HOMOTOPY PROBABILITY WORKSHOP, SAARBRÜCKEN

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1. September 4: Owen Gwilliam: Formal matrix integrals via a quantization of the Loday–Quillen–Tsygan theorem

Thank you for the chance to talk and participate. My goal in this talk is to advertise a theorem from the 1980s that has some applications to large N limits. It relates cyclic homology of algebras to Lie algebra homology. It's not obvious that it has anything to do with probability and physics. My goal is to look at a concrete examples. If you have any ideas here I'd love to discuss. My assumption is that everyone knows what is the homology of a chain complex but not necessarily Lie algebra homology. I'll start with a (very incomplete and biased view) of what that is. Everything here is joint with M. Zeinalian and G. Ginot.

Let me start with the key example. I'll let $\mathcal{H}_N$ denote $N \times N$ Hermitian matrices. This supports a Gaussian measure $e^{\operatorname{tr}(X^2)/\hbar}dX$ where $X = X_{ij}$ is a variable matrix.

The goal of the talk is to give a different way of understanding the associated probability measure (I'm just lazy and not normalizing things) especially in the large N limit.

Let me make some remarks. First of all, really cool stuff happens when N gets big, like the distribution of eigenvalues by the Wigner semi-circle law. If you're a physicist it looks like you're studying a Euclidean path integral with action $S(X) = \operatorname{tr}(X^2)$, this is the simplest matrix model, and it's clear that this admits generalizations, you could have the action start with this quadratic piece and have a cubic piece and so on, $S(X) = \operatorname{tr}(X^2) + c\operatorname{tr}(X^3) + \cdots$

So I'll use a physical motivational approach to move to Loday–Quillen. It's pretty clear that these functions admit a natural symmetry by the unitary group by conjugation, and the questions I want to focus on include like computing the expected values of invariant observables (functions on the vector space $\mathcal{H}_N$)

If you do read what the physicists talk about in this context, they have this Feynmann diagrammatic expansion which let them give asymptotic expansions. In the large N case, there is this beautiful ribbon graph expansion.

In these kinds of matrix models, the Feynmann graph expansion leads to a different, very beautiful and combinatorial expansion called the ribbon graph expansion, and you don't need to know it for this talk, but let me give a fact that comes out of it.

$$\int_{\mathcal{H}_N} \operatorname{tr}(X^{2n}) e^{-\operatorname{tr}(X^2)/2} dX = \sum_{g \geq 0} c_g(m) N^{m+1-2g}$$

where $c_g(m)$ is the number of ways to glue a $2m$-gon to a surface of genus g. The thing I want you to take away is that there is a connection between something you compute for surfaces and computing this expectation. There's a beautiful exposition of this by Etingof, very short.

What I'd really like is to give you a homological perspective on this computation.

Now I'll jump into the heart of the talk. Like I said I want to use some ideas from physics. There's a formalism known as the Batalin–Vilkovisky formalism, sort of a fancy version of the BRST procedure. I have no expectation that you know what any of this is, I'll call it BV and give a gloss of the essentials. The idea is to shift from studying the measure to studying the expected value map, and in fact to a chain complex whose zeroth homology is what the expected value is supposed to factor through.

So we have a measure $\mathbb{E} : \mathcal{O}(V) \to \mathbb{C}$ and in particular we want to characterize the kernel of the expected value map in a natural way. Let's think of this as being the Gaussian measure for instance. What sort fo things integrate to zero? Total derivatives integrate to zero. I'm thinking about Stokes' theorem. Let's see what that means here.

I'll think here about $\mathcal{O}(V) = \mathrm{Sym}(V^*)$, and expected value will just be reading off moments. I claim that if I think about polynomial vector fields $\mathrm{Sym}(V^*) \otimes V$, and there's an operator called divergence against the measure, $\mathcal{X} = f\frac{\partial}{\partial x_i}$, So suppose I have a map $S : V \to \mathbb{C}$ which is, lets say quadratic in x, and the fact is that divergences of polynomial vector fields against a Gaussian measure always integrate to zero, have zero expected value, and in fact span the kernel of this expected value map.

