INTERNATIONAL CONFERENCE ON OPERAD THEORY AND RELATED TOPICS

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[N.B.: I do not take notes at slide talks]

1. Nov 6: Sasha Voronov: Algebras over operads and BV-algebras

It's an honor to come to this conference in China, Anhui, and Hefei, and also an honor to be a first speaker. I've never been the first speaker so please bear with me.

I'm reporting on joint work with Lucy Yang, who was an undergraduate student at the university of Minnesota and did an undergraduate thesis with me and now went off to Harvard.

So BV algebras arose as one component of the BV-formalism, created mostly by Batalin and Vilkovisky to compute path integrals, at least in principle.

Let me start by telling you the definition of what a BV-algebra is.

Definition 1.1. A *BV-algebra* is a graded commutative associative algebra V (it has a product $\cdot$ from $V \otimes V$ to V) and it has a second order differential $\Delta : V \to V$, called the *BV operator*. What it means is this, I'll use the convention that $|\Delta| = -1$, that $\Delta^2 = 0$, that $\Delta(1) = 0$, and the second order condition, that $[[[\Delta, L_a], L_b], L_c]$ are zero for all a, b, and c in V (these are graded commutators, everything in my talk is graded) where L_a takes x to ax.

This is the condition of being a second order differential operator, first introduced by Grothendieck. With just two commutators, this would mean that the operator was a derivation, taking into account that $\Delta(1) = 0$. With one commutator it would mean being linear with respect to the product. You can write down seven terms, but an equivalent way of wording this, of writing down this equation, turns out to be pretty useful, with the BV bracket.

This identity which I promised to give an equivalent version to, is maybe written down as follows. You define a bracket (sometimes called an anti-bracket), you define $\{a, b\}$ to be

$$(-1)^{|a|}\left(\Delta(ab) - (\Delta a)b - (-1)^{|a|} a\Delta(b)\right)$$

and the conditions are equivalent to saying that $\{,\}$ is a Lie bracket of degree -1 and Δ being a derivation with respect to the bracket, this is the same as saying that Δ is a second order operator.

Remark 1.1. There are generalizations,

(1) a *dg BV-algebra* is a BV-algebra equipped with another differential d which is a derivation of first order of $\cdot$ and which commutes with Δ.
(2) a *commutative BV_∞-algebra* where you have a formal power series of differential operators $\hat{d} = \Delta_1 + \hbar\Delta_2 + \hbar^2\Delta_3$, where Δ_n is a differential of order

n, the whole operator is degree +1, (where $|\hbar| = 2$) and the whole $\hat{d}$ squares to zero.

A dg BV algebra is one of these where $\Delta_1 = d$ and $\Delta_2 = \Delta$.

For simplicity I'll only talk about BV-algebras but most of the things that I'll say have analogs with commutative BV_∞-algebras.

Theorem 1.1 (Folklore). *Suppose that $\mathfrak{g}$ is a graded Lie algebra. Then the symmetric algebra on the suspension of $\mathfrak{g}$, that is, $S(\mathfrak{g}[-1])$, carries the canonical structure of a BV-algebra.*

This is folklore because you all know this structure even if you don't know BV-algebras. So $\mathfrak{g}[-1]^n = g^{n-1}$. The dot product is the standard product on the symmetric algebra. If you started with a Lie algebra, you get the symmetric algebra on the shift, and this is the Chevalley–Eilenberg complex with coefficients in the trivial module, and the BV operator is the Chevalley–Eilenberg differential:

$$\Delta(x_1 \cdots x_n) = \sum_{i<j} \pm x_1 \cdots [x_i, x_j] \cdots \hat{x}_j \cdots x_n$$

and this is the Chevalley–Eilenberg differential.

I should have said earlier, I'm working over a ground field of characteristic zero.

The second example, another source of BV-algebras, is the following somewhat surprising theorem:

Theorem 1.2 (Terilla–Tradler–Wilson). *If A is graded associative, then $T(A[-1])$ carries a canonical BV-algebra structure. The product, we want a commutative associative product, it's not the standard product on the tensor algebra but the shuffle product, and*

$$\Delta(x_1 \otimes \cdots \otimes x_n) = \sum_i x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n$$

These are both known as bar complexes, this is more specifically known as the bar complex of the associative algebra A, and this is another sort of amazing theorem, that things that are familiar to us become, from a different perspective, different objects, with different algebraic structure.

This is a conference not so much on mathematical physics or algebras, but on operads, so let me bring this to the operad perspective, and recall the operad perspective on this. Let me recall, as Martin Markl calls them [Markl: this is from Loday, I learned it from Loday] — oh I thought he learned it from you. So the three graces are the Lie operad, Koszul dual to the commutative operad, and the associative operad which is self-dual. So we expect a natural generalization to algebras over (at least Koszul quadratic) operads.

