else, you have an isomorphism after applying a functor, it's better to have it beforehand.

Let me top rambling and start doing stuff. All I'm going to say about ∞-categories is that I can replace a set of morphisms with a space or ∞-groupoid of morphisms.

So I told you how to construct pro-objects, let me do this for an ∞-category. I can produce another ∞-category $\mathrm{Pro}(\mathcal{C})$, *I really only care about this for spaces or π-finite spaces. When I had finite limits I could do a simpler description, let me just jump to that in this ∞-categorical setting.*

Let $\mathcal{S}$ be the ∞-category of spaces (or ∞-groupoids), then $\mathrm{Pro}(\mathcal{S})$ *will be a subcategory of* $\mathrm{Fun}(\mathcal{S},\mathcal{S})^{\mathrm{op}}$, *and something in the functor category will be in* $\mathrm{Pro}(\mathcal{S})$ *if it preserves finite limits (and under my breath, is locally accessible).*

Now let me do the other example that I care about, let $\mathcal{S}^\pi$ be the ∞-category of π-finite spaces, and let me define $\mathrm{Pro}(\mathcal{S}^\pi)$, *let me call this* $\mathrm{Prof}(\mathcal{S})$, *the ∞ category of profinite spaces.*

The nice thing is, before it was very difficult to say what the profinite completion is, let's do profinite completion. I have to produce a functor from $\mathrm{Pro}(\mathcal{S})$ *to*

$\mathrm{Prof}(\mathcal{S})$. *So suppose I have* $F : \mathcal{S} \to \mathcal{S}$, *then* $\mathcal{S}^{\pi} \xrightarrow{i} \mathcal{S}$, *this preserves finite (homotopy) limits, and then I take* F, *and the composition* $F \circ i$, *this is the profinite completion.*

Moreover you actually have the universal property that the profinite completion is left adjoint to the canonical inclusion $\mathrm{Prof}(\mathcal{S}) \to \mathrm{Pro}(\mathcal{S})$. *In particular, you want this to be left adjoint, in the homotopy category it will be killed. Now we can prove things by looking at simple things and gluing them together.*

Let $\mathbf{k}$ *be a commutative algebra, and let* $\mathrm{Aff}_{\mathbf{k}}$ *be the category of affine* $\mathbf{k}$*-schemes (of finite type, I'm only doing this for some size issues, this is a convenient choice but not the only one). Now let* $\mathrm{Sh}_{\infty}(\mathrm{Aff}_{\mathbf{k}}, \acute{e}t) \subset \mathrm{Fun}(\mathrm{Aff}_{\mathbf{k}}^{\mathrm{op}}, \mathcal{S})$ *be those functors satisfying étale descent, that is,* ∞*-stacks with respect to the étale topology.*

The inclusion, that is, has a left adjoint, a sheafification functor a *which preserves finite limits. Since I care about this kind of preservation, that might be nice to know.*

Some examples:

(1) *Any algebraic stack* $\mathcal{X}$ *is a sheaf of groupoids, which are* 1*-types in* $\mathcal{S}$. *If you're worried about using the Grothendieck topology I'm using, these embed fully faithfully in the étale topology.*
(2) *Suppose I have* $\mathcal{Z} \in \mathcal{S}$, *and you consider the constant presheaf* $\Delta_{\mathcal{Z}}^{\mathrm{pre}} : \mathrm{Aff}_{\mathbf{k}}^{\mathrm{op}} \to \mathcal{S}$. *This is stupid, but I can turn anything into a sheaf, so* $a(\Delta_{\mathcal{Z}}^{\mathrm{pre}})$ *is the constant stack with value* $\mathcal{Z}$, *called* $\Delta_{\mathcal{Z}}$.

I'll smash these together to give myself étale homotopy type.

