CGP DISTINGUISHED VISITORS SERIES DMITRY KALEDIN: HIGHER ALGEBRAIC STRUCTURES AND THE SEGAL APPROACH

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1. April 6

This is all joint work in progress with E. Balzin. This goes back to the 70s, and what people were studying was infinite loop spaces, suppose you have X_0, a topological space with a basepoint, and suppose that this is the loopspace of something else ΩX_1, and suppose $X_1 \cong \Omega X_2$, and so on $X_n \cong \Omega X_{n+1}$. An example is BU. This is equivalent to what people call a connective spectrum.

If you know that a topological space is a loop space, you have an operation of composing loops. It's automatically a monoid with respect to composition of loops. Normally you take loops of length one and compose and reparameterize, and this is associative up to homotopy. There is a way out in this situation. You can use loops of varying length and you can compose those so that they are associative. But in X_2, a double loop space, then this composition operation becomes commutative, but it's already hopeless to have a strictly associative and commutative operation, just at the level of homotopy groups. Then an infinite loop space is associative and commutative up to all higher homotopies. This means that if you just have a associative commutative group, [unintelligible]an infinite loop space, but the converse is not true.

There are several approaches to working with this, what is the precise meaning of "up to higher homotopies?" Informally, the meaning is clear. It's not associative, but there are some homotopies between two compositions. Then this should satisfy something, and it does, but only up to higher homotopies, and so on. So there is May's approach, based on the notion of operads. The observation is the following. For something strictly commutative and associative, if X is strict, then you have a single map $X^n \to X$ which just composes, this factors through the quotient by the symmetric group X^n/Σ_n. You introduce some additional bunch of topological spaces that you call an E_∞ operad, for any n you have a space $\mathcal{O}_n$, and there are operations I won't recall, but you also have $\mathcal{O}_n \times X^n \to X$. These have to be compatible with compositions in a sense that can be made precise. This takes some writing down but it can be done. May has some specific topological space that you can put here, but you can prove that this doesn't matter, any set of topological spaces such that these guys are contractible, and the symmetric group action is free, this is good enough. You still have to choose it. When you have a specific X, you have to construct all of these. Nobody uses this approach, it's impractical.

There's an alternative due to Segal. The package is similar data in a way that is much more efficient to do topology. So I need some notation. Let's denote by Γ_+ the category of finite sets with partial maps, the maps are maps from a subset of

S_1 to S_2. You can describe this alternatively as the category of finite pointed sets. You define this on a subset, and you take the complement to the fixed point.

Then one has the definition.

Definition 1.1. A functor $X : \Gamma_+ \to \mathrm{Top}$ is *special* if the following is true. If we have two finite sets we can take disjoint union. There is a canonical map $S_1 \sqcup S_2 \to S_1$ and likewise for S_2. Since X is a functor, we have

$$X(S_1 \sqcup S_2) \to X(S_1) \times X(S_2).$$

These maps always exist, and the requirement is that these maps are homotopy equivalences for any S_1 and S_2 in Γ_+.

You can produce an infinite loop space, Segal tells us, from a special functor. There is one point, the difference between monoids and groups. But let me neglect that.

Theorem 1.1. *E_∞ topological spaces are equivalent to special functors $\Gamma_+ \to \mathrm{Top}$.*

Let me make this more precisely later. I want to omit the proof, this is nontrivial, but let me explain what is going on. Why does this work? First simplify the problem. First consider topological spaces which are discrete, just sets. Assume we have a special functor $X : \Gamma_+ \to \mathrm{Top}$. You can decompose any finite set into its elements, and so you have $X([n])$ (where $[n]$ is the set with n elements), we have a projection to X^n (where $X = X([1])$), and there is also a map to X which corresponds to sending all elements to 1. The former map is a homotopy equivalence. If these are discrete then the homotopy equivalence is an isomorphism and you can invert it, then you get $X^n \to X$ for every n. You don't have to require anything. More generally, if you have an E_∞ algebra, if X is an E_∞ space, we have a diagram

$$\begin{tikzcd} & X_n \times \mathcal{O}_n \arrow[dl, "\cong"'] \arrow[dr] & \\ X^n & & X \end{tikzcd}$$

so if X has the structure of an E_∞ space, you get something of this nature, but it doesn't actually have to be $X_n \times \mathcal{O}_n$, it can be just $X([n])$. This extends to a correspondence from E_∞ spaces to Segal spaces, not surjective on the nose but up to homotopy.

Okay. So that's how it works. Because of this flexibility. In topology the application is to construct a spectrum, like K-theory. So you have a space and you want to prove it has an infinite loop space structure, and this is hard to do by hand so you need a machine.

You can play the operad game in any monoidal category, like vector spaces or chain complexes. The May approach works here.

In topology, Segal's approach is standard, but operads make sense in any symmetric monoidal category. This was resurrected in the early 90s and this became a huge industry immediately. Many structures fitting this came from physics. This turned out to be the only technology. This is not enough, there are structures that don't come from operads, like Hopf algebras, so there are extensinos. So why is this not common in topology? Well you need much less to check in the Segal approach. I didn't give a definition of an E_∞ space or operad but I've given an honest expression. So you'd like something like this for chain complexes.

Does the Segal approach work for these other categories? No. I gave a product of two canonical maps in the Segal approach. In a general monoidal category, we have two separate maps $X(S_1 \sqcup S_1) \to X(S_1)$ and $X(S_1 \sqcup S_2) \to X(S_2)$. But we don't have what we actually want, a map from $X(S_1 \sqcup S_2)$ to the *product* of these. In topological spaces I used the Cartesian product, but here I'm using a tensor product with no universal property.