Theorem 1.3 (main theorem of the talk). *The statement of the previous two theorems can be generalized to other operads under mild assumptions. Let $\mathcal{O}$ be an operad of graded vector spaces which is*

(1) *connected (the zeroth component is the ground field) and*
(2) *of finite type (that is, the dimension of each graded component in each arity $\mathcal{O}(n)^p$ is finite), and*
(3) *the operad is quadratic or what is sometimes called* binary *quadratic, i.e., $\mathcal{O}(\geq 1)$ is generated by binary generators and quadratic relations, via a presentation $\mathcal{F}(E)/(R)$, that is, $E(n) = 0$ for $n \neq 2$ and R is an ideal generated by relations sitting in $F(E)(3)$, and*

(4) *$\mathcal{O}$ is cocommutative Hopf—this means that it's an operad in the category of cocommutative coassociative coalgebras—in other words, it has certain coproducts in each component.*

Under those assumptions on $\mathcal{O}$, let's take its Koszul dual (I'm not assuming Koszul duality), and then suppose I have V a (graded) $\mathcal{O}^!$-algebra. Then the cofree conilpotent $\mathcal{O}$-coalgebra $\mathcal{F}^c_{\mathcal{O}}(V[-1])$ has a canonical BV-algebra structure.

Moreover, the structure is compatible with the $\mathcal{O}$-coalgebra structure.

Here $\mathcal{F}^c_{\mathcal{O}}(V[-1])$ is
$$\bigoplus_n \mathcal{O}(n)^* \otimes_{S_n} V[-1]^{\otimes n}.$$

Examples of this include the associative operad, the commutative operad (covered by the two theorems that I gave)—here the Hopf structure on $\mathcal{O}$ is given by $\mathcal{O}(n) \to \mathcal{O}(n) \otimes \mathcal{O}(n)$, so in the associative case $\delta(\sigma) = \sigma \otimes \sigma$ where $\sigma \in S(n)$, writing $\mathrm{As}(n)$ as $\mathbf{k}[S_n]$, and in the commutative case, $\delta(e_n) = e_n \otimes e_n$. Then the Poisson operad is an example, is also Hopf, you can say it's generated by μ and $[\,,\,]$, and $\delta(\mu) = \mu \otimes \mu$ whereas $\delta([\,,\,])$ is $\mu \otimes [\,,\,] + [\,,\,] \otimes \mu$. Then graded versions of Poisson, like the Gertsenhaber operad, are Hopf.

Part of this theorem is actually a result of Livernet–Patras of 2006, but dualized. This is related to the fact that the commutative algebra and the $\mathcal{O}$-coalgebra structures on $\mathcal{F}^c_{\mathcal{O}}(V[-1])$ are compatible.

Let me give you a glimpse of the construction in the remaining couple of minutes, the BV structure on this cofree coalgebra. We need two things, the BV operator and the graded commutative multiplication.

So the BV operator Δ is something we are all familiar with, which is the codifferential on the conilpotent cofree $\mathcal{O}$-algebra structure coming from the $\mathcal{O}^!$-algebra structure on V, The multiplication is obtained by playing around with the Hopf structure on the operad $\mathcal{O}$. Some shuffles are involved even for a general operad.

I don't want to say more because I'm already a little over time, but one thing about this is that I actually believe that this way, we get a huge supply of BV algebras. If you start with the dg category, you get dg BV algebras, and I think that dg BV algebras are related to a quantized kind of deformation theory, and I suspect that these constructions for more general $\mathcal{O}$ will provide some kind of quantum deformation theory over more general rings that commutative rings, with bases not being commutative rings but associative rings or $\mathcal{O}$-algebras. My hope is that these structures will play off in deformation theory basically.

2. David Gepner: Analytic monads and ∞-operads

It's a pleasure to be here. This is a paper joint with Rune Haugseng (who is here in the audience) and Joachim Kock. I thought I'd structure my talk around describing with these words mean, what an analytic functor, a monad, and an ∞-operad are.

What's an analytic functor? First we should think about what an analytic function is. An analytic function is one that locally has a power series expansion, $f(x) = \sum_{n=0}^{\infty} f_n x^n/n!$. This is an analytic function, something like the real numbers or the complex numbers to itself. If instead of a number we have a category, if $\mathcal{C}$ is a category and for now I'll let myself also consider an ∞-category, then we can consider what is an analytic functor, which in this case, our analytic functions went from, say, the real numbers to the real numbers or the complex to complex numbers.

Monads have underlying endofunctors so I'll pick endofunctors but I could do this more generally changing categories. So in this case, an analytic functor is one of the form

$$F(X) = \coprod_{n\in\mathbb{N}} F_n \times X^n/\Sigma_n.$$

Here, the coefficients are objects of $\mathcal{C}$. I'll need my category to have products or some kind of tensor. I'll ignore that distinction for now. I'll need Cartesian powers and quotients of group actions. I won't go into detail on this for now. There are other notions of analytic functor in the literature, there is one due to MacLane and one due to Goodwillie. In Goodwillie it will be more like a convergent limit. Note that not all the use of the term analytic functor is consistent. This is what I mean.

What is the main example of a category where we might consider these. The main one is $\mathcal{C} = \mathcal{S}$, this is again an infinity-category of spaces, topological spaces but up to weak homotopy equivalence. This is the main example because this is where operads were originally born. These are things like the little n-cubes operad, which exists on spaces.

We'll also need other important examples, like where $\mathcal{C}$ is $\mathcal{S}^T$ where T is a space. This is, $\mathcal{S}^T$ is the category of functors from T to $\mathcal{S}$, and I'll say what it means to think of a space as a category. If I have a space T, then it's an infinity category via its singular complex, what is the same, its fundamental ∞-groupoid, where morphisms are paths and objects are points. This is the same as the slice of spaces over T. This is either a T-indexed family of spaces, either living over T or as a functor from T to spaces, given a point it returns the fiber over that point.

Feel free to interrupt if you have any questions. The first thing to notice about an analytic functor is that it is completely determined by its coefficients.

First let's give an example of an analytic functor. The primary example of an analytic function is the exponential where the coefficients are all 1. So $\mathrm{Sym}(X)$ is the analog, where this is $\coprod X^n/\Sigma_n$, the free symmetric algebra on X.

This is in general the quotient in the ∞-categorical sense, aka, the homotopy quotient. So I could change my coefficients to be $E(\Sigma_n)$, free contractible Σ_n-spaces.

If I have an arbitrary one $F(X) = \coprod (F_n \times X^n)/\Sigma_n$ (and the bracketing is important) then you can recover $F(X)$ from the value of F on a point from Sym because

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & \mathrm{Sym}(X) \arrow[d] \\ F(\mathrm{pt}) \arrow[r] & \mathrm{Sym}(\mathrm{pt}) \end{tikzcd}$$

is always a pullback (in the ∞-categorical setting I'm doing the homotopy pullback).

If I do this on a point I get the disjoint union of pt/Σ_n, which is $B\Sigma_n$, and in the upper right I get $\coprod X^n/\Sigma_n$, and the bottom left is $\coprod F_n/\Sigma_n$. To see that it's the pullback you want to see that the fibers are the same, and the fibers, you can see, in both cases, is the nth value I wanted. So the key piece of data is $F(\mathrm{pt}) \to \mathrm{Sym}(\mathrm{pt})$, so what is a space with a map to $\coprod B\Sigma_n$, so to give something over the disjoint union is to give, for all n, a space with an action of Σ_n. So this data is classifying the associated symmetric sequence, and therefore remembering all the data of the analytic functor. So this thing is called a symmetric sequence, which is the first indication that this is related to operads. Traditionally operads were defined as something like monoid objects in symmetric sequences under something like the composition product.

In continuing with the structure of my talk, what's a monad? I'll tell you historically they arose from Godemont's "standard resolutions" which is something like an injective resolution to compute the cohomology of a sheaf. But they're implicit earlier in Kan's notion of an adjoint functor. If I have a free-forgetful adjunction, $\mathcal{C}$ might be some sort of category which has a forgetful functor to $\mathcal{D}$, so $\mathcal{C} \xrightarrow{U} \mathcal{D}$ might have a left adjoint F, and if you have a left adjoint to a functor you always have a monad lying around. A good example of a monad is the free Abelian group monad on sets, so I take an Abelian group and freely determine an Abelian group on the underlying set. So the monad is UF, which goes from $\mathcal{D}$ to itself. Then by how adjoints work, I have a unit from id to UF which I call M, and also something μ from $M^2 = UFUF$ to UF using the counit of the adjunction. This is subject to some relations.

So a monad is an endofunctor M from some category $\mathcal{D}$ to $\mathcal{D}$ equipped with stuff like this unit map η, the multiplication $\mu : M^2 \to M$, and I'll say et cetera because in the infinity context, I need to specify a lot more information and conditions.

There is a handy way of summing up in any context what a monad is, all it is is a monoid in the category of endofunctors, this is a definition that works in complete generality, as long as you have the theory to make sense of what these things are.

Definition 2.1. A *monad* M in an $(\infty-)$category $\mathcal{D}$ is a monoid in the monoidal $(\infty-)$category $\mathrm{End}(\mathcal{D}) = \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ under composition.

We'll write as notation, $\mathrm{Mnd}(\mathcal{D}) = \mathrm{Mon}(\mathrm{End}(\mathcal{D}))$. You can think of lots of examples, an adjunction gives you an example of a monad. Now what is an ∞-category? I won't say what a category is, but I'll give an analogy to say what an ∞-category is relative to an ordinary category.

An ∞-category $\mathcal{C}$ is (roughly—I don't want to be precise and my model won't be relevant) a category where the hom sets (usually $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set) are instead spaces $\mathrm{Maps}_{\mathcal{C}}(A, B)$. We should have a composition, if I have a map from A to B and a map from B to C I should be able to compose them to get a map from A to C, and this should be a continuous map of spaces, but it really only has to be continuous up to homotopy. We don't care about the spaces themselves, only their homotopy type. We need some rules, like associativity. We don't need it to compose strictly, because we're always concerned with homotopy types, and so you say "up to coherent homotopy" and that's where you need a machine or a model.

Let me give you some examples that will be relevant for this talk.

First, if I have an ordinary category, I can take its nerve N to get an ∞-category. This is via taking a set to a space via the discrete topology.

There's also a way to go back, the homotopy category, I can go from a space to a set by taking π_0.

What ∞ means here, let me explain by what I wanted to give as my next example. If I have T a topological space, I get an ∞-category T where the objects are the points of T and the morphisms are the paths and what's ∞ about this is that I need 2-morphisms, and I'll get paths between paths etc.