Let $\mathcal{X}$ *be any* ∞*-stack. Then define the following functor* $\pi_{\infty}^{\acute{e}t}(\mathcal{X}) : \mathcal{S} \to \mathcal{S}$, *which takes* $\mathcal{Z} \mapsto \mathrm{Hom}_{\mathrm{Sh}_{\infty}(\mathrm{Aff}_{\mathbf{k},\acute{e}t}, \Delta_{\mathcal{Z}})}$. *Limits are computed objectwise,* a *preserves finite limits, and so the whole thing preserves finite limits. This thing we will call the* étale homotopy type *of* $\mathcal{X}$. *This can be defined for* any *kind of stacks, doesn't really have to be finite type, this is like the most general you could actually do, and now you have control over this construction actually. This construction that preserves colimits, this preserves colimit-preserving functors from* ∞*-sheaves or* ∞*-stacks* $\mathrm{Sh}_{\infty}(\mathrm{Aff}_{\mathbf{k},\acute{e}t}) \xrightarrow{\pi_{\infty}^{\acute{e}t}} \mathrm{Pro}(\mathcal{S})$.

I want to tell you a profinite comparison theorem. The first thing I'll do is to tell you how to produce analytification as well. This is also my construction, it's not that bad. There's a functor $(\quad)_{\mathsf{T}} : \mathrm{Sh}_{\infty}(\mathrm{Aff}_{\mathbf{k},\acute{e}t}) \to \mathrm{Sh}_{\infty}(\mathrm{Top})$ *where* Top *is the* 1*-category of topological spaces. This is totally categorical. This is the unique colimit-preserving functor such that for* X *an affine scheme,* $X_{\mathsf{T}} = X_{\mathrm{an}}$. *This shows it's unique but not that it exists. The theorem is the existence of this thing.*

If $\mathcal{X}$ *is an algebraic stack, then* $\mathcal{X}_{\mathsf{T}}$ *is the underlying topological stack in the sense of Nouhi, this agrees with previously done constructions. So if you had a groupoid presentation in schemes and you take the underlying topological stack, this [unintelligible].*

Okay now I want to extract a homotopy type. I just need to describe, now what it does on my site.

Theorem 25.2. *(C., '15) Let* $\pi_{\infty} : \mathrm{Sh}_{\infty}(\mathrm{Top}) \to \mathcal{S}$ *be the unique colimit preserving-functor such that* $\pi_{\infty}(T) = T$. *This is about the homotopy colimit of a hypercover of a space being equivalent to it*

Let me put this together and unwind what it means, safely removing almost all of the adjectives that we put.

Theorem 25.3. *(C., 15) The following diagram commutes in the ∞-categorical sense (up to equivalence). This will capture anything that is* locally *finite type.*

$$\begin{tikzcd}
\mathrm{Sh}_\infty(\mathrm{Aff}^{\mathrm{ft}}_{\mathbb{C},\acute{e}t}) \arrow[r, "\pi_\infty^{\acute{e}t}"] \arrow[d, "(\ )_\top"] & \mathrm{Pro}(\mathcal{S}) \arrow[drr, "\widehat{(\ )}"] & & \\
\mathrm{Sh}_\infty(\mathrm{Top}) \arrow[rr, "\pi_\infty"] & & \mathcal{S} \arrow[r, "\widehat{(\ )}"] & \mathrm{Prof}(\mathcal{S})
\end{tikzcd}$$

In particular, for any algebraic stack LFT/$\mathbb{C}$, we have $\widehat{\mathcal{X}}_\top \cong \widehat{\pi}_\infty^{\acute{e}t}(\mathcal{X})$.

So how do you do this? All the functors preserve colimits, so you just need to prove it for affines.

Now I have to go really fast, this was my plan all along, to give a nice talk and then scream at you guys for five minutes.

I lied to you, this is not how to define étale homotopy type. First go to ∞-topos. If I have a ring, I can take ∞-sheaves on its small *étale type. There's again a (nonobvious) colimit-preserving functor. You do it, done. That's hard, whatever. done. Let's assume we all know what an ∞-topos is. Think of a sheaf on a site. We know how to take the constant stack. I can take global sections, that preserves limits, and that gives me a prospace, called the* shape *of this sheaf. This gets me to* $\mathrm{Pro}(\mathcal{S})$. *There's a theorem, that the shape of* Sh_∞ *is the same as* $\pi_\infty^{\acute{e}t}$. *So I didn't lie.*

Something here has to be nonformal. First let me tell you one more thing that's formal, so for X a separated scheme of finite type over $\mathbb{C}$, you can prove that $\pi_\infty(X_{\mathrm{an}})$ *is the same as the shape of* $\mathrm{Sh}_\infty(X_{\mathrm{an}})$ *in prospaces. Then analytification produces a geometric morphism which goes from ∞-sheaves on* X_{an} *to* $\mathrm{Sh}_\infty(X_{\acute{e}t})$. *Finally this is a profinite homotopy equivalence if and only if for all π-finite V,*

$$\Gamma_{\acute{e}t}\Delta_{[unintelligible]}(V) \xrightarrow{\sim} \Gamma\Delta(V)$$

Grothendieck hit this with a hammer for me. By GAGA, if this map induces an isomorphism on π_1 and on cohomology with local coefficients in any finite Abelian group, then I can do it by induction on Postnikov towers and I'm done. There's actually a lot more but I'm out of town. I hope you liked it up to my shouting and then hated me for the ending.

26. Christian Haesemeyer: On the K-theory of monoid algebras

There won't be much higher structure in this talk although there will be some under the hood. I'll try to use Dave's language. The goal of what I want to talk about, this is joint project with Gordiñas, Walker, and Weibel, studying the algebraic K-theory of singularities.

The goal of what we are trying to do, R or $\mathbf{k}$ will be a commutative ring, and I actually mean a (discrete) commutative ring, and A or B or M will be a coummutative monoid, for example the natural numbers. Normally I will write them multiplicatively and they will be pointed, so they will have 0 *with the usual properties. So for instance, lattice points in a cone. This is what toric varieties are built out of, in particular because they model simple types of singularities.*

We'd like to study the algebraic K-theory $K_*R[A]$, and I want to address how much is contributed by R and how much by A.

I give talks about algebraic K-theory all the time, I never say what it is, it's really the stable homotopy groups of some spectrum built out of the category of finitely generated projective modules over the ring, $\pi_* KR[A]$. So π_0 classifies these things, π_1 gives automorphisms, and so on.

As an example, if R is a regular ring, so every module has a finite projective resolution and A is the monoid $\mathbb{N}^r$, then what is $R[A]$? It's the polynomial ring in r variables, and a theorem of Quillen says that

Theorem 26.1. *(Quillen)*

$$K_* R \xrightarrow{\sim} K_* R[A]$$

So for a monoid like this, this rational cone, the K-theory doesn't see the monoid at all. To see the monoid, it should be uglier than this.

Here is something true for a lot of monoids, surprising if you think about it.

Theorem 26.2. *(Gubeladze) If $\mathbf{k}$ is a regular domain and $M \subset \mathbb{Z}^n$ is finitely generated (this is not really necessary, but it's a way to reduce the number of hypotheses) normal (this means for a monoid that if $x \in M$ and $\frac{1}{n}x$ is in $\mathbb{Z}^n$ then $\frac{1}{n}x$ is in M) which does not contain a line, then every projective $\mathbf{k}[M]$-module is induced from $\mathbf{k}$.*

This is a generalization of a famous theorem of Quillen and [unintelligible], doing Serre's conjecture, which is this for a field and A as above.So this led to

Conjecture 26.1. (Gubeladze) Let $\mathbf{k}$ be a regular ring and A a submonoid of a torsion-free Abelian group not containing a nontrivial unit. Let $c = (c_1, c_2, \ldots)$, with $c_i \geq 2$ and $\mathcal{O}_{c_i} : K(\mathbf{k}[A]) \circlearrowleft$ the "dilation" induced by $a \mapsto a^{c_i}$. Then the map

$$K(\mathbf{k}) \xrightarrow{\sim} \operatorname*{colim}_{\mathcal{O}_{c_i}} K(\mathbf{k}[A])$$

or equivalently, "dilations act nilpotently on $K(\mathbf{k}[A])/K(\mathbf{k})$."

That's a conjecture, I could be more precise. I will sketch the proof of this today, special cases were done before. Gubeladze proved this in some cases, say if $\mathbb{Q} \subset \mathbf{k}$. His methods, first he proved it for fields and then noticed it was the same for regular rings. His methods are totally affine, using methods from polyhedral geometry. He says I can put this inside, let's say we have a monoid where we can take sufficiently many roots. Then I can approximate the monoid by something like $\mathbb{N}^r$ with sufficiently many roots added. Then I can understand the difference with trace methods (like cyclic homology). He uses a continuous theorem about excision for the fiber of the map from K-theory to cyclic homology. This was done in 2005–2008.

When $\mathbf{k}$ contains a field, then we proved this, somewhere between 2007 when [unintelligible]gave a talk about it and 2014 when it finally appeared. Using what I'll call global methods (and I'll say more about those because basically I'll only sketch the case where $\mathbf{k}$ contains a field).

I wont' stay in the affine world. I'll glue them together to get schemes. You take spectra of the rings and localize, but I'll only [unintelligible]glue together monoid schemes. I'll explain what these are and how to deal with them later on.

The characteristic here, if you want something fancy, will be 1.

I should make a statement of the conjecture that applies in the non-affine case, and even when M doesn't satisfy these annoying conditions. The statement uses *[unintelligible]version of [unintelligible]theory called homotopy K-theory which is forced to satisfy Quillen's theorem no matter what R is.*

Definition 26.1. Let R be a ring over a scheme X, as noncommutative as you like. Define $KH(R)$ to be the realization of a simplicial spectrum

$$KH(R) = \underset{\Delta^{\mathrm{op}}}{\operatorname{hocolim}} R[\Delta]$$
$$KH(X) = \underset{\Delta^{\mathrm{op}}}{\operatorname{hocolim}} K(X \times \Delta)$$

where $\Delta^n = \operatorname{Spec} \mathbb{Z}[t_0, \ldots, t_n]/\sum t_j - 1$.

I won't prove this but if you think about it you can see it must be true.

Lemma 26.1. *If A is as in the conjecture and $\mathbf{k}$ is any commutative ring then*

$$KH(\mathbf{k}) \xrightarrow{\sim} KH(\mathbf{k}[A]).$$

Okay in another words, the conjecture is completely trivial if you replace K with KH.

Theorem 26.3. *(CHWW) Let $\mathbf{k}$ be a regular Noetherian commutative ring (of finite Krull dimension—I think this is not actually necessary) and let X be a separated pctf (I'll explain in a second) monoid scheme of finite type [this is not a scheme in the usual sense, it's built out of monoids; it's completely combinatorial] and c_i are integers infinitely many of which are at least 2. Then the dilations $\mathcal{O}_{c_i}$ act on monoid schemes and the theorem says there is an equivalence*

$$\underset{\mathcal{O}_{c_i}}{\operatorname{colim}} K(X_k) \xrightarrow{\sim} \underset{\mathcal{O}_{c_i}}{\operatorname{colim}} KH(X_k).$$

Note that this implies the conjecture, and also let me note that this should work whenever $K(\mathbf{k}) \cong KH(\mathbf{k})$. This should work, for instance, for commutative C^* algebras. For now this is work in progress and we need a regular ring (of finite Krull dimension).

Okay, how would you prove this? You reduce to the case that you know is true, the theorem of Quillen that I just erased.

In order to tell you how it works, I should tell you something about monoid schemes. These are built out of pointed commutative monoids, I'm thinking multiplicatively and then there's an element 0 that kills everything. An ideal in a monoid is a subset mapped by multiplication to itself. So we build this the way you build schemes out of rings. A lot of people who want to do characteristic 1 want to do this. This is nothing to do with that. One way to think about things in characteristic 1 are geometric objects that are completely combinatorial, so their behavior does not depend on the coefficients at all.

Now I'll write $\mathcal{M}$ for the category of monoid schemes and if you have a commutative ring $\mathbf{k}$, you have a realization functor $\mathcal{M} \to \mathrm{Sch}_{\mathbf{k}}$ which sends $X \to X_k$. This is adjoint to the forgetful functor $\mathrm{Sch}_k \to \mathcal{M}$ that forgets the addition. For properties like separation, it's important that it's an adjoint.

I won't really explain separated, I'll say one thing that held us up for years; if you have a surjective monoid map, it might not be a quotient by an ideal. In general that creates problems, it's not easy to think of what separatedness is.

Now pctf, the tf means torsion free, that means there are no roots of unity. The pc means partially cancellative. This is not necessarily, in general, quotients or cancellative monoids by an ideal, but that's the condition to be partially cancellative.

Definition 26.2. X is smooth if and only if all stalk monoids are equivalent to $\mathbb{N}^r \times \mathbb{Z}^s$.

You should not be surprised by this if you have experience with toric varieties, you want to be able to complete extremal rays to a $\mathbb{Z}$-basis of the lattice.

There's an easy lemma. If X is smooth then $K(X_k) \xrightarrow{\sim} KH(X_k)$ and then the theorem follows in this case.

So the plan is to reduce to this using resolution of singularities, the way I think about this, this is a process that allows you to take an algebraic variety and associate to it smooth manifolds and a recipe to put those together and contract pieces of them to regain your variety. The recipe follows fairly strict rules, otherwise it wouldn't be very useful.

Resolution of singularities is [unintelligible], and even more unknown in positive characteristic, and even more in mixed characteristic. Here's the plan. We'll take some liberties with notation, I'll write $\mathcal{M}_{pctf}$, and a functor on schemes will give me a functor on monoid schemes by fixing $\mathbf{k}$ and composing with $\mathbf{k}$-realization. If $\mathcal{E} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sp}$, then I'll write $\mathcal{E}$ for $\mathcal{E} \circ (\quad)_{\mathbf{k}}$. I'll fix c and write $\mathcal{E}^c$ for the appropriate homotopy colimit, computed objectwise.

We need to show that the homotopy fiber of $K^c \to KH^c$, let me call this fiber G, vanishes.

Now I'll introduce a topology. If X is smooth, then I know this is true. I'll introduce a topology where I know everything is smooth and then I'll show [unintelligible]in that topology. That's the goal.

Definition 26.3. A Cartesian square of monoid schemes in $\mathcal{M}_{pctf}$ is an *abstract blowup square* if p is proper and surjective (proper is not easy to define), i is a closed immersion (these are the maps to the target), and p is an iso outside $p^{-1}(\mathrm{im} i)$.

For example, this will only make sense if you've seen toric varieties before. One way to get p from a cone is to subdivide [pictures].

Definition 26.4. The cdh-topology on $\mathcal{M}_{pctf}$ is generated by

(1) the Zariski topology and
(2) $\{i, p\}$ where you have an abstract blowup square.

In characteristic zero, you can resolve by blowing up, replacing a point in the plane by a tangent vector. In characteristic zero everything is locally smooth.

Theorem 26.4. *(classical) Every object in $\mathcal{M}_{pctf}$ admits a smooth cover in the cdh-topology.*

You do this with toric varieties by subdividing cones. It's nasty combinatorics but classical.

If you believe this, then it suffices to show that for an abstract blowup p (I'll ignore i), if we apply G, then $G(X) \xrightarrow{p^*,\sim} G(X')$.

Theorem 26.5. *KH is a sheaf in the sense of, it's an (∞-)(hyper-)sheaf.*

We work just with the model structure on presheaves of spectra. This is similar, but now you have to do something. Using a Chern character:

Theorem 26.6. *the fiber of the map to negative cyclic homology (over $\mathbb{Q}$)*

$$K \otimes \mathbb{Q} \xrightarrow{\mathrm{ch}} HN(quad \otimes \mathbb{Q}/\mathbb{Q})$$

is also an ∞-sheaf as is the fiber of the cyclotomic trace

$$K/p \xrightarrow{\text{cyc}} TC/p$$

for p a prime.

I have two minutes left. These are hard theorems that come down to understanding not just that there is a resolution of monoid schemes but that there is one not given by subdivisions. Why would this in any way tell you anything? Now you can separate the part that comes from $\mathbf{k}$*, that's the whole point. We apply Hesselholt–Madsen to say that if you have a commutative monoid, you can compute*

$$TH(\mathbf{k}[A]) \cong_f TH(\mathbf{k}) \wedge |N^{\mathrm{cy}}(A)|.$$

Now we've separated out the $\mathbf{k}$*, now the negative cyclic homology can be computed from Hochschild homology. This reduces us to showing that if you take the cyclotomic nerve and promote it to a functor on monoid schemes, a Zariski [unintelligible] space with values in [unintelligible], then this is a cdh ∞-sheaf in $\mathbb{S}$-spaces. This is basically where the separating out part interacts with monoid schemes.*

That's a direct computation (sort of). A direct calculation plus some hard stuff in monoid schemes.

27. Mark Weber: Operads and polynomial 2-monads

[I do not take notes on slide talks.]