Let me do the one dimensional example. Say that the measure is $e^{-ax^2/2}dx =: \mu$ and suppose I have a vector field $x^m\frac{\partial}{\partial x}$. The definition is that

$$\mathrm{div}_\mu(x^m\partial)\mu = L_{x^m\partial}\mu = d(x^m e^{-ax^2/2}) = (mx^{m-1} - ax^{m+1})e^{-ax^2/2}dx$$

so the divergence is $mx^{m-1} - ax^{m+1}$.

Let's think abotu the cokernel of the divergence map. If I have a polynomial x^{m+1}, its cohomology class is the same as $\frac{m}{a}[x^{m-1}]$. This is the recursion relation satisfied by moments of a Gaussian. If $m+1$ is even, you go down by 2 and the expected value $\langle x^{2n}\rangle = \frac{(2n-1)!!}{a^n}$, the usual Wick expansion or whatever you call it in probability theory for the moments of a Gaussian.

So the BV formalism tells you how to extend to a cochain complex, you could view this as the first part of a cochain complex which in the Gaussian case recovers the usual expected value by looking at zeroth cohomology.

The BV complex, let's say, will look like

$$\mathrm{Sym}(V^*) \otimes \wedge^k V \to \cdots \to \mathrm{Sym}(V^*) \otimes V \to \mathrm{Sym}(V^*)$$

and I claim that this lets you read off a lot of the data about the expected value map.

There's really no reason we can't apply this to the matrices, so that's what I'm going to do. I'll introduce some notation. If you don't want to do a Gaussian measure, the recursion relations are more complicated, it doesn't respect degrees, and this becomes an asymptotic expansion and not an actual integral.

Let me introduce a little notation. I want to write this BV complex as $(\mathrm{Sym}(V^* \oplus V[1]), \mathrm{div})$. This is a commutative *graded* symmetric algebra, which gives me the wedge k powers. For us, we want to use $V = \mathfrak{gl}_N$ and divergence against $e^{-\operatorname{tr}(X^2)/\hbar}dX$. If we view $\hbar$ as a parameter, as a variable, then we get

$$BV_n = (\mathrm{Sym}(\mathfrak{gl}_N^* \oplus \mathfrak{gl}_n[1]), d^{\mathrm{cl}} + \hbar\Delta).$$

The part d^{cl} is classical and doesn't depend on $\hbar$ and a "BV Laplacian" Δ.

I don't have time to unravel this. I claim that this is how a physicist might want to study a Gaussian integral. I also want to consider $(BV_N)^{\mathfrak{gl}_N}$, the invariant subcomplex (so in degree zero the invariant functions on $\mathrm{Sym}((gl_N)^*)$).

Now I want to reinterpret using Lie algebra cohomology.

Let me explain Lie algebra cohomology. Homology is easier, first, say I have a Lie algebra $\mathfrak{g}$. What is a representation M, it's a map $\mathfrak{g}\otimes M \to M$, and the coinvariants $\mathrm{coinv}(\rho) = M/\mathfrak{g}\cdot M = M_{\mathfrak{g}}$. This will be the zeroth Lie algebra homology. Homology extends this to the left, looks like

$$\cdots \to \wedge^k \mathfrak{g} \otimes M \to \cdots \to \mathfrak{g} \otimes M \to M$$

and what is the differential? I can bracket two elements of $\wedge^k\mathfrak{g}$, or I could have elements of the Lie algebra act on M. You can check here that, well, anyway, that's homology, this complex I'll denote by $C^{\mathrm{Lie}}(\mathfrak{g}, M)$, and it looks like $(\mathrm{Sym}(\mathfrak{g}[1]) \otimes M, d_{\mathrm{Lie}})$, a symmetric thing with an interesting differential.