There are a lot of models for the theory of ∞-categories, but some key things where the model of Boardman–Vogt and developed by Joyal and then Lurie, it's easy to construct $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. If I took topologically enriched categories. I would not create the correct ∞-category of functors, which would only be the correct thing if these were defined on the nose. The whole theory needs to account for laxness. In many models it's difficult to construct this, I'd have to resolve $\mathcal{C}$ in some way. But in quasi-categories this is the exponential simplicial set.

That was a note for the experts, if you don't know what those words mean, don't worry about it.

So what about operads now? Or ∞-operads? I would be so bold as to say that there, the notion of a 1-operad is not well-behaved, because even little n-cubes are all being considered up to homotopy. So there is not much difference between operads and ∞-operads, you want to consider this up to homotopy anyway. The way this is developed in Lurie's "Higher Algebra" — and I mean coloured operads, a.k.a. operads with many objects — which gives some technicality. Let me skip the example of little n-cubes. The motivating example for many-object operads corresponds in a natural way to ∞-(symmetric) multicategories.

What's a multicategory? This is a category where you have a list of inputs into your hom. You instead have $\mathrm{Hom}(A_1, \ldots, A_n, B)$. If I wanted to compose in a multicategory, this is asymmetric, I have a whole bunch of inputs and only one output. If you think about what you're forced to do you're naturally led to think about trees. I might have a whole list of things that I want to compose. Oh, let me say, in an ∞-multicategory I'll have a whole space of maps and the composition only has to be homotopy-coherent. If I think of having B_1 through B_n mapping to C and if I had a thing from say $A_{11} \cdots A_{1m_1}$ to B_1, a list of things that maps to B_1, and then I can compose to get to B_n, say $A_{n1} \cdots A_{nm_n}$ to B_n, and the way you compose in a multicategory is with this sort of data, basically if I have a map from each A-list to B_j and then a map from the list of Bs to C, then I get a map from the entire list of As to C. If you remember that these are unordered, then you can write down the list of axioms of a (coloured or many-object) operads. If you remember that they are spaces of maps, then that's what an ∞-operad is.

I told you what an analytic endofunctor was but not an analytic monad.

Definition 2.2. An *analytic monad* on an ∞-category $\mathcal{C}$ (with some properties I'll ignore) is a monad on $\mathcal{C}$ whose underlying endofunctor $M : \mathcal{C} \to \mathcal{C}$ is analytic.

I haven't, the beauty of this definition as opposed to definitions of ∞-operads in the literature is that I haven't specified any models here. I can say what it means for an endofunctor to be analytic. The theorem of the talk is that ∞-operads are precisely analytic monads. It doesn't mention trees or finite sets explicitly, but that's hidden under the hood in what it means to be analytic and how you compose maps of multicategories.

Let me give a precise statement of a theorem and then I'll finish my talk.

You have an ∞-category Op_∞ of ∞-operads (à la Lurie, in Higher Algebra [James: what page?] —around page 780, I don't know [laughter]) and you also have dendroidal sets, or dendroidal complete Segal spaces, and what is that, you have Ω the category of trees (roughly) and this is some, let me say the theorem.

Theorem 2.1 (Heuts–Hinich–Moerdijk)**.** *∞-operads embed fully faithfully into functors from Ω^{op} to spaces, this is dendroidal spaces.*

The complete Segal condition is describing the image of this embedding. The two definitions respectively take the ∞-categorical perspective and work directly with trees, and they characterize the image as the complete Segal ones.

The theorem we prove, let me do it in two steps. The annoying thing is that often you have to work one space of colors at a time. So I have all ∞-operads, and as just noted that includes into dendroidal spaces, and I can evaluate this at the trivial tree, and that gives me a space, and that space tells you what the underlying space of colors of your ∞ operad. I can take a particular space I like T and can take $\mathrm{Fun}^T(\Omega\mathrm{op}, \mathcal{S})$ as the pullback, the dendroidal spaces which have color space T. That's ∞-operads with objects/colors T, in ad hoc notation Op_∞^T. This is just notation, there is no theorem yet here. I chose this notation because $\mathcal{S}^T$ is $\mathrm{Fun}(T, \mathcal{S})$ but also $\mathcal{S}_{/T}$. The theorem I'll list in two parts. The first part characterizes the ∞-operads with objects T and morphisms the identity on T.

Theorem 2.2 (G.–Haugseng–Kock).

- *The ∞-operads with object space T are equivalent to analytic monads on $\mathcal{S}^T$.*

  Then I can string this all together, it's not quite as easy as I'm making it sound, string these together over all spaces and get the real theorem, that
- *analytic monads on $\mathcal{S}_{/*}$ (spaces with a varying base) is equivalent to ∞-operads*

I'll stop there.

3. Nov 8.: Chris Rogers: homotopical applications of convolution

Thanks to the organizers, it's been an enjoyable and productive conference in my opinion. This is a vague title, on purpose. As you move toward the end of the conference you might want to talk about something different from what you wanted to talk about when you first arrived. I want to talk about the homotopy theory of filtered L_∞ algebras and some interesting examples of these. The examples that show up will come from operadic convolution. The things I'll talk about will be well-known to the experts, and it's sort of designed to be that way because there are supposed to be some students in the audience. I think this is important, because these filtered L_∞ algebras are what govern deformation theory in characteristic zero. Benoit, this morning, was talking about various things and this is touching on some of the machinery that arose in that context.

I'm going to talk about this homotopy theory and our goal at the end of this talk is to solve the following problem, to solve the homotopy transfer problem. Let's consider the following setup to get us started. Let's take a cochain complex (A, d) and a cochain map ϕ between it and an A_∞-algebra (B, d, m_k^B), this is, let's say, in cochain complexes, let's say everything in characteristic zero. Let's say that a solution to the homotopy transfer problem is a pair that consists of two things, (m_k^A, Φ), an A_∞-structure on A and an $A\infty$-morphism that is a lift of ϕ, so that the restriction to the linear piece is just ϕ.

The goal here is to produce a space at the end of the talk, we've constructed a space S, actually an ∞-groupoid (but you can think of it as a topological space), or in reality a Kan complex. The objects, or the vertices, or the points, will be solutions to the homotopy transfer problem.

We'll prove two things. The first thing that we'll show is that S is non-empty. The second thing we'll show is that it's a contractible space. The first thing is existence, it says that there's a structure. The second statement is less commonly talked about, that's a uniqueness statement, saying that the structure in the transfer is unique up to an isomorphism which is unique up to a 2-morphism, and everything is unique up to one morphism higher.

If this were somehow a 1-groupoid, then $\pi_0(S)$ being a point means you have one isomorphism class. Then π_1 being zero says that any isomorphism is unique. That's uniqueness in the strongest sense possible.

This A_∞ thing is totally just for exposition purposes. Here we can replace A_∞ with *any* $\mathrm{Cobar}(\mathcal{C})$-algebra where $\mathcal{C}$ is a dg cooperad, conilpotent or complete with respect to the coradical filtration, and we need $\mathcal{C}(0) = 0$. Examples of $\mathrm{Cobar}(\mathcal{C})$-algebras include L_∞, or A_∞ or C_∞ or something.

This will all spill out of the homotopy theory for these gadgets. I'll say that these are not L_∞ but shifted L_∞ algebras. You have a cochain complex (L, d) and a shifted L_∞ structure is a system of degree 1 symmetric brackets on L, $\{\cdots\}_{k\geq 2} : \bar{S}^k L \to L$, satisfying a coherent version of the Jacobi identity up to homotopy.

Such a structure is equivalent to a degree one codifferential Q on $\bar{S}(L)$, the reduced cofree cocommutative coalgebra such that $Q|_L = d$. The restriction of Q to any $S^k(L)$ will give our kth bracket.

One more thing you could write here is that this is the same as putting and L_∞-structure on the suspension of L.

So now we want a filtration on the chain complex we're talking about, which should start at $L = F_1 L \supset F_2 L \supset \cdots$ with $L = \lim_{\leftarrow} L/F_k L$ and brackets are compatible with the filtration.

Let me give the example of one of these gadgets in case it looks like we're talking about nothing. Take a chain complex (A, d_A) in Ch^*. I can look at linear maps $\mathrm{hom}_{\mathbb{K}}(\bar{T}A[-1], A[-1])[-1]$, which we've promoted to a shifted Lie structure. I'll take $\overline{\mathrm{Coder}(A)}$ inside here as the maps f which, restricted to A, give zero. This is the Gerstenhaber bracket and all the higher brackets are zero, so this is just a shifted Lie algebra. This comes from the preLie structure (up to shifts) of a certain convolution operad, $\mathrm{Conv}(s^{-1}\,\mathrm{coAss}, \mathrm{End}_A)$, and the filtration is by the filtration on the tensor. This is supposed to be something that everyone is familiar with, maybe not at this late stage of the day but tomorrow morning, maybe it'll pop into your head. This is basically the Hochschild complex. A second example, also coming from a convolution point of view, let's say A and B are A_∞-algebras. I'll form an L_∞-algebra which will be, as a complex, $\mathrm{map}(A, B)$, linear maps from tensors in A into B, with some shifts,

$$\mathrm{map}(A, B) \coloneqq \mathrm{hom}_{\mathbb{K}}(\bar{T}(A)[-1], B[-1])$$

and the differential will be $\partial(f) = f \circ D_A \pm d_B f$ where D_A is the full A_∞ structure and d_B is just the first piece.

The brackets, let me just tell you what those are and this is probably as explicit as I'll get for the whole talk. It'll be $\{f_1, \ldots, f_n\} = \mathrm{pr}_B\, D_B(f_1 \otimes \cdots \otimes f_n)\bar{\Delta}_n$, symmetrized to give a symmetric bracket.

Again this is a filtered L_∞-algebra. Again it comes from an operad convolution origin. There is a nice paper by Martin's postdoc Felix Weierstra and he has a very nice interpretation of where this L_∞-structure comes from. This is an algebra over a certain convolution operad. He produces a map from the shifted L_∞ operad into the convolution operad. It's a nice conceptual thing, this kind of idea has showed up a lot and this operadic idea of its origin is due to him.

Let's quickly talk about morphisms in the filtered L_∞ category, sometimes I use a hat to let me know we're talking about complete things, $\widehat{\mathrm{sLie}_\infty}$, and $F : L \to \tilde{L}$ is a dg coalgebra morphism $\bar{S}(L) \to \bar{S}(\tilde{L})$ that respects the filtration in the right way. The emphasis is that these are the weak morphisms. These are not morphisms of algebras over an operad. These are the weaker ∞-morphisms.

Let's let f be the restriction of F to cogenerators L, and we'll say F is a weak equivalence if f is a quasi-isomorphism of all subcomplexes in the filtration. It's not exactly the same thing as an L_∞-quasi-isomorphism but it's close, it's the natural filtered analog. The other class is fibrations, let's define what we mean by fibrations, these are surjective at all stages in the filtration.

One of the things that's important in applications is a Maurer–Cartan element. We can be sure that the things that we want to be convergent are convergent, this was addressed in Damien's talk at the end.

The Maurer–Cartan elements of L are going to be the degree zero elements α in L^0 with curvature zero,

$$\sum_{k=2}^{\infty} \frac{1}{k!}\{\alpha, \dots, \alpha\}_k + d\alpha = 0.$$

I require this to be complete and I start at an element α in $F_1 L^0$. What else to say about this? Just to point out, morphisms are required to be filtered in the right way so we get a map between Maurer–Cartan sets, and get a weak map between Maurer–Cartan sets, $F_* : MC(L) \to MC(\tilde{L})$, I won't give a formula now but we can talk about it later.

One thing that this lets you do is twist your L_∞-structure. I'll say very briefly what that twisting is. If you take $(L, d, \{,\}_k)$ then you get a twisted structure $(L^\alpha, d^\alpha, \{,\}_k^\alpha)$, where with dg Lie algebras you turn d into $d + [\alpha, -]$ and it's some L_∞-version of this.

Okay, the simplicial Maurer–Cartan functor goes from filtered $\widehat{\mathrm{sLie}}_\infty$-algebras to simplicial sets. Then $MC_n(L)$ will be $MC(L \hat{\otimes} \Omega^*_{\mathrm{poly}}(\Delta^n))$. This is the main player in the game to get the solution space to the homotopy transfer problem.

Now let me put some results on the board. The first theorem is that this category that I've been talking about $\widehat{\mathrm{sLie}}_\infty$ has finite products (direct sums) and functorial factorizations, any morphism can be factored into a weak equivalence followed by a fibration. Then pullbacks of fibrations and acyclic fibrations exist and are themselves fibrations and acyclic fibrations. This has a category of fibrant objects structure in the sense of Brown. If you know what model categories are, this is like saying you have the right half of a model category. It's open, as far as I know, it's not clear to me what, if any, model categories are around where these are the fibrant objects. They'd have to be some kind of filtered coalgebras, maybe something that Damien was talking about. So this is what he was calling ad hoc, we were joking about this.

The last bit of the main theorem is that the simplicial Maurer–Cartan functor preserves weak equivalences and fibrations, sends fibrations to Kan fibrations, and so $\mathrm{MC}(L)$ is a Kan complex, i.e., an ∞-groupoid. There are variants of this kind of result. This is an exact functor between categories of fibrant objects. I'll say some names because I'm running out of time. The statement about being a category of fibrant objects is due to Getzler (unpublished) and will appear in a paper of mine by Christmas. The Maurer–Cartan theorems have versions due to Hinich, Getzler, and Yalin. For the fully weak case one part is due to Dolgushev and myself and the part about being a Kan complex is in a paper from last year.

Okay, so let me solve the homotopy transfer problem. What I'm going to do is construct (this example is due to Dolgushev and Willwacher), they construct, given (A, d_A) and (B, d_B), and they construct $\mathrm{cyl}(A, B)$, an L_∞ structure, this is $\mathrm{coder}(A) \oplus \mathrm{map}(A, B) \oplus \mathrm{coder}(B)$. The L_∞-structure is not just the sum but is mixed around. The Maurer–Cartan elements are an A_∞-structure on A, an A_∞-structure on B, and a map Φ between them. In particular, if you have a quasi-isomorphism of complexes $(A, d_A) \xrightarrow{\phi} (B, d_B)$, then $(0, \phi, 0) = \alpha$ is an example of such a Maurer–Cartan element. Then I can twist this and tell you what Maurer–Cartan elements in $\overline{\mathrm{Cyl}(A, B)}^{\alpha}$. The Maurer–Cartan elements here are just going to be a subset of the Maurer–Cartan elements where Φ lifts ϕ. A fact is that ϕ being a quasi-iso implies you have a map from the twisted L_∞-algebra $\overline{\mathrm{Cyl}}(A, B)^\alpha$ to $\mathrm{Coder}(B)$, this is a strict L_∞-map. If ϕ is a quasi-isomorphism, then this projection is a quasi-isomorphism of L_∞-algebras. It's an acyclic fibration in my category of L_∞-algebras and so I can hit this with my functor MC and get an acyclic fibration $\mathrm{MC}_\bullet(\overline{\mathrm{Cyl}}(A, B)^\alpha) \to \mathrm{MC}(\mathrm{Coder}(B))$. Then I can use a point to hit $\{m_k^B\}$, and I can take the pullback and Kan complexes are a category of finite objects, so the resulting set S is contractible (and non-empty). This is the proof of the homotopy transfer theorem that is not only fast but easy.

I'd be happy to talk about other applications mentioned in my abstract but I'll stop for now. Thank you very much for your attention.

4. Nov 9: Martin Markl: Operadic categories

I was really delighted about how you took care of us foreigners in China for the first time.

Half of this is based on my paper with Batanin, "Operadic categories and duoidal Delign's conjecture."

One sign of advance in age is instead of doing useful things, is starting to philosophize. All people who work with operads, have some feeling, this is something that looks like an operad or a prop. So the question is how to make a formalization that encodes all examples we know but also things that have a similar flavor like permutads or n-operads. The theory that we were after must be strong enough to reproduce all constructions available for classical operads. We're writing up a paper that puts some basics of this theory that already has 100 pages. Then once that's done we think all this kind of thing will be easy to prove.

All right, so there are already some approaches that tried to say what an operad is in a general setup. One started with pasting schemes, which I introduced in 2008, which I wrote for the handbook of algebra. Let me give you a brief idea of what a pasting scheme is.

A pasting scheme is graphs that are "of some type." I'll give you some examples. Pasting schemes have a property call hereditarity, meaning that if I take a graph of my family of graphs and contract a subgraph, I'll get a graph of the same type. This concept was later taken up by Borisov, Manin, and most recently by Ralph Kaufmann who generalized it to Feynman categories. Let me recall the notions used in this approach.

One example is "planar rooted tree" where this clearly has the property of hereditarity, where if I contract an edge I get a tree of the same type. As an exercise, you can think of a class of graphs that is not hereditary. I'm sure you will easily find one.

These govern operads, or non-symmetric operads. If I am given such a tree and decorate the vertices with elements of a collection [pictures] then if I paste everything together, I'll get something in this picture in $P(5)$. There might be more than one way to do this but however I do it I get the same thing, that's the axioms of a non-symmetric operad.

All of these operations are generated by a single edge contraction, say, a $\circ_i$ operation, which generates my non-symmetric operad operations as well.

If I take a planar tree with no orientation, decorating as follows, say, here by $P(4)$, here by $P(3)$, here by $P(4)$, then by contracting you get something in $P(7)$ if I'm not mistaken. This is the pasting scheme for cyclic operads, or planar cyclic operads.

Let me give one more example, which I will use again, nothing deep is going on on this level, are, of course, modular operads. So I'll talk about a version without genus grading. Let me draw a pasting scheme for modular operads. Let me draw a picture, It's a graph which has a loop, and a double edge if I'm adventurous, and here I decorate by $P(4)$, here by $P(5)$, and here by $P(4)$, and contraction should give me something in $P(5)$.

So let me give you the following ingenious notion of Batanin, even if you get nothing else from my talk, if you get this then your time will not be wasted. I'm going to define an n-operad, and I'll actually do $n = 2$, which is not trivial. So first I should tell you what is a 2-tree.

So I have $[k]$ the finite ordinal $\{1, \dots, k\}$, and $[\ell]$, and you have a map between these things, and $[\ell]$ maps to $[1]$. This is a "tree with two levels", the maps are order preserving. [pictures] And everything goes eventually to the terminal ordinal downstairs. This is Batanin's 2-tree. The elements of k are called "tips" in his language. It's pretty obvious what it is. So what are morphisms of these things? Suppose I have one 2-tree and another one, order-preserving maps. A morphism is a couple of maps φ_1, φ_2, not necessarily order-preserving, but the square must commute, so these are maps of trees, and there's a condition. If I take a point in $[\ell]$ and its preimage in $[k]$. Then φ_1 should preserve the order of the preimage. You have a category of 2-trees in this setup.

Here is an example that I prepared: $(1,2);(3)$ maps to $(12,3)$ and also maps to $(1,23)$. It's an exercise to find a map of trees that isn't a map in this category. A 1-tree is just the category of ordinals, and it's just Δ, a skeletal category of ordinals and order-preserving maps.

The next important thing is that each such map has a fiber. Let me modify the picture slightly to explain what the fibers are. For each tip of the second tree, I have a line that joins the tip with the terminal ordinal, and now I take a preimage of this under the map, so you can take $\varphi^{-1}(i)$, which is not the set theoretic inverse. So in my examples, what are the fibers over (12) and (3)? What is mapped to the line from (12)? It's this picture [pictures]. It's the same kind of picture for (3).

Very good. So now I can give you the following.

Definition 4.1 (Batanin)**.** A 2-operad in your favorite symmetric monoidal category, is a collection $A(T)$ for T in $\mathcal{T}_2$ along with maps

$$\gamma_\phi : A(T) \otimes A(F_1) \otimes \cdots A(F_n) \to A(S)$$

and here ϕ is a map of 2-trees, and $F_1, \ldots, F_n$, are the fibers.

This doesn't correspond to any pasting scheme as far as I know.

There's a theorem saying that a space is an E_n-space if and only if it's acted on by an n-operad. People in higher category theory know this is close to globular category theory. So it has many many applications.

Let me give an example that is easier than this one. I'll write in more detail what is a 1-operad.

Recall that T_1, this is just Δ, so what is Δ? It's finite ordinals, so for S and T, I have a map $\phi : S \to T$. The fibers are preimages, so to be more specific say that this was $[k] \to [\ell]$, and this is order-preserving, so this is a division into subintervals, and these are the fibers. So the structure map is

$$\gamma_\phi : A(\ell) \otimes A(k_1) \otimes \cdots \otimes A(k_\ell) \to A(k_1 + \cdots + k_\ell).$$

This is then the data of an operad. Everyone knows that a space is a loop space if it's acted on by a contractible non-symmetric operad, a 1-operad.

Okay, very good, What about operadic categories? Not much needs to be changed here. This is the definition of an operadic category. This is a category $\mathcal{O}$ equipped with a functor $|-|$ from $\mathcal{O}$ to finite sets (or a skeleton) which we call *cardinality* and it has the following properties.

(1) Each $f : S \to T$ in this category has fibers $f^{-1}(i)$ for $i = 1, \ldots, |T|$ (this needn't be a set theoretic inverse).
(2) This says something about local terminal objects in $\mathcal{O}$, but as most of you know, units and constants usually make a lot of trouble. Even though this is dull it's somewhat sensitive and subtle.

So what is an operad, I have things for each object in your category, and maps for the morphisms, and there should be units, this requires the choice of terminal objects.

So three examples that we saw of this type of structure. We saw n-trees and finite sets. Actually a 0-tree is just the terminal category and operads over them are monoids and so monoids fit into the theory.

Let me give you an example with some flavor that will explain the basic pieces of the theory. Modulo details I'll define A category of graphs Gr. So a graph may have loops or double edges, there are some labels and local and global orders, I put them under the carpet. Benoit would probably call these hairs, but it has legs, it has edges and vertices, and what is a morphism of graphs? Modulo some details a morphism is given by a contraction. Let me draw the pictures from my notes. [pictures]

This is a category. To make it an operadic category, I have to give a cardinality function. The cardinality of the graph is the set of vertices. The fibers are the thing that you've contracted.

So what is the operation related to this morphism? What are the operads in this category? We have $\mathcal{O}$ which is labeled or indexed by graphs, and the structure operation related to this morphism, according to this morphism, it's γ_f from $\mathcal{O}(T) \otimes \mathcal{O}(F_1) \otimes \mathcal{O}(F_2)$ into $\mathcal{O}(S)$. If you read carefully Getzler and Kapranov on modular operads, then the operads in Gr are the hyperoperads, at least morally.

I also have to tell you what an algebra in an operadic category is. So first of all I need an object, for T in $\mathcal{O}$, I have $t(T)$, the target is simply $[T]$, the isomorphism class of T in the set of connected components of my category $\pi_0(\mathcal{O})$, I say that my category is small. So what is $s(T)$? It's the sequence of classes in $\mathcal{P}_0$, $[F_1], \dots, [F_k]$, all in π_0, where F_j are the fibers of $1' : T \to T$. Connected components are indexed by corollas with n spikes or whatever, again I'm not taking genus into account. [pictures]

What is my target? I get a corolla with eight, or whatever. [more pictures]. This is similar to the terminology of globular categories. This analogy is not a coincidence.

So what is an algebra over an operad. So let P be an $\mathcal{O}$-operad. A P-algebra is a collection of objects X_c, with c the connected components of $\mathcal{O}$ and $\alpha_T : P(T) \to \mathrm{Hom}(\bigotimes_c X_c, X_{t(T)})$, which is $\mathrm{End}_X(T)$. You need some axioms, including ones involving the terminal object that I won't write.

If $\mathcal{O}$ is Δ, then the correction is $1, 1, \dots 1, 2$ and you immediately get algebras over non-symmetric operads. Also the n-trees and n-operads are just one thing. What about for my hypersurfaces example? Then the algebra is given by natural numbers. If I had genus I'd also have genus here. The structure operations for this map T are like

$$\alpha_T : P(T) \to \mathrm{Hom}(X(5) \otimes X(5) \otimes X(4) \otimes X(5), X(8)).$$

If P is the terminal Gr-operad, then for each graph I get morphisms of this type, $\alpha_T(1)$. The models follow Getzler and Kapranov, more or less. So now things are arising as algebras over operads, so for instance we have $\mathbf{1}$, the terminal Gr-operad, which is $\mathbb{K}$ in each thing. This is quadratic and its quadratic dual, I'd call it "odd" meaning not that it seems weird, but that there is a shift, this arises in the cyclic and modular case.

Let me finish the talki by mentioning what I call the opetopic principle. When I realized how Jacobi was abused, I tried to rename it Jacob's ladder. You have the structures you want, which appear as algebras over $\mathcal{O}$-operads, so e.g., modular operads arising as algebras over Gr-operads. What about $\mathcal{O}$-operads? They appear as algebras over $\mathcal{O}^+$-operads. This is another operadic category constructed from $\mathcal{O}$. It goes on, so you have $\mathcal{O}^+$-operads that are $\mathcal{O}^{++}$-algebras, and so on. If I have time, I can even explain what $\mathcal{O}^+$ means for Gr. Everything improves. As we "suspend" we get more and more duality.

Finally, it goes up without bound. Thank you very much. Once again, I have printouts.