Okay, solution. The solution requires several ideas. First, one thinks, one reason it doesn't work is you're trying to combine things for two different worlds, symmetric monoidal categories and [unintelligible]. So restart. Rewrite the definition of symmetric monoidal categories using the Segal formalism. This is possible to do because the category of categories is Cartesian. I'll define by Cat the category of small categories, it has a Cartesian product, so I can try defining a symmetric monoidal category, it's basically, you can define a commutative associative monoid in Cat by functors Γ_+. This is not quite the correct notion, there is one problem, namely that what we usually mean is not a commutative monoid, because in nature thing are never commutative or associative precisely, always up to a canonical isomorphism. In practice you cannot achieve this. So we have the problem that a symmetric monoidal category is not strictly commutative. So this would work but it describes the wrong notion. Fortunately the solution to this has been known for a long time. We change the following.

Definition 1.2. A *pseudofunctor* A from some category $\mathcal{C}$ to Cat is the following:

(1) for an object c of $\mathcal{C}$ a category $A(c)$,
(2) for a morphism $f : c \to c'$ a functor $f_! : A(c) \to A(c')$,
(3) for a pair of composable morphisms f and f', there are the functors $f'_! \circ f_!$ and $(f' \circ f)_!$, and if this were a category these would need to be strictly equal, but we want a natural isomorphism $(f' \circ f)_! \to f'_! \circ f_!$
(4) (plus something for identity maps)

This data should satisfy that for triple compositions there is some condition.

The whole point of my exercise is to cut down on the amount of data. But here I have already a lot of stuff, functors, morphisms, so on.

There is a solution to this invented by Grothendieck, these days it's called the Grothendieck construction, in SGA I, Exposé VI. So what do you do to package this concisely. Assume you are given a pseudofunctor $A : \mathcal{C} \to \mathrm{Cat}$. Then from this we can construct one more category, define a category $\mathrm{Tot}(A)$ (similar to describing sheaves by the étale covering space) whose objects are pairs $\langle c \in \mathcal{C}, a \in A(c)\rangle$ and morphisms from $\langle c, a\rangle \to \langle c', a'\rangle$ are given by $\langle f : c \to c', f_!(a) \to a'\rangle$. To define composition you need the pseudofunctor map in the direction I gave.

Okay, we have a projection $\mathrm{Tot}(A) \to \mathcal{C}$, just forget the second factor, and the miracle is that you can recover the pseudofunctor and all the maps from this. This is a Grothendieck cofibration. Let me explain this. I'll need a couple of abstract definitions.

Definition 1.3. Assume we have a functor $\pi : \mathcal{C}' \to C$. Then a morphism $g : c'_1 \to c'_2$ is called coCartesian with respect to π if for any map $c'_1 \xrightarrow{g} \tilde{c}$ in $\mathcal{C}'$ which project to the same thing as f, $\pi(f) = \pi(g)$, there exists a unieque diagram

$$\begin{tikzcd} c_1' \arrow[rr, "g"] \arrow[dr, "f"'] & & \tilde{c} \\ & c_2' \arrow[ur, "i"'] & \end{tikzcd}$$

So I take all sorts of arrows that project downstairs to the same morphism downstairs, then I want f to be initial in the category of these (with fixed initial point). I will illustrate this for the $\mathrm{Tot}(A)$ case.

Definition 1.4. π is a cofibration if

(1) For any $f : c_1 \to c_2$ and c_1' in $\mathcal{C}'$ such that $\pi(c') = c_1$, there is a coCartesian map $f' : c_1' \to c_2'$ such that $\pi(f') = f$
(2) Composition of coCartesian maps is CoCartesian

This is analagous to fibration, the covering homotopy property. This is more precise. You not only can do it in some way but you can do it in this unique way.

Now the theorem.

Theorem 1.2. *(Grothendieck) There is a one to one correspondence between pseudofunctors* $\mathcal{C} \to \mathrm{Cat}$ *and cofibrations* $\mathrm{Tot}(A) \to \mathcal{C}$*.*

Let me explain how this goes. One direction is obvious, a pseudofunctor gives a projection. Why is this a cofibration? This is because, we have to look at the definition of maps. What are the maps with a given projection downstairs? A map $g : \langle c, a\rangle \to \langle c', a'\rangle$ with $\pi(g) = f$, what remains is just the second factor, is the same thing as a map $f_! a \to a'$ Any map, we can always just take $\langle c, a\rangle \to \langle c', f_! a\rangle \to \langle c', a\rangle$, there is a canonical factor like this, and this is exactly the coCartesian morphism. So then every other guy factors through this.

Conversely, assume that we have a cofibration $\mathrm{Tot}(A) \to \mathcal{C}$ where this is notation, I want to recover A. Then $A(c) = \pi^{-1}(c) \subset \mathrm{Tot}(A)$. Assume $f : c \to c'$, any $a \in A(c)$ gives a coCartesian lift $f' : a \to a'$ such that $\pi(f') = f$, in particular, a' sits in the fiber over c', and by the universal property, such a lifting is unique up to unique isomorphism. There are choices but they can be made consistently by this uniqueness. So let $f_!(a) = a'$. Finally you need the other maps, but these follow from the coCartesian condition and the composition property.

Every second year, someone asks me what a group acting on a category is, you can package it in terms of such cofibrations, this is what this was invented for. It minimizes the amount of data you have to keep track of. Conditions are easy.

I explained what we change the functor to, let's go back to monoidal functors. Symmetric monoidal categories are in one to one correspondence with special pseudofunctors from Γ_+ to Cat, which are in one to one correspondence with cofibrations $\mathcal{M} \to \Gamma_+$ which are special. Now we can say what this means: $\mathcal{M}(S_1 \sqcup S_2) \to \mathcal{M}(S_1) \times \mathcal{M}(S_2)$ is an equivalence of categories. You don't need to say anything about compatibilities, it's just a condition.

Maybe it's good to rewrite this explicitly if I have a usual monoidal category. Given a symmetric monoidal category M, then $\mathcal{M}$ is the following. Objects are pairs $\langle S, \{A_s\}\rangle$ where $A_s \in M$. Morphisms, there are two types in Γ_+. There are those which are maps of sets, and there are those which are subsets. Lurie calls these *active* and *inert*. For active maps $f : S_1 \to S_2$, to lift this, I need $f_! a \to a'$, and if you spell this out in terms of our description, for any s in S_2, well, I have a morphism $\langle S_1, \{A^1_s\}\rangle$ to $\langle S_2, \{A^2_s\}$, for all s in S_2 I have $\otimes_{[unintelligible]} A^1_{s'} \to A^2_s$.

Proposition 1.1. *Commutative associative algebras in M are in one to one correspondence with sections of the projection $\mathcal{M} \to \Gamma_+$ satisfying the condition that for any inert map f in Γ_+, $\alpha(f)$ is coCartesian with respect to π.*

What's going on here? I don't give a proof but I'll give the correspondence. How do I get α given A? For a one point set, this is just $\alpha(S) = \langle S, \{A_s\}\rangle$ with $A_s = A$. A section is a functor. For every morphism of sets, I should have a collection of maps $\otimes A \to A$, these are just the product maps.

This looks like a lot of data $\langle S, \{A_s\}\rangle$ but the proposition tells you that you never need to see this again.

Okay. Now in the last ten minutes I want to give you a final definition which is new. I don't expect you to have any feeling so far. I just want to have it on the blackboard. Then in the next lecture I'll recall model stuff that will put this on more solid ground.

Did we find some sort of solution to the Segal problem, a good version of Segal's approach for general monoidal categories, commutative associative objects in them. I want chain complexes, but it's simpler to do it in general. No, well, we did, but it's not homotopic. We have M which is chain complexes. M is a monoidal category, and I want to say it has a notion of weak equivalences, quasi-isomorphisms or something. These should satisfy axioms but more about this next time. We obtain $\mathcal{M} \to \Gamma_+$, and we know that commutative associative algebra objects correspond to sections $\alpha : \Gamma_+ \to \mathcal{M}$. What I started with, I did not need a commutative associative algebra, but something slightly less. I needed, for any n a diagram

$$\begin{tikzcd} & A_n \arrow[dl, "\cong"'] \arrow[dr] & \\ A^{\otimes n} & & A \end{tikzcd}$$

You better have a packaging that provides a composition, but anyways. This formalism with α does not give it. You can relax coCartesian to make the other map in the factorization a weak equivalence, but then [unintelligible]is still a tensor product. You can't just use the tensor product. This gives a nonhomotopical answer. There is one more thing to do. I won't give a precise definition. We have some general cofibration $\pi : \mathcal{E} \to \mathcal{C}$ and the fibers have some notion of homotopy in them. What one does is redefine sections. One can (and should) define "derived sections" of π (I won't give a definition today). One needs simplicial technology and the notion of the nerve of a category. I recall that for any small category you have this simplicial set the *nerve* of $\mathcal{C}$. You have this category Δ of finite nonempty sets, and on Δ^n you have $c_0 \to \cdots \to c_n$. So one applies the Grothendieck construction and obtains $\Delta\mathcal{C}$ which is cofibered over Δ^{op}. The general slogan is start with a cofibration, produce this cofibration, and consider sections of that. In $\mathcal{C}$ you get two objects and a morphism. Here you get an object corresponding to the morphism and two maps which are the inclusions of the domain and codomain objects.

2. April 11

Last time I described an approach that was supposed to be "up to something" homotopy or something, and today I want to talk about what that means precisely.

The title is something like "Generalities on abstract homotopy theory." There is a book of Quillen, "Homotopical algebra," there is a rumor he wasn't happy about it, but I think it's pretty good, there are flaws in the theory but we know how to fix a lot of them. There are two main ideas here.

(1) The first is that the subject of abstract homotopy theory is localization, and not abstract homotopies. This originates in topology, where you think about homotopies of maps. Homotopy classes of maps is what you're working with. Already in topology, though, and definitely in algebra, it turns out it's better not to identify things but to invert things. You get the same category for topological spaces, but this is more natural in many other situations, differential graded algebras.

(2) Then there's a technical point, which says to control this localization, you need some technical things, and Quillen developed a technology he called model categories.

These days people sort of conflate these but I think it's important to think of the second as a technical tool and the first as the fundamental idea.

2.1. **Localization.** So what's localization? We start with a category $\mathcal{C}$ and a class of maps W in $\mathcal{C}$. This should be closed under composition, contain isomorphisms, and it's useful to impose an axiom called 2-out-of-3. If you have two maps f and g which are composable, so you have the composition $g \circ f$, if two of these are in W, then the third one is.

Given this, from this data, one defines the homotopy category of $\mathcal{C}$ by formally inverting W. There is $\mathcal{C}$, there is $\mathrm{Ho}(\mathcal{C})$ and a functor $\mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ such that for any functor F from $\mathcal{C}$ to $\mathcal{E}$ which inverts W, you get a functor F' :

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{E} \\
\mathrm{Ho}(\mathcal{C}) \arrow[ur, dotted, "F'"'] &
\end{tikzcd}$$

How do you do this? Explicitly, objects of $\mathrm{Ho}(\mathcal{C})$ are objects of $\mathcal{C}$ and morphisms are chains like this

$$\begin{tikzcd}
& \bullet \arrow[dl, "\sim"'] \arrow[dr] & & & \bullet \arrow[dl, "\sim"'] \arrow[dr] & & & \bullet \arrow[dl, "\sim"'] \arrow[dr] & \\
C_0 & & \bullet & {} & \bullet & {} & \bullet & & C_1
\end{tikzcd}$$

where the maps marked by $\sim$ should be W. And you have to do something, take some equivalence relation maybe on these.

So the problem here is that this might not be a set, the category could be large, so there might be too many of these, and you have to solve this problem. There are some categories where you can do this for trivial reasons, dg modules over dg algebras, then you don't need model categories. Then it's completely abstract and you have no control but you can at least define it.

One concept you can already define is the concept of homotopy limits and colimits. So you have I a small category, and you have the category $\mathcal{C}^I$ of functors from I to $\mathcal{C}$. Here you have an obvious class of weak equivalences induced by W, we say a map is a weak equivalence if it as weak equivalence pointwise. We have this class

W^I, where a map $f : A \to B$ is in W^I if for all i in I, the map $f : A(i) \to B(i)$ is in W.

We have a tautological pullback functor $\tau : \mathcal{C} \to \mathcal{C}^I$, you can consider the constant functor with value that object. If you have a functor that sends weak equivalences to weak equivalences, you know it induces a functor on homotopy categories, so you have $\tau : \mathrm{Ho}(\mathcal{C})$ to $\mathrm{Ho}(\mathcal{C}^I)$. Then $\mathrm{hocolim}_I$ and holim_I are left and right adjoints to τ (between the homotopy categories).

Hirschhorn makes a mess out of this in his book, he makes a definition that computes it. This requires a bunch more structure. This definition doesn't give you existence but it gives you a definition.

Now a thing that exists in full generality, I won't tell you about this today but just mention it, for any two objects c_0 and c_1 in $\mathcal{C}$, you have, well, you have the space of maps in the homotopy category, but you have a homotopy type $\mathcal{H}om(c_0, c_1)$, so that homotopy classes of maps are components of this, but you have higher homotopy information. This was done by Dwyer and Kan. You can not just glue the zigzags stupidly but you can keep some information about the gluing. Anyway this exists in full generality.

In my mind the thing to compare this to is homological algebra. Homological algebra you work with derived categories. You either take resolutions and then resolutions up to homotopy or you can invert quasi-isomorphisms. The latter gives you more freedom but is more difficult to work with explicitly. In sheaves, you always do this by adjunction. For example, pullback of sheaves is obvious. The nontrivial fact is to define direct images, but you know that you get [unintelligible]. This adjunction is very important and this is the way to get into this world.

This is much as you say in full generality. Let me now go to point two, to model categories or model structures.

2.2. **Model structures.**

Let me start with a definition and then give an example. Maybe it's useful to repeat the definition. This is due to Quillen except he called it closed model category.

Definition 2.1 (Quillen)**.** The category $\mathcal{C}$ is a closed model category if

0 The category $\mathcal{C}$ has finite limits and colimits (you want to avoid the limits but you can't)

1 You have three classes C, F, and W, called cofibrations, fibrations, and weak equivalences, closed under composition, and retracts (this means if you have a map and a direct summand, then the direct summand is also contained) (and W satisfies 2-out-of-3).

2 Assume you have a diagram like this

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "a"'] & \bullet \arrow[d, "b"] \\ \bullet \arrow[r] \arrow[ur, dotted] & \bullet \end{tikzcd}$$

where either $a \in C$ and $b \in F \cap W$ or $a \in C \cap W$ and $b \in F$, then there exists a dotted morphism making the diagram commute.

3 Assume you are given a map c, then there exist a and b such that c factorizes as the composition $b \circ a$ where $a \in C \cap W$ and $b \in F$. Also there is a factorization where $a \in C$ and $b \in F \cap W$.

I'll tell you why this is useful and then give an example.

Definition 2.2. For a model category $\mathcal{C}$, an object c is called *cofibrant* if the map from the initial object 0 to c is a cofibration and *fibrant* if the map from c to the terminal object 1 is a fibration.

Proposition 2.1. *(Quillen) Morphisms in* $\mathrm{Ho}(\mathcal{C})$ *are all represented by diagrams of the form*

$$\begin{tikzcd}
& P \arrow[r] \arrow[dl, "\sim"'] & I & \\
c_0 & & & c_1 \arrow[ul, "\sim"]
\end{tikzcd}$$

where P is cofibrant and I is fibrant, and the maps marked ~ are weak equivalences.

Then there is a big theorem, Quillen adjunction, which gives you some way to construct homotopy adjoint functors, adjoint functors from the homotopy categories. This does not give you a way a priori to construct homotopy limits and colimits.

Let's do an example. It's too hard to understand what's going on. My example will be linear. Take a ring R and take the category of bounded above R-modules. My class W will be quasi-isomorphisms, F will be termwise surjective maps. There is a general fact about model categories, saying that if you know W and C then you can recover F, because the lifting property gives you this. In order to specify the model structure you can always give W and either C or F. The fact or claim is that this indeed defines a model structure on $C^*(R)$. I'll describe cofibrant objects are complexes of projective R-modules.

Fine, now, examples where this doesn't work, where there is no model structure. The first is a trivial example. Assume you have two complexes M and N, and a map f between them which is a quasi-isomorphism. This is too stupid to be interesting, but you could add data and then make it interesting. Then this category is not a model category for a stupid reason, axiom 0 breaks down. When you have a limit or colimit of such a thing, the condition of being a quasi-isomorphism is not stable under limits or colimits. This is maybe what Quillen was unhappy about. For a long time people wanted axiomatics that didn't require this condition.

A second example, the first was about a homotopical condition. This is a question of size. Complexes of constructible sheaves on a complex manifold, here what breaks down is, you want to have those factorizations, fibrant and cofibrant replacements. In Abelian categories you have this one I described (projective) and the dual, injective. For sheaves you don't have projectives, so you use injectives, but those are huge and take us out of the constructible world.

So this is why people started talking about derived categories, [unintelligible].

A general slogan for dealing with both is that the essential data is the weak equivalences, and for most purposes, you don't need $\mathcal{C}$ itself to have a model structure, but you usually need to put $\mathcal{C}$ in some bigger category $\mathcal{C}'$ that already has a model structure fully faithfully and you want the embedding on the level of homotopy categories to also be fully faithful. So you put a condition on a model category stable on objects under weak equivalences. If you do something on the bigger category it gives you the thing on the subcategory. You can first make a model category of all maps and then if your condition is homotopy invariant, then you are good there.

Okay, now let me, now I want to illustrate this in one problem, computing limits and colimits, this is known and due to Bousfield, and then I'll go back to what I started with and show how this can give a very nice description of [unintelligible]. I don't know what I'll get to today, but I'll start.

2.3. **Application to** holim **and** hocolim**.** Let us try to construct these. Let me start with a model category $\mathcal{C}$ (let's say with all limits and colimits). Ideally I'd like to prove that for any small category I, we have $\mathrm{hocolim}_I$ and holim_I from $\mathrm{Ho}(\mathcal{C}^I) \to \mathrm{Ho}(\mathcal{C})$.

There is this great Quillen adjunction theorem that says a functor between model categories with a nice enough adjoint, you can get adjoint functors on the homotopy category level, but there is a problem that $\mathcal{C}^I$ has no model category structure. You can try, you know weak equivalences. You could say on a hunch maybe cofibrations are termwise cofibrations. Fibrations would be something horrible. Under some assumptions this works but in general no. So this is imposing conditions on $\mathcal{C}$. There is a huge industry for contstructing cofibrantly generated model structures and these are basically of the form I had here for complexes. You have weak equivalences. You give yourself some class of cofibrations and then you generate things and see what works. What this gives you in practice, typically cofibrant objects are free resolutions of some sort. I had projective resolutions in my example, but [unintelligible]. I could have considered dg algebras. F is something simple, W is quasi-isomorphisms, and then cofibrants are like free algebras. This doesn't work for Hopf algebras, there are no free Hopf algebras. We need some easy gadget that controls localizations, free objects is too much to assume on a category. This takes us back to the 50s where we can't compute homology without choosing a free resolution.

There is one general theorem about existence of model structures in situations like this that impose no conditions on $\mathcal{C}$ but a strong condition on I. This is kind of the approach I like. This is called the Reedy model structure. The approach is two steps. You do this for some very special category I, and for any category you can find a different one that satisfies the condition, the nerve of I. So

Definition 2.3. I is a *Reedy category* if, well, I is small, we have a degree function that assigns a non-negative integer to any object $\mathrm{Ob}\ I \to \mathbb{N}$, and any map $f : i_0 \to i_1$ has a unique factorization into $f_1 \circ f_0$ such that the middle trem is less than or equal to the the degree of i_0 and i_1. These should be in a subcategory of latching and matching maps.

This is an extremely strong condition, if there are isomorphisms, you can factor and get two different decompositions of the same map.

Let me give examples. These are strong axiomatics but there are examples. Take Δ or Δ^{op}, finite totally ordered sets (or the opposite category). I think Reedy did this in his thesis and then didn't do much after this. I think he only did it for Δ and Δ^{op}. For any category I you have its simplicial nerve and this is also a Reedy category. This I'll explain in the next lecture.

Theorem 2.1. *For any Reedy category I, the category $\mathcal{C}^I$ has a model structure with the specified weak equivalences.*

The price you pay is that both the fibrations and cofibrations are hard to describe.

How does this work? You do it by induction. Say you have some object in I of some degree n.

Definition 2.4. The *latching category* is the category of all latching maps (which increase degree) into i and the *matching category* is the category of matching maps (which decrease degree) out of i.

Now assume, denote by $I_{<n}$ the full subcategory of objects of degree less than n. Assume you are given a functor $A : I_{<n} \to \mathcal{C}$. I want to extend this to the object i. I want to add the object i to the category, describe extensions. There is an easy way to do it by the axiomatics. For any such guy, we can consider first of all, i' is degree strictly less, so A is defined on i', then you let $L_i(A)$ be the colimit over $L(i)$ of $A(i')$ and $M_i(A)$ is the limit over the matching category $M(i)$ of $A(i')$. An observation is that we have a natural map from $L_i(A) \to M_i(A)$. Why is this? Say we have an object in the latching category and an object in the matching category. You have $i'_0 \to i \to i'_1$. The composition has a factorization that lowers and raises. This one raises and then lowers, so you have another factorization that stays in $I_{<n}$. You glue these together and they give a map from the colimit over one to the limit over the other, giving the canonical map t. So extending A to $I_{<n} \sqcup i$ is equivalent to giving $A(i)$ in $\mathcal{C}$ and maps $L_i(A) \to A(i) \to M_i(A)$ whose composition is t. Now the latching gives exactly what you need.

Now you get an inductive definition. I defined a functor by induction, and now for any object i in I and $f : A \to B$ you have a diagram

$$\begin{tikzcd} L_i(A) \arrow[r] \arrow[d] & A(i) \arrow[rr] \arrow[dr] & & M_i(A) \arrow[d] \\ L_i(B) \arrow[rr] & & B_i \arrow[r] & M_i(B) \end{tikzcd}$$

and then I can factor through a pushout on the left and a pullback on the right,

$$\begin{tikzcd} L_i(A) \arrow[r, "f^R"] \arrow[d] & A(i) \arrow[r] \arrow[d] \arrow[dr] & B(i)_f \arrow[r] \arrow[d] & M_i(A) \arrow[d] \\ L_i(B) \arrow[r] & A(i)_f \arrow[r, "f^L"'] & B_i \arrow[r] & M_i(B) \end{tikzcd}$$

Then we say f is a Reedy fibration if f^L is a cofibration for all i and a fibration if f^R is a fibration for any i. You need lifting properties and factorization. We prove lifting by induction. You add the objects step by step, and easily show that extension requires the condition at level i. Factorization is similar.

You can say what is a Reedy fibrant object. Each term has to be fibrant, and each i, you have a map from $A(i)$ to the matching object is a fibration. When you study Hodge theory by Deligne, there's the notion of a hypercovering, and you have a simplicial scheme with a properness condition at every point, and this is precisely this map. These are coskeletons if your I is Δ^{op}. This is the generality in which things work.

Okay, my time is up, this is where I stop today. In the last lecture I explain how this special theorem gives us homotopy limits and colimits in full generality, and then I give the Grothendieck construction. Applications if I have time.

3. April 15

I discussed some generalities about model structures last time and talked about Reedy model structures which I claimed to be useful. So today I want to present applications and generalizations of those. I won't remind you of the general definition but in particular situations things will come up and I'll remind you.

So I'm interested in homotopy limits. So $\mathcal{C}$ is a model category, it has cofibrations, fibrations, and weak equivalences (C, F, W) and let's say it has all limits and colimits. I is a small category, and in this generality, the theorem, in this generality due to Bousfield, is, well, as a reminder,

Theorem 3.1. *(Bousfield) The pullback functor $\tau : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C}^I)$ has left and right adjoints* hocolim *and* holim.

The difficulty is that this diagram category is not a model category so you can't use Quillen. The idea of the proof is to consider the nerve ΔI, this is a category whose objects are sequences of composable maps in I and whose morphisms are maps from $[m]$ to $[n]$ and then an isomorphism $i_m \to j_{f(m)}$ so that everything commutes. This thing projects to Δ by forgetting the diagram and the fibers are just sets.

We also have the projection $t : \Delta I \to I$ which projects to the last element. Then there's a natural map that might involve the maps in the objects.

Definition 3.1. A map in ΔI is *special* if f sends the last element to the last element.

Then the projection t sends a special map to the identity. What matters for us is that it's invertible.

So this I was an arbitrary category. Moreover, the nerve ΔI is a Reedy category. Objects should have a degree, and there should be some sort of lowering maps and some sort of raising maps and every map should decompose uniquely into a lowering map followed by a raising map. Here this just comes from Δ, the degree is just the length of a sequence, and then you can split into a lowering followed by a raising. Then $\mathcal{C}^{\Delta I}$ has a model structure.

Proposition 3.1.
$$\mathrm{Ho}\,\mathcal{C}^I \to \mathrm{Ho}\,\mathcal{C}^{\Delta I}$$
is fully faithful, and in fact you can characterize the image, $A : \Delta I \to \mathcal{C}$ is in the image if and only if it sends all special maps to weak equivalences.

If A sends every special map to an isomorphism, then you can see that it factors through t and at the level of homotopy categories this is true except with iso replaced with weak equivalence.

Let me look at what we can do with it, let's first do homotopy colimits. I have $\tau : \mathcal{C} \to \mathcal{C}^{\Delta I}$ by pullback, and then I have the colimit in the other direction. Then the claim is that τ is right Quillen, that is, that it has an adjoint, sends fibrations to fibrations and trivial fibrations to trivial fibrations, F to F and $F \cap W$ to $F \cap W$. Then the general machinery of Quillen tells you that these induce an adjunction at the level of homotopy categories. This claim is nontrivial. On $\mathcal{C}^{\Delta I}$ you have a nontrivial description of fibrations, let me remind you.

So let us recall the fibrations. We have some map $f : A \to B$ in $\mathcal{C}$ which is a fibration, and then we consider its pullback, the condition for Reedy fibration is pointwise. We have some object $[i]$ in ΔI, and the definition is that

$$\begin{tikzcd}
\tau^* A_{[i]} \arrow[r, dotted] \arrow[dr, "f"'] & A'_{[i]} \arrow[r] \arrow[d] & M_{[i]}\tau^* A \arrow[d] \\
& B_{[i]} \arrow[r] & M_{[i]}\tau^* B
\end{tikzcd}$$

where M denotes the matching object and $A'_{[i]}$ is the pullback in its square. Then the condition is that the dotted arrow is a fibration.

Okay, let's compute. On the left side we get just A and B. But what about for the matching objects? If I had just Δ, then M would be formed by surjective maps out of this, all degeneracies. If I have i, it's similar, I have the diagram, and I have to contract terms, and in my definition, I need this to be only identities to contract. If there are no objects, the matching object is empty. If I have an identity, I can contract such terms. So this category is easily described, and it has a terminal object. If I contract all maps, I get a terminal object. This then is either empty (if there are no identity maps) or has a terminal object. Then the matching object is the limit over this category of the constant functor. So then the matching objects are either both 0 or A and B respectively. In both cases the statement is trivially true to make this preserve fibrations. This basically proves the claim that τ is right Quillen and gives the construction of colimits.

I don't want to explain the proposition but it uses a similar trick, you take a derived fuctor of the projection and then pull it back. $\mathcal{C}^I$ is not a model category but you can still do the procedure, that turns out not to matter.

Remark 3.1. I used the nerve ΔI instead of I, but instead of this, you can also use $(\Delta I)^{\mathrm{op}}$. Now this projects to I, sending a chain of maps to its first objects. Now we say a map is special if it takes the first element to the first element. Then the proposition is still true. You can use either of these categories and still get this fully faithful functor. So the proposition still holds but now $\tau : \mathcal{C} \to \mathcal{C}^{(\Delta I)^{\mathrm{op}}}$ is no longer right Quillen. Now the latching and matching category switch places. Now instead of surjections I get injections, and then I have two injections from i_0 and i_1 to $i_0 \to i_1$. This is two objects, not one. I get $B \times B$ and $A \times A$ instead of B and A, and so it all breaks down.

However, it is now left Quillen, so it doesn't send fibrations to fibrations, but it sends cofibrations to cofibrations, because it's completely dual. So if I want to construct limits, I should use this model.

Now I finally come to the new stuff. I want to show how to generalize this and combine it with what we had in the first lecture. What happens to be true is a certain generalization of the original theory of Reedy.

Definition 3.2. (Bolzin) A Grothendieck cofibration $\mathcal{E} \to I$ is a *model cofibration* if

(1) the fibers are model categories,
(2) $f_! \mathcal{E}_{i'} \to \mathcal{E}_i$ for $f : i' \to i$ sends F to F and $F \cap W$ to $F \cap W$.

This is like right Quillen but without the condition of having an adjoint, preserving limits.

For example, consider $\otimes : \mathrm{Vect}^n \to \mathrm{Vect}$. This certainly does not preserve sums or products. So $(V_1 \oplus W_1) \otimes (V_2 \oplus W_2)$ certainly is not $(V_1 \otimes V_2) \oplus (W_1 \otimes W_2)$. But (if I take complexes) this satisfies the conditions of the definition. But fibrations are surjections, you can see that the canonical map in this case is still going to be a surjection, and this is true generally.

Okay but there's no chance that sections will have a model structure so we need to use nerves now.

Now assume we are given a model cofibration $\mathcal{E} \to I$, consider the nerve ΔI, with the projection t to I, and we have, we can pull back this to $t^*\mathcal{E} \to \Delta I$, and now one needs to do a categorical procedure to get the arrows in the right directions. We can take a covariant functor (cofibration) and make it a contravariant functor from the opposite category (fibration). This is totally tautological. This is the transpose. So let $\mathcal{E}^T \to (\Delta I)^{\mathrm{op}}$ be the transpose, with the same fibers and the same transition functors, just now interpreted differently.

This is not merely the opposite because that would also give me the opposite on the fibers, which I don't want. I want the base to be opposite but the fibers to be the same.

Then the main definition

Definition 3.3. (Balzin) A *derived presection* of $\mathcal{E}$ is a section of $\mathcal{E}^T \to (\Delta I)^{\mathrm{op}}$ (that is, a functor in the opposite direction so that the composition is the identity of $(\Delta I)^{\mathrm{op}}$). The category of presections is $\mathrm{PSect}(I, \mathcal{E})$.

A derived presection is a *derived section* if the following holds. I wanted, remember to distinguish functors from the nerve that came from I. I want to say that for every special map $f : [i'] \to [i]$ (in the opposite nerve), I have a natural map $A([i']) \to f^*A([i])$, and I want this map to be a weak equivalence.

This is kind of an abstract definition but let me give you some idea of what this is, in practice you look just at two objects. A presection corresponds to a trivial diagram. For any i we have $A(i)$ in $\mathcal{E}_i$. What do we have for maps? Assume given a map $i_0 \to i_1$.

This also gives an object in the nerve of course and I get maps

$$\begin{tikzcd} & {[i_0 \to i_1]} \arrow[dl] \arrow[dr] & \\ {[i_0]} & & {[i_1]} \end{tikzcd}$$

$$\begin{tikzcd} & A_{[i_0 \to i_1]} \in \mathcal{E}_{i_0} \arrow[dl] \arrow[dr] & \\ f^*A([i_0]) & & A([i_1]) \in \mathcal{E}_{i_1} \end{tikzcd}$$

then the section condition is that this map $A([i_0 \to i_1]) \to f^*(A_{[i_0]})$ is a weak equivalence. This is exactly the kind of thing I wanted. Say these are the one element and two element set. Then what do I have? I have $A^{\otimes 2}$ and A and I have an $\tilde{A}$ about which I know nothing, but it has maps to both and the one to $A^{\otimes 2}$ is a quasiisomorphism.

There's higher things for composition (you can't compose directly) but they are nice.

Theorem 3.2. *(Balzin)* $\mathrm{PSect}(I, \mathcal{E})$ *is a (Reedy type) model category.*

This is surprising, you don't even know if this has limits and colimits. You have to prove this by a different procedure, but it's still true. Weak equivalences are pointwise, both fibrations and cofibrations are nontrivial, they're something like the Reedy cofibrations and fibrations but you need transition functors.

There is also one theorem which I maybe will give you in a while but let me know try to explain how to apply this in our real lives.

In fact there are many applications, like in a linear situation. You can construct homotopy limits and colimits in many different ways in that kind of situation, but I expect this might sometimes be useful. But our applications that we started with was to nonlinear structures, starting with E_∞ algebras. This gives a choice-free definition of E_∞ algebras similar to Segal's.

We start with Γ_+, consider a monoidal category which we turn into a cofibration over Γ_+, you give this a kind of model structure, check that the transition functions satisfy the right condition, which they do, and then you consider $\mathrm{DSect}(\Gamma_+, M^0)$. This gives an answer. But who cares about E_∞ algebras other than topologists? But this isn't all you can do. You can replace Γ_+ with other categories. One way to do this,

Definition 3.4. (Barwick) An *operator category* is a small category I with the following conditions. It has a terminal element 1 and some pullbacks, pull backs along morphisms out of the terminal object. A map is an *admissible monomorphism* if it is a composition of such pullback maps. Then i can define I_+ to have the same objects but morphisms are diagrams $i_0 \leftarrow i \to i_1$ where the maps $i \to i_0$ are admissible monomorphisms.

This is like what we did before with partially defined maps. Then surprisingly this simple situation helps many situations. Then you can define special objects and play the Segal game.So examples.

Example 3.1.
(1) Γ, then you get infinite loop spaces.
(2) Δ, now you don't have pullbacks. You do, though, you can check, have pullbacks like this special kind. This gives loop spaces.
(3) There is a category of n-ordinals due to Batanin that gives n-fold loop spaces. It's not even Batanin, it's a short paper, again, by someone who never did anything else but very useful.
(4) There's a version B which gives just 2-fold loop spaces. This B is a fashionable category, for instance there was a talk about factorization algebras that used this on Monday.

So the objects of B, you fix a disk D and an object of B is a finite set S and an injective map $S \to D$, basically a point in the configuration space. The morphisms are, $\langle S, i\rangle$ and $\langle S', i'\rangle$, and these are first of all a map $f : S \to S'$ and a homotopy between these two $S \times I \sqcup_f S' \to D$ but I need an injectivity condition I don't want to state, if the map is injective, there's no condition. But if it's surjective. This is n distinct points and some come together to the diagonal. This should be stratification-compatible in the sense that once you hit a stratum you never come out of it. This is the stratified fundamental groupoid and it's well-defined.

For instance, the automorphisms of a guy is just the braid group. I'm not allowed to hit the diagonal, any other stratum.

The observation is that this is an operator category and you can play the Segal game.

You can do things other than topological spaces, you can do categories. So the first example gives symmetric monoidal categories, the second gives monoidal categories, and B gives braided monoidal categories. In fact B_+ gives a model for something like the Ran space. The n-ordinals give something like n-fold monoidal categories. You can see this in Deligne's [unintelligible]conjecture, you have n different maps. This is more technical. The B case gives you something that looks more interesting.

If one writes down the definition of derived sections of $(B_+, C^*(\mathbf{k}))$, one sees that these are more or less the same as the factorization algebras that people were talking about. More or less means that there are the higher coherences that no one gives you and those might be different, I haven't checked. What's a factorization algebra? You take powers of the disk, and complexes of constructible sheaves locally constant with respect to the stratifications. The strata are $K(P,1)$ so you get representations of the fundamental group, and the whole thing is representations of the stratified fundamental groupoid. Then there are the factorization isomorphisms which tell you that if you separate your configuration and project into different pieces, these are the maps you have in the definition of derived sections.

This is then basically the same thing, up to higher homotopies which nobody writes down. The question is then, how to construct such a thing? More specifically, there is this Deligne Hochschild homology conjecture, the testing stone for machinery in this field, which claims that for an associative algebra A, the Hochschild cohomology complex should have this structure. Originally this meant precisely the algebra over the chain complex of E_2. Morally this was the same, but these days they like this more because it involves less choices. How to construct this is nontrivial still.

There is a theorem of [unintelligible]that gives you a procedure for doing this, maybe I'll explain this and that will be the end.

This uses the main theorem fairly heavily. You can take different I and $\mathcal{E}$ and do things with them. So what you observe looking at proofs of Deligne's conjecture, it's very easy to construct a section of a different category.

For any finite set S, the S power of the disk D^S has a cellular decomposition, elements are numbered by marked stable planar trees, this occurs in the original paper of McClure and Smith (well trees appear) and this decomposition is in Kontsevich–Soibelman. There are marked and unmarked vertices, one root, it's in the plane. The vertices are marked by elements of S, and the stability condition is that every unmarked vertex has valence at least three. You can get multiple markings on one vertex. This is a stratification of the configurations plus the diagonal. This is a regular cell decomposition so the strata are just cells. Morphisms of these (they form a category, an operator category T) are subtractions and contractions of edges.

We have a functor $T \to B$. This is something like a resolution. You have a nice topological space, finite dimensional, maybe $K(P,1)$. Then a local system on this guy is just a representation of the fundamental group. These are really hard to construct. But say you have a cellular decomposition, cells give you a partially ordered set, representations of this give you sheaves which are constructible with respect to this stratification. If you assume that the maps are quasi-isomorphisms, this gives a description of the category of local systems in terms of less data. Here we have all configuration spaces together.

Definition 3.5. (Balzin) $F : I' \to I$ is a *resolution* if for any object $[i]$ in the nerve of I, you consider the diagrams in I' mapped to $[i]$, you consider the category $F_{[i]}$ of all diagrams $i'_0 \to \cdots \to i'_n$ such that F takes them to $[i]$, and the condition is that this should have contractible nerve.

For example, an object should have contractible nerve as its preimage. This is what you get when you try to compare [unintelligible]. The example which is not a difficult exercise is that $T \to B$ is a resolution. The theorem

Theorem 3.3. *For any resolution $I' \to I$ and any model cofibration $\mathcal{E}/I$, the pullback functor is fully faithful at the homotopy level and has a left adjoint.*

You can characterize the image as well.

This is completely general, nothing about operator categories. So if you have a T-algebra, you can make a B-algebra. This is the most canonical approach to the Deligne conjecture. The idea that this should be the most canonical way to do the Deligne conjecture is in my lectures in Seoul from some years ago but now this is fully realized.

If you look at people who construct quantum field theories mathematically these days, you have some moduli spaces or something and then some sheaves and then some factorization conditions, how does Kevin Costello do this directly? You can't just construct a representation of a braid group. You need a bunch of spaces and maps between them, he constructs something like this functor by hand, constructs adjunctions and stuff by brute force, and everything works. I think that this theory gives a conceptual framework, explains in what kind of situation you should expect this behavior. It also gives the same bar complex, but without thinking about it.

The big breakthrough is the theorem of Balzin about the model structure on $\mathrm{PSect}(I, \mathcal{E})$, and this is just an application of it.