The cohomology is the dual, and hopefully you see where I'm going, reinterpreting the BV complex in this way. So I can get a map $M \to \mathrm{Hom}(\mathfrak{g}, M)$. I can ask what the kernel of this map is, it's the m such that $\mathfrak{g} \cdot m = 0$, this is called the *invariants* and is the zeroth Lie algebra homology. You extend, taking linear duals (continuous linear duals is best)

$$C^*_{\mathrm{Lie}}(\mathfrak{g}, M) = (\widehat{\mathrm{Sym}}(\mathfrak{g}^*[-1]) \otimes M, d_{\mathrm{Lie}}).$$

I can define $C^*_{\mathrm{Lie}}(\mathfrak{g}) \coloneqq C^*_{\mathrm{Lie}}(\mathfrak{g}, \mathbf{k})$. Let $\mathfrak{g}$ be the following very silly Lie algebra, where $\mathfrak{g}$ is $\mathfrak{gl}_N$ in degrees 1 and 2, no bracket, and there's a differential which is the identity. I said this for regular Lie algebras but we can do this for Lie algebras with differentials by taking a total complex of a double complex. Then $BV_N \mod \hbar = C^*_{\mathrm{Lie}}(\mathfrak{g})$, the "classical BV complex."

Let me point out something. I'm leaving lots of holes in this exposition, talking about BV itself is an involved process that I want to avoid. The crucial thing is that there is something related to the one we want coming from Lie algebra homology. So $\mathfrak{g}_N$ is $\mathfrak{gl}_N(A)$ where A has generators a and b in degrees 1 and 2, with all products zero.

Probably everyone who hasn't done BV kinds of stuff is confused. I want to massage this question of studying these integrals into a very funny framework of Lie algebra cohomology because of this amazing theorem. Maybe you'll find a better use for it than I have. Tsygan stated this theorem first in one of these Soviet journals, but didn't give the proof, and Loday–Quillen did it (independently?) a few years later.

Theorem 1.1. *(Loday–Quillen–Tsygan) Let k be characteristic zero and let A be a unital dg $\mathbf{k}$-algebra (it can be A_∞). There is a natural quasi-isomorphism*

$$\lim_{N\to\infty} C^{\mathrm{Lie}}_*(\mathfrak{gl}_N(A)) = C^{\mathrm{Lie}}_*(\mathfrak{gl}_\infty(A)) \xrightarrow{\cong} \mathrm{Sym}(\mathrm{Cyc}_*(A)[1]).$$

So the large N limit of the Lie algebra homology is the symmetric thing on the cyclic homoology. We want the dual statement

$$\widehat{\mathrm{Sym}}(\mathrm{Cyc}^*(A)[-1]) \xrightarrow{\cong} C^*_{\mathrm{Lie}}(\mathfrak{gl}_\infty(A)).$$

which tells you that stable classes come from cyclic cohomology. I didn't tell you what cyclic cohomology is. Tsygan called this "additive K-theory."

I need to tell you what cyclic cohomology is, very quickly. I'll recapitulate the kind of description I gave for the Lie algebra case. Suppose A is an algebra an M an $A-A$-bimodule. You have a map $A\otimes M\to M$, called the action, I am thinking $A\otimes M\mapsto a\cdot m - m\cdot a$. The cokernel is $M/[A,M]$, which is called 0th Hochschild homology, and traces factor through this.

So Hochschild homology extends to the left. I get

$$\cdots\to A^{\otimes k}\otimes M\to\cdots\to A\otimes M\to M,$$

and this looks like $(\mathrm{Hoch}_*(A,M) = \mathrm{Tens}(A[1])\otimes M, d_{\mathrm{Hoch}})$. If $M = A$, this gives the cyclic bar complex

$$\cdots\to A^{\otimes k+1}\to\cdots\to A^{\otimes 2}\to A.$$

If you rotate, the differential descends, and so you can look at

$$\cdots\to A^{\otimes k+1}/(\mathbb{Z}/k+1)\to\cdots\to A^{\otimes 2}/(\mathbb{Z}/2)\to A.$$

This is something very natural if you're thinking about associative algebras. The cyclic cohomology is the linear dual, and let me sketch the idea, let me say just a little more here.

You can imagine arranging the elements of your algebra on a circle, and then the differential runs through all the ways of taking two in a row and tensors them together. [picture]

The idea of the proof of Loday–Quillen–Tsygan, remember the homology version, one piece are these k-polynomials in the Lie algebra, $\mathrm{Sym}^k(\mathfrak{gl}_N(A)[1])$, and you can symmetrize, getting $M_N(A)^{\otimes k}[k]$ and so this determines a map from $C_*(\mathfrak{gl}_N(A))\to\mathrm{Cyc}_*(M_N(A))[1]$, and a great property of cyclic homology is that it's Morita invariant, there's an equivalence to $\mathrm{Cyc}_k(A)[1]$, and then you unravel things to show tha there's a quasi-isomorphism, once you take primitives. The extra juice is the invariant theory that show that you get a quasi-isomorphism after taking invariants.

In my last three minutes, I told you what the algebra was, this funny algebra with generators in degree 1 and 2 with a differential between them that is an isomorphism, you can work out that $\mathrm{Cyc}^*(A)[-1]$ is concentrated in non-positive degrees. In degree 0 it's $\mathbb{C}[x]$ and in degree $-k$ it's the vector space of cyclic words in $\{x,\xi\}$ with precisely k copies of ξ. In degree -1, I can always make this $\mathbb{C}[x]\xi$. The Loday–Quillen–Tsygan map takes $\mathbb{C}[x]$ to the degree 0 piece of BV_N modulo $\hbar$ which is $\mathrm{Sym}(\mathfrak{gl}_N^*)$ which takes $x^m\mapsto\mathrm{tr}(X^m)$. The question is, is there a way to go from this relationship modulo $\hbar$ to the relationship involving $\hbar$? Can I quantize the Loday–Quillen–Tsygan map. I don't want to give succinct formulas, but let me give an abstract description that may not be helpful for anyone except Jae-Suk. I want to say, as Roland said, it's exciting that there's a collision among homotopy theorists, physicists, and probabilists, but there's not a common body of knowledge yet so it's hard to communicate in a talk.

So suppose I have a finite dimensional A_∞ algebra with a non-degenerate cyclic pairing of degree -3. Then $\mathfrak{gl}_N$ of this thing is an L_∞ algebra with the same kind of thing. Then $LQT_N:\widehat{\mathrm{Sym}}(\mathrm{Cyc}(A)[-1])\cong C^*_{\mathrm{Lie}}(\mathfrak{gl}_N(A))^{\mathfrak{gl}_N}$. You can lift this to a P_0-algebra map senting the variables ν to N. The great thing about having a map of Poisson algebras determines a BV quantization on the other side. This is uniform in N, and the assertion is that if you cherry-pick some A, you can get some large N limit of Chern–Simons, and in particular there's a distinguished thing on the one side coming from an involutive Lie bialgebra structure. I think this has potential in physics and hopefully also in probability theory.

2. Carlos Vargas-Obieta: Non-commutative distributions for simplicial complexes

Thank you very much, today I want to talk about a simple example within the realm of noncommutative probability theory. We'd like to show that some algebraic topological features of spaces can be encoded in noncommutative distributions. For this we need not just scalar-valued noncommutative probability but also operator-valued noncommutative probability. One motivation is using noncommutative probability to understand random matrices.

We start, one of the easiest ways to produce a random matrix is to pick a normalized matrix Z_N with entries independently and in a complex normal distribution. The question of computing the eigenvalues is a bit hard. Wigner considered one way of modifying this matrix to get a self-adjoint matrix, consider $X_N := \frac{1}{\sqrt{2}}(Z_N + Z_N^*)$, and his pioneering result is that $\Lambda(X)$, the value of a single eigenvalue, the distribution is semicircular, from -2 to 2, just pushed down so that the integral is 1. Later the Wishart model was considered, instead of adding the adjoint, you multiply by the adjoint, $Z_N \cdot Z_N^*$, this is a random positive matrix. Call this W_N, and then the random eigenvalue here $\Lambda(W_n)$ is a positive distribution, the Marcenko–Pastur distribution (1967). They also considered taking Z_N and conjugating some D_N by it, there have to be some regularity conditions on D. We ask that $\frac{1}{N}Tr(D_N^k) \to m_k$, the moments of some probability measure μ.

These were the main results for a long time. Some people summed some of these ensembles, but it was really hard to do some new models. In 1991, Professor Voiculescu came up with the following idea, I'll rephrase these results. The main result is that "the involved matrices satisfy an independence relation defined for noncommutative random variables."

At the beginning this allowed to understand the Wigner and Wishart matrices from other points of view, but then it allowed more general matrices, such as a combination of these.

Some nice facts, moreover: the semicircular law plays the role of the Gaussian distribution in probability theory. If you consider $\frac{1}{\sqrt{N}}\sum_{i=1}^N a_i$, this converges to some semicircular distribution.

Also in the same sense, the Marcenko–Pastur distribution plays the role of the Poisson distribution in classical probability theory.

Then, maybe at the beginning it was maybe just recording known results. Then this idea thinking of these as random variables with some notion of independence led to new tools to let you compute new things. One can also say that the Cauchy–Stieltjes transform plays the role of the Fourier (characteristic function) and here I mean that if we know the distributions of random variables then we can use Fourier to compute invariants for combinations of them. An important development later in 1995, there is also the notion of freeness with respect to a conditional expectation, and this allowed, the relation between the random matrices is that they are free. There is an objection, maybe this is too specific. But if we allow ourselves to consider freeness with respect to something conditional, then we can treat more models. Then a year later, immediately, it was noticed that this notion of freeness can be applied to random matrices. There was a long list of developments, Voiculescu, [unintelligible], Biane, many more, and there's a nice paper by Belischi–[unintelligible]–Speicher where they compute for any polynomial evaluated in these matrices.

I hope that this convinces you that non-commutative probability is important to study [unintelligible]distributions. The moral here is that it's very important to think of random matrices as non-commutative random variables satisfying an independence condition.

Outside the world of random matrices, Roland Speicher introduced the notion of cumulants, and this led to new notions of independence, Boolean independence, and also the story of how to produce independent random variables in a simple way, and this is related to spectral graph theory, this is the simplest example of independent noncommutative random variables. I'll try to give some basic examples, but especially in spectral graph theory.

Before doing this I want to tell you something about topological data analysis. You are comparing data sets through their topological invariants and at diferent scales. I gave you a picture, you have two donuts, on the left they are randomly distributed and on the right they have some repulsion property so that they are more evenly distributed.

I'm going to make a simplicial complex where I add a simplex among a collection if the balls of radius r around each of them intersect. If you have enough points you will eventually recover the homology of the torus, but the main idea is that using the Betti numbers let us use the shape, the idea of topological data analysis is to do this with data sets, and somehow these Betti numbers capture important information about the shape of the data.

The main question of my talk (and the main goal) is: can we get the Betti numbers from the noncommutative distributions?

Let's start now with main definitions.

Definition 2.1. An operator-valued non-commputative probability space is a triple $(A, B, \mathbb{F})$, where B is a unital $*$-subalgebra of a star algebra A and $\mathbb{F}$ is a conditional expectation to B, that is, $\mathbb{F} : A \to B$ satisfies $\mathbb{F}(bab') = b\mathbb{F}(a)b'$ and $\mathbb{F}(1_A) = 1_B$.

The main example is conditional expectations in probability theory. So the first example is in the context of a classical probability space $(\Omega, \mathcal{F}, P)$, we define A to be the algebra of complex $\mathcal{F}$-measurable random variables and ask for $\mathbb{F}$ to be the expectation. Any sub-σ algebra of F gives one of these: if $\mathcal{H}$ is a a subalgebra of $\mathcal{F}$, then we have a unique map $\mathbb{F} : A_{\mathcal{F}} \to A_{\mathcal{H}}$. So if $\mathcal{H} = \{\varnothing, \Omega\}$ then $\mathbb{F}$ is the regular expectation. On the other hand, if $\mathcal{H} = \mathcal{F}$ then the expectation is the identity.

One of the reasons to do this is that the algebra of matrices with the trace gives an operator-valued probability space.

The second example is, take $\mathcal{A}$ to be the complex matrices, $N \times N$, and take $B = \mathbb{C}$ and $\mathcal{F}$ is the normalized trace $\frac{1}{N}\operatorname{Tr}$. Now the main idea is that we will be interested in the distribution of the elements of our algebra. We call, let me fix the case $B = \mathbb{C}$ with $\mathbb{F} = \tau$, this is scalