MINI-WORKSHOP ON LOW DIMENSIONAL TOPOLOGY

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1. Sept. 14: Min Hoon Kim: A family of freely slice good boundary links

It's a pleasure to be here. This is joint work with Jae Choon Cha and Mark Powell.

This is about links with certain properties, which are important in our field because they link to a major conjecture.

A *boundary link* is a link $L = L_1 \cup \cdots \cup L_m$ in the three-sphere such that there is a map f from the fundamental group of the link complement to the free group on m generators such that the meridians are mapped to the generators.

There are some equivalent conditions. The first one is that there is a map of link complements (really from the complement of a tubular neighborhood), well in here there is the boundary which looks like $L \times S^1$, and this can be mapped to the wedge of m circles (the Eilenberg–MacLane space for the free group)

Tis is also the same as the existence of disjointly embedded surfaces V_1 through V_m in the three-sphere so that the boundary of V_i is L_i.

So for example the linking numbers between each component should be trivial for boundary links. We'll think of a chosen map f from this fundamental group to F_m.

We say that a boundary link is *good* if $\ker(f)$ is perfect, the Abelianization of this kernel is trivial. This is the same as saying that the first homology of $\widetilde{S^3 \setminus L}$ is trivial.

This condition might look peculiar but I'll explain why it's important.

Let me give some exampls to give some feeling.

(1) A knot K is a good boundary link if and only if its Alexander polynomial is trivial. Every knot is a boundary link because it bounds a Seifert surface. If you Abelianize the complement you get $\mathbb{Z}$. But trying to make it a good boundary link, you need the homology of the cyclic cover to be trivial, that's the Alexander module.
(2) The Whitehead double of L is a good boundary link if and only if the linking numbers of L are all trivial. The kernel of this map is given by a Seifert surface. The Seifert form for the obvious surface for the Whitehead double is the same as the Seifert form of the finite double of the unlink. Then it should have a trivial Alexander module.

Now I should explain freely slice. A link L is *freely slice* if there exists topologically locally flat disjointly embedded disks $D_1, \ldots, D_m$ in D^4 so that $\partial D_i = L_i$ and $\pi_1(D^4 \setminus \sqcup D_i)$ is freely generated by meridians.

Freedman showed in 1983 that knots with Alexander polynomial 1 are freely slice. This follows from Freedman's topological surgery for $\pi_1 = \mathbb{Z}$. It's still open to extend this to general fundamental group, and that's why the following conjecture is important.

Conjecture 1.1. All good boundary links are freely slice.

Conjecture 1.2. All surgery problems have solutions.

I'll explain what a surgery problem (and solution) are. A surgery problem is a degree 1 normal map f from $(M, \partial M)$ to $(X, \partial X)$ where M is a topological 4-manifold and X is a Poincaré complex (a CW-complex with Poincaré duality) such that

- $f|_{\partial M} : \partial M \to \partial X$ is a $\mathbb{Z}[\pi_1(X)]$-homology equivalence.
- the obstruction class $\sigma(f)$ in $L_4^h(\mathbb{Z}[\pi_1(X)])$ is trivial.

A *solution* is a topological 4-manifold N with the same boundary as M and a homotopy equivalence g from $(N, \partial N)$ to $(X, \partial X)$ such that g restricted to ∂N is f restricted to ∂M.

Remark 1.1. The first conjecture holds if and only if the second conjecture holds.

I'd like to explain one direction of this remark, assuming the surgery conjecture and showing that boundary links are freely slice.

The first step, a good boundary link gives a surgery problem. Then the second step is to say that a solution gives a freely slice complement. That's what I want to sketch.

So for the first step, let M_L be a surgery manifold of L, and since L is a good boundary link, you can get a map to a wedge of circles, which we can promote to a map to $\#(S^1 \times S^2)$ which is the boundary of $\natural\, S^1 \times D^3$, and so you look at $[M_L, i \circ f]$ in $\Omega_3^{\mathrm{Spin}}(\natural\, S^1 \times D^3)$ and this is zero. So this is the surgery problem.

Why is [unintelligible]true? Because $\Omega_3^{\mathrm{Spin}}(\natural\, S^1 \times D^3) \cong \bigoplus \Omega_3^{\mathrm{Spin}}(S^1) \cong \bigoplus \mathbb{Z}_2$; since L is a good boundary link, the Arf invariant of L_i is zero.

I didn't check the conditions to be a surgery problem. The fundamental group for me is a free group, so I think of the universal cover, so I look at $H_1(\widetilde{M_L}) \to H_1(\#(\widetilde{S^1 \times S^2}))$ which is zero. Since L is a good boundary link $H_1(\widetilde{M_L})$ is zero, and then actually $M_L \to \#(S^1 \times S^2)$ is a $\mathbb{Z}[F_m]$-homology equivalence in higher degree as well.

For the surgery obstruction, $L_4^h(\mathbb{Z}[F_m]) \cong L_4^h(\mathbb{Z}) \cong 8\mathbb{Z}$ by Cappell-splitting. Then you can assume that $g : W \to \natural(S^1 \times D^3)$ has trivial surgery obstruction.

This good boundary link gives a surgery problem. Now I'd like to explain why a solution gives a freely slice complement.

A solution is a homotopy equivalence h from V to $\natural\, S^1 \times D^3$ with boundary f. So you attach two-handles to V along the meridians of L. Then π_1 of the resulting thing B is trivial. Then there is no second homology, and this implies that B is contractible. The surgery diagram looks like [pictures] which is just the three-sphere. So the boundary of B is the three-sphere, so B is homeomorphic to the 4-ball. Now the 2-handles are the slice disks we were looking for. And the complement is V, drilling out the cocore disks.

That's why a solution gives a free slice complement.

Remark 1.2. The surgery conjecture and the S-cobordism conjecture is the same as the disk embedding conjecture.

The best result on this is due to Freedman and Teichner in 1995. This says the disk embedding conjecture is true for 4-manifolds with π_1 with subexponential growth. So not for the free group.

Freedman conjectured in 1984 that surgery fails for $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. Specifically, the Whitehead double of the Borommean link is not freely slice.

There are some partial positive results on this conjecture that I want to state now.

In 1985, Freedman showed that if L is a boundary link then the Whitehead double of L is freely slice. In 1988, he showed that if L is two-component with linking number zero, then the Whitehead double of L is freely slice. In 1993, he showed that ∂^2-links are freely slice. I don't want to describe these explicitly. The best result on Whitehead doubles, of Freedman and Teichner in 1995, is that if L is homotopically trivial-plus, then the Whitehead double is freely slice.

I'd like to remark that boundary links and two component links with trivial linking are homotopically trivial plus.

Our theorem,

Theorem 1.1 (Cha–K–Powell)**.** *Suppose L is a good boundary link that has a Seifert surface admitting a homotopically trivial plus good basis. Then L is freely slice.*

This theorem recovers all the positive results, i.e., Freedman–Teichner.

It looks a bit technical, so I should explain some technical stuff from now. I should remark that we couldnt' get new examples of freely slice finite doubles, but could get some new freely slice good boundary links. There are some chances of getting new freely slice Whitehead doubles by looking at non-standard Seifert surfaces.

A *Seifert form* θ on $H_1(V)$ is an integer valued form $(x, y) \mapsto \mathrm{lk}(x^+, y)$, you push x and then count the linking number. A *good basis* is a symplectic basis $\{a_i, b_i\}^g$ of $H_1(V)$ such that the Seifert matrix of V with respect to this basis can be reduced to the null matrix via a sequence of elementary S-reductions.

I think I should give a matrix. A matrix is like this

$$
\begin{array}{c|ccccccc|}
 & a_1 & b_1 & a_2 & b_2 & \cdots & a_g & b_g \\
a_1 & 0 & \epsilon_1 & 0 & * & \cdots & 0 & * \\
b_1 & (1-\epsilon_1) & 0 & * & \cdots & 0 & * & \\
a_2 & 0 & 0 & 0 & \epsilon_2 & \cdots & 0 & * \\
b_2 & * & * & (1-\epsilon_2) & 0 & \cdots & 0 & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_g & 0 & 0 & 0 & 0 & \cdots & 0 & \epsilon_g \\
b_g & * & * & * & * & \cdots & (1-\epsilon_g) & 0
\end{array}
$$

where $*$ is arbitrary but ϵ_i is either 0 or 1.

Here is a little lemma using linear algebra.

Lemma 1.1. *A boundary link L is good if and only if it has a Seifert surface admitting a good basis.*

Maybe I should give an example. I told you that a Whitehead double is an example of a good boundary link. [pictures]

So what is HT^+? we say that alink L is homotopically trivial if L can be changed to an unlink by changing crossings of the form [picture]. A link L is homotopically trivial plus if $L_i \cup L_i^+$ is homotopically trivial for any i, where L_i^+ is a 0-framed parallel copy of L_i.

[pictures] So the Whitehead link is homotopically trivial-plus. But if you take this link, the resulting four-component link is not homotopically trivial.

I should still define a HT^+ good basis but I'm running out of time. A good basis $\{a_i, b_i\}$ is HT^+ if $K \cup b_j, K \cup b_j$ are HT where $K = \sqcup b_i'$ and b_i' is a praallel copy of b_i so that the linking number of a_i and b_i' is trivial. The result is K, and take a parallel copy, take the union with a_j and with b_j, for any j.

In this example [pictures] So then $\{a_i, b_i\}$ is HT^+ if and only if L is HT^+. This is why we recover Teichner and Freedman's result.

Lemma 1.2. *If $\{a_i, b_i\}$ is HT^+, then there exist $2g$ immersed disks Δ_i^+ and g immersed disks Δ_i in D^4 so that $\partial\Delta_i^+ = a_i$ and $\partial\Delta_{i+g}^+ = b_i$ and $\partial\Delta_i = b_i'$ for $1 \le i \le g$, so that $\Delta_i \cap \Delta_j = \varnothing = \Delta_i \cap \Delta_j^+$ (this latter even for $i = j$), and the signs of all the self-intersections add up to zero.*

Here is the real thing that we actually prove.

Lemma 1.3. *Let L be an m-component good boundary link with a Seifert surface admitting an HT^+ good basis. Then there exists a smoooth four-manifold W whose boundary is M_L and so that*

(1) *$\pi_1(W) = F_m$ and $H_2(W) = \mathbb{Z}^{2g}$ and*
(2) *there are immersed spheres $\Sigma_1, \ldots, \Sigma_{2g}$ which form a basis of $H_2(W)$ such that*
   - *for any i, Σ_{2i-1} and Σ_{2i} have a distinguished intersection point*
   - *$\bigcup \Sigma_*$ is an immersion of $\bigcup(S^2 \wedge S^2)$.*
   - *Undistinguished intersection points are paired up by Whitney disks.*
   - *The union of the Σ_i to W is trivial on π_1 for any choice of basepoint*

[pictures]

2. BoGwang Jeon: Volumes of hyperbolic 3-manifolds

Thank you to the organizer for giving us a chance to give a talk. I think I'm the only one who will talk about geometry. I think you all know about hyperbolic 3-manifolds, but let me go over it. So the typical example of a non-compact 3-manifold is a knot or link complement.

Theorem 2.1 (Thurston, 80s)**.** *Most link complements are hyperbolic.*

Most of them are a quotient of $\mathbb{H}^3$ by a discrete group action.

How about closed manifolds? First, how do we produce closed 3-manifolds? The typical way is Dehn filling. What is Dehn filling? It's a way to produce a closed manifold from a knot complement. First we truncate the boundary of the knot complement and then attach a solid torus by a homeomorphism from the boundary torus to the boundary of the knot complement. The topology is determined by the image of the meridian curve. For instance, [pictures].

Theorem 2.2 (Lickorish–Wallace)**.** *Any closed 3-manifold can be obtained by Dehn filling on a link complement.*

Theorem 2.3 (Thurston, 1980s)**.** *If M is hyperbolic then Dehn fillings are hyperbolic for almost all coefficients.*

Theorem 2.4 (Mostov, 1960s)**.** *If M is hyperbolic then its hyperbolic structure is unique*

Then the corollary is that voluem is an invariant of hyperbolic 3-manifolds.

Theorem 2.5 (Thurston)**.** *The volume of hyperbolic Dehn fillings approach the volume of M if $|p_i| + |q_i| \to \infty$.*

Corollary 2.1. *There exist only finitely many hyperbolic Dehn fillings of M having the same volume.*

Then we can ask the following question. Does there exist $c > 0$ such that the number of Dehn fillings of M having the same volume is less than c?

I'm sure Thurston was aware of this question, but today the theorem I'd like to talk about is:

Theorem 2.6 (J.)**.** *If M is a 1-cusp hyperbolic three-manifold, then the answer is yes.*

How do you answer this question? First let me explain why this question is not trivial.

Theorem 2.7 (Neumann-Zagier, 1985)**.** *There exists an explicit volume formula, for M a 1-cusped manifold, the volume of $M_{p/q}$ is $\mathrm{Vol}(M) + \Phi(p,q)$.*

For example if M is a figure eight complement, then

$$\Phi(p,q) - \frac{2\sqrt{3}\pi^2}{p^2+12q^2} + \frac{4\sqrt{3}\pi^4(p^4+72p^2q^2+144q^4)}{3(p^2+12q^2)^4} + \cdots$$

So we're counting when $\Phi(p,q) = \Phi(p',q')$.

So first, the volume formula is given as follows. So suppose $\Phi(p,q) = -\frac{1}{p^2+12q^2}$, and so for this case this is counting lattice points and the answer will be no, this is not uniformly bounded.

We don't believe that this is possible. The conjecture is that this is an infinite series which is transcendental. But if the volume form is simple then the answer to the question is no, there is no uniform bound.

Even if the volume formula is of the form, is some power series in terms of $p^2 + 12q^2$, for instance, then the answer is no.

Definition 2.1. Let $p_1/q_1, \ldots, p_n/q_n$ be a tuple of rational numbers. Then the *height* of this tuple is the maximal integer in $\{p_i, q_i\}$.

Theorem 2.8 (Bambieri-Pila, 1988)**.** *Let C be a curve defined by $y = f(x)$ in $\mathbb{R}^2$. Suppose that $f(x)$ is analytic but not algebraic. For example, we can take 2^x or e^x. For any ϵ there exists a $c(\epsilon, f)$ so that the number rational points of height less than T on C is bounded by the growth rate, by $c(\epsilon,f)T^\epsilon$.*

For example, rational points on 2^x grow logarithmically.

For the surface case we have to be careful. It's not enough to say non-algebraic. For instance, $z = x^y$, this is not algebraic, but if $y = 1$ or any rational y, this has many rational points. Or similarly 2^{x+y} has many rational points for $x+y$ rational.

Definition 2.2. Let S^{alg} be the union of all connected algebraic components embedded in S.

Definition 2.3. $S^{\mathrm{trans}} = S \smallsetminus S^{\mathrm{alg}}$.

So for instance, if S is algebraic then the complement is empty.

Theorem 2.9 (Pila, 2003)**.** *Let S be an analytic surface. Then for any ϵ, there exists $c(\epsilon, S)$ so that the number of rational points on S^{trans} whose height is less than T is bounded by $c(\epsilon, S)T^{\epsilon}$*

Later in 2007,

Theorem 2.10 (Pila–Wilkie)**.** *Let X be an analytic manifold, then there eists $c(\epsilon, X)$ so that the number of rational points on X^{trans} of height less than T is bounded by $c(\epsilon, X)T^{\epsilon}$.*

So let's think about $\Phi(x, y)$ as an analytic function in two variables. Consider $\Phi(x, y) = \Phi(z, w)$, this is a 3-dimensional analytic real manifold. Finding these points $\Phi(p, q) = \Phi(p', q')$, so this is finding rational points.

But we have to remove the algebraic part.

Theorem 2.11 (J.)**.** *If and only if X is algebraic, then and only then $\Phi(x, y)$ is of the following form, an analytic function $h(t)$ and a polynomial $f(x, y)$, and $\Phi(x, y) = h(f(x, y))$, which is the case I mentioned where there are infinitely many lattice points.*

Theorem 2.12 (J.)**.** *If $\Phi(x, y)$ arises as a volume function of a hyperbolic three-manifold then $\Phi(x, y)$ is not of the form $h(f(x, y))$ for h analytic and f polynomial.*

Then X^{trans} is non-empty.

Theorem 2.13 (J.)**.** *X has only finitely many algebraic components.*

For instance, go back to the figure eight knot complement. There are a few honest examples, $x = \pm z, y = \pm w$.

Then in general it should lie over this part, so there should only be finitely many.

3. Se-Goo Kim:Integer valued knot concordance invariants

Thank you, I'm pleased to be here, I'm, I was, I'm a graduate of POSTECH, that was almost 23 years ago now. The math department was not in this building at that time. POSTECH has many new buildings, and I'm pleased to see that the math department is bigger and has its own building.

Anyway my talk is about integer-valued knot concordance invariants. In fact, this is joint work with Mi Jeong Yeon and my work, everything in my work is combinatorial. It's a topological thing but all the techniques are combinatorial and very easy.

Let me give you a few things. The most famous invariants that are integer-valued knot concordance invariants are signature σ and around the 21st century, the τ-invariant introduced by Ozsváth-Szabó and the Rasmussen s-invariant. I'll mostly talk about these three invariants.

These three invariants satisfy many things. Say $\nu = \tau$ or $\nu = -\frac{s}{2}$. Then

(1) $\nu(K \# J) = \nu(K) + \nu(J)$,
(2) $|\nu(K)| \leq g_4(K)$, the minimal genus of a surface bounding K in B^4,
(3) $\nu(K) = -\frac{1}{2}\sigma(K)$ if K is alternating,
(4) $\nu(T_{p,q}) = \frac{(p-1)(q-1)}{2}$ for $p, q > 0$.

When Rasmussen first introduced the Rasmussen invariant, he conjectured that they were the same, $\tau = -\frac{s}{2}$. It turned out that it's not true, Hedden–Ording found examples of a knot K for which these two invariants are not equal.

So the question we could ask is why they are different (or when are they the same)?

So the goal of today's talk is that there exists some kind of local move so that if these three invariants are equal, then they remain equal.

Let ν be an integer-valued knot invariant satisfying the connect sum, genus, and alternating signature properties.

If K is alternating, then the signature can be written (this is well-known), if D is a reduced alternating diagram of K, then $-\sigma(K)$ is $1 - s_A(D) + n_+(D)$. Here $s_A(D)$ is the number of circles after A-smoothing of all crossings, and $n_+(D)$ is the number of positive crossings.

We let $t_A(D) = \frac{1}{2}(1 - s_A(D) + n_+(D))$.

One condition that will be necessary is an *almost alternating* condition on diagrams, this is a diagram which becomes alternating after one crossing change.

Theorem 3.1. *Let D' be a reduced almost-alternating diagram with negative alternation crossing d, representing a knot K' with $\nu(K') = t_A(D')$. Suppose D is obtained from D' via [pictures], and you get a reduced almost alternationg diagram with alternation crossing d, then $\nu(K) = t_A(D)$.*

Consider (generalized) pretzel knots [pictures].

Since ν is a concordance invariant, then ν of this pretzel knot

$$P(a_1, -a_1, \ldots, a_k, -a_k, b_1, \ldots, b_m)$$

is the same as for $P(b_1, \ldots, b_m)$ which is $t_A(P(b_1, \ldots, b_m))$, and if you isotope your negative crossings, [pictures]. By flyping you can make an almost-alternating diagram [pictures and examples].

Let me explain why this theorem is true. Livingston tells us that if K_+ and K_- differ at only one crossing in the evident way, then $0 \le \nu(K_+) - \nu(K_-) \le 1$.

Lemma 3.1. *Let D be reduced and almost-alternating. Then $t_A(D) - 1 \le \nu(K) \le t_A(D)$.*

The proof of this lemma is, D_a is the alternating diagram of D after changing the alternation crossing. Then you can give a checkerboard coloring on D_a. Since D is reduced the regions are separated. Then $s_A(D_a)$ is the number of unshaded regions. But the type A-move will separate, connect two white regions, so $s_A(D)$ is one less than $s_A(D_a)$. Let's suppose that the alternation crossing is negative in D, then $n_+(D) = n_+(D_a) - 1$. Then $t_A(D) = t_A(D_a)$. Now you can apply the Livingston equality here, we get

$$\nu(K_a) - 1 \le \nu(K) \le \nu(K_a)$$

but because K_a is alternating, then we get

$$t_a(D) - 1 \le \nu(K) \le t_a(D).$$

Other cases are similar. By this Lemma, $t_A(D) - 1 \le \nu(K) \le t_A(D)$, and $s_A(D') = s_A(D)$ whereas $n_+(D')$ and $n_+(D)$, we have to divide into two cases, if one of the crossings is minus then they're the same, and moreover we can use Livingston's criterion on one of the crossings involved, and we get

$$t_A(D) \le \nu(K) \le t_A(D) + 1.$$

The other cases are similar but a little bit complicated but quite easy.

So we could prove the theorem in a very similar manner.

This is basically, as you can se, the pretzel knots of the type I outlined, you can increase the negative strands to keep the same μ. That's about it. Thank you.

4. Peter Teichner: Topological phases, field theories and manifold invariants

[I do not take notes at slide talks]

5. September 15: Motoo Tange: Ribbon disk diagram in handle decomposition of B^4

Definition 5.1. A *ribbon disk* is an immersion from a two-dimensional disk D to S^3, with singularities which are all double points satisfying the ribbon condition.

[picture]

The boundary of such a disk is called a *ribbon knot*.

Definition 5.2. A *slice disk* is an embedding $D \to B^4$, and the boundary of such a disk is called a *slice knot*

A fundamental remark is that a ribbon disk can embed in B^4 as a slice. The consequence is that a ribbon knot is a slice knot, and the conjecture is that a slice knot is a ribbon.

I want to consider some generalized ribbon disk diagram. [picture]

We'd like to generalize the singularity diagram of a ribbon disk to any slice disk.

Definition 5.3. A *perforated ribbon diagram* in a handle decomposition $H^1 \cup H^2 \cup H^3$ (where H^1 is a circle, H^2 is a framed link, and H^3 is some surface) is an immersion $i : d_0 \to (S^3, HD)$, where $\Im(i) = d = \partial d + \mathring{d}$.

The boundary of d is $K \cup h_1 \cup \cdots \cup h_n$, then (S^3, HD, d) is called a PR-diagram in HD if it satisfies the conditions

1. K is embedded in $S^3 \setminus HD$.
2. $h \in H^1$ means $h \cap \mathring{d} = \varnothing$
3. $\mathring{d}$ transversally intersects $H^2 \cup H^3$
4. $hi = h$ is a component of $H^1 \cup H^2$ and the surface framing aronud hi coincides with the smooth framing of h.
5. Separating d and H^3 from $H^2 \cup H^1$ then we obtain a ribbon hole disk.

[pictures]

Definition 5.4 (Regular)**.** Let (S^3, HD, d) be a perforated ribbon diagram. If $H^3 \cap S(d)$ is empty, then we call the diagram *regular.*

[pictures]

This is a cancelling pair of HD [pictures] and this is a modified cancelling pair [pictures]

Theorem 5.1. *Let K in S^3 be slice. Then there exists a perforated ribbon diagram satisfying the following:*

1. *The boundary of d is K*
2. *HD is a modified cancelling pair.*
3. *HD is regular.*

[discussion]

For example, for any slice disk in B^4 we can get a perforated ribbon diagram, where HD_0 is a handle decomposition of $B^4 - D_0$ and $HD_0 \cup h^2$ is B^4. [pictures]

Let K be slice. If there is a perforated ribbon diagram (d, HD) satisfying $\partial d = K$, that HD is a *real* cancelling pair, and HD is regular, then K is ribbon.

Let me prove this remark. [pictures]

6. Taehee Kim: The 4-genus of knots and links

I am the last speaker so I want to thank Jae Choon for organizing this nice workshop. I'll give some new examples for the stable 4-genus for knots and 4-genus for links.

The 3-genus (or just genus) of K in S^3, denoted $g_3(K)$ is the minimal genus of a surface, compact, connected, and oriented, in S^3, which is bounded by K.

The 4-genus of a knot is $g_4(K)$, the minimal genus of a surface, the same except in the four-ball.

As you know, S^3 is the boundary of the four-ball. We measure the complexity of the knot by taking this minimal boundary. I'm allowing topologically locally flat embeddings so this surface has a topological normal bundle.

Because a surface in S^3 can be thought of in the four-ball, we have $g_4 \leq g_3$. But for the three-genus we have an additive property: $g_3(K\#J) = g_3(K) + g_3(J)$. But for the four-genus we only have subadditivity. Sometimes we have a trivial four-genus, $g_4(K\#(-K)) = 0$ for all K because this is a slice knot.

If g_3 is zero then K is the unknot, while if g_4 is zero then K is slice. So compared to the 3-genus, finding the 4-genus is much harder.

If you have the unknot, then its 3-genus is zero. The trefoil has genus 1. How do you find it. You have many lower bounds, [unintelligible]and the Floer homology, and even the [unintelligible]of a 3-manifold which is a generalization. But determining the 4-genus it's much harder. So what are the lower bounds that we have?

The elementary lower bound comes from the signature. Let V be a Seifert matrix for K. Then we can think of the signature $\sigma_K = \mathrm{sign}(V + V^t)$, and if $g_3(K) = g$, then we can find a Seifert matrix which is $2g \times 2g$, and this gives a lower bound trivially for the three-genus. But $\sigma_K \leq 2g_4(K)$. This is not trivial.

When $\omega \in S^1$, then $\sigma_K(\omega)$ is the signature of $(1-\omega)V + (1-\bar{\omega})V^t$, and this is also less than or equal to $2g_4(K)$. So e.g. $\sigma_K = \sigma_K(-1)$.

Maybe the strongest classical lower bound is the following, $m(K)$ which is half the dimension of V less half the dimension of the maximal null space of V. This gives a lower bound. These all vanish if K is algebraically slice (meaning there exists a Seifert matrix with top left corner 0).

There are some non-slice knots which are algebraically slice. So then it's much harder to find the 4-genus. Using Casson–Gordon invariants there are examples of algebraically slice knots with 4-genus at least 1 (and arbitrarily high).

Theorem 6.1 (Gilmer, 82)**.** *There is an algebraically slice knot K with $g_4(K) = g_3(K) = g$ for arbitrary g.*

Casson and Gordon in the 70s gave the first examples of algebraically slice knots that are not slice and this was a generalization.

Then we see that 4-genus is related to slice. Around 2000 Cochran, Orr, and Peter Teichner gave the first example of a non-slice knot with vanishing Casson–Gordon invariants.

Theorem 6.2 (Cochran–Orr–Teichner 03)**.** *There exists a nonslice knot with vanishing Casson–Gordon invariants.*

After their discovery we were interested in finding the behavior of knots with vanishing Casson–Gordon invariants.

Theorem 6.3 (Cha 08)**.** *There is an algebraically slice knot with vanishing Casson–Gordon invariants such that $g_4 = g$ (for arbitrary positive g).*

The Cochran–Orr–Teichner theorem uses L^2-signatures, which I'll talk about later.

Now it's time to state our theorem, which is about the *stable* 4-genus.

Definition 6.1. The *stable* 4*-genus* of K is $g_{\mathrm{st}}(K) = \lim_{n\to\infty} \frac{g_4(nK)}{n}$.

This limit is a positive real number. Since this is subadditive, then definitely this is bounded above by the four-genus of the knot.

For example, the stable four-genus of the trefoil, it turns out, that the 4-genus of the trefoil can be determined from the signature which is additive, and from that and some work we can see that g_{st} of the trefoil is 1.

The figure eight knot, if you take two copies of it, it's amphichiral so $4_1\#4_1$ is slice, so $g_4(4_1\#4_1) = 0$ which implies that g_{st} of the figure eight is zero.

One interesting question is what are the possible values of the stable genus?

Theorem 6.4 (Livingston 10)**.** *Let $\epsilon > 0$. There exists an algebraically slice knot K with stable genus between $\frac{1}{2}(1-\epsilon)$ and $\frac{1}{2}$. At this moment we don't have a knot with a non-integer stable genus but anyway we get an estimate.*

We wanted to use more modern tools to find out more refined behavior, so let me write down our first main theorem.

Theorem 6.5 (Cha–M.H. Kim–K. 18)**.** *Let $g \geq \mathbb{N}$ and $\epsilon > 0$. There exists an algebraically slice knot K such that*

(1) *K has vanishing Casson–Gordon invariants,*
(2) *$g - \epsilon \leq g_{\mathrm{st}}(K) \leq g$.*

This depends on both g and ϵ. For the proof we used L^2-signatures, of Cochran–Orr–Teichner.

For the proof I'll give the key ingredients. How do we get the classical lower bounds? People started using Morse approaches, and then branched covers. Nowadays people use bordism.

Suppose K is the boundary of a surface in the four-ball of genus g.

Then for M_K, a 0-framed surgery on K in S^3, there exists a four-manifold W^4 such that

- $\partial W = M_k$
- $H_1(M) \cong H_1(M_K)$ (induced by the boundary inclusion), and
- $\beta_2(W)$ the second Betti number is $2g$.

This implies that the ordinary signature of W is less than or equal to $2g$.

We obtain W by taking the 4-ball, and attaching 0-framed 2-handles and cap off the surface, taking off a tubular neighborhood of the surface. This was a surface of genus g, and we attach a handlebody of genus g, so I'll write $H_g \times D^1$.

Then we get the bordism whose first homology is that of the surgery.Then $\pi_1 M_K \to \pi_1 W^4$ agree over $\mathbb{Z}$. But in the algebraically slice case the signature is 0. So then we use the L^2 signature.

So we have the fundamental group of the 0-framed surgery and the 4-manifold, and we go to the commutator subgroup of $\pi_1 W$ quotiented by *its* commutator subgroup, $G^{(1)} = [G, G]$ and $G^{(2)} = [G^{(1)}, G^{(1)}]$, so this is

$$\left(\pi_1 W^{(1)}/\pi_1 W^{(2)} \otimes_{\mathbb{Z}} \mathbb{Q}\right) \rtimes \underbrace{\mathbb{Z}}_{\pi_1 W/\pi_1(W)^{(1)}}$$

which I'll call Γ. Then we have $\mathbb{Z}\Gamma \to \mathbb{C}\Gamma \to N\Gamma$, the *group von Neumann algebra*. Then we have $H_2(W, N\Gamma) \times H_2(W, N\Gamma) \to N\Gamma$, and it turns out that for any $N\Gamma$-modules we can take the L^2-signature with values in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ and we can take $\mathrm{sign}^{(2)}_\Gamma(W)$ in $\mathbb{R}$.

How do we approximate the signature, we use the second Betti number, that's the classical case, and it turns out that

Lemma 6.1. *The second L^2-Betti number of W with respect to Γ, which is the L^2 dimension of $H_2(W, N\Gamma)$, this is less than or equal to $\beta_2(W)$.*

I won't prove this, but [quick argument]. It's not hard but just believe me.

The corollary is then that the L^2 signature is at most the second L^2 Betti number so at most $2g$.

This is how we give a lower bound on g using the L^2 signature.

Due to lack of time I won't give other ingredients, but it turns out we need to look at the difference between these two, and the lower bound we get is $2g + 2g = 4g$.

Okay. I have twenty minutes so let me change the subject. We measure how close a knot is to being slice to see its complexity. We can also use gropes to decide how close a knot is to being slice. I'll use only symmetric gropes with a height in this talk. What is a height 1 grope? It's a surface with one boundary component. This is a surface of arbitrary genus. A grope with height 2 is a genus 1 surface (which thus has two symplectic basis curves) with a surface with one boundary component attached along the two symplectic basis curves. These are all disjoint except at attaching circles. Continuing this you get gropes of different heights.

What do we do with these gropes?

Let me denote by G_n the knots which bound gropes of height n in B^4. Then it turns out that if $\mathcal{C}$ is the knot concordance group, that this gives a filtration $0 \subset \cdots \subset G_{n+1} \subset G_n \subset \cdots \subset \mathcal{C}$. If you have a grope of height $n+1$ then ignore the top surface and you get a grope of height n. Shen 0 is the slice knots, a grope of any height. A disk, pick any point, and a small neighborhood you can just replace. The height gets bigger and bigger, then this gets closer and closer to being slice.

A big open question, we know that G_n/G_{n+1} has a big subgroup $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty$. All of these have $g_4 \geq 1$ but that's all we know. If K is in G_4 then K is algebraically slice and has vanishing Casson–Gordon invariants. It's hard to find the information about what happens in height four. So what baout K in G_n for $n \geq 4$, one with $g_4(K) > 1$.

Today I can't give examples of knots, but I can give examples of links. We can think of the same question for links. Suppose L is an m-component link. We can define the 4-genus in a strong sense, there are two versions but for me the 4-genus is the minimum of the sum of genus of surfaces Σ_i, $\partial\Sigma_i$ is L_i, this is in B^4 and these are nonintersecting, compact, and oriented. We have the same kind of grope filtration, $G_n(m)$, here m is fixed and is the number of components.

Theorem 6.6 (Cha–M.H. Kim–K. 18)**.** *Let $n \geq 4$ and let $m \geq 2g \geq 4$. Under this condition, there exists an n-component link L which satisfies the following conditions:*

(1) *L is a boundary link (has vanishing Milnor concordance invariants in particular) and has unknotted components*
(2) *L is in $G_n(m)$ but not $G_{n+1}(m)$.*
(3) *(this is our new condition, the other two were known) and $g_4(L) \geq g$.*

In links the connected sum is not well-defined, but with choices we can make a connect sum, and we can come up with a generalization about the stable case but let me not say that in this talk.

I'll give two sentences about the proof, only. We proved this using two different methods. The first method is algebraic n-solution, which is due to Cochran–Teichner and generalized by Cochran and myself. The second method is using iterated covers. You have Hirzebruch-type invariants from iterated covers (this is by Cha). Okay so I have five minutes so I think I'll stop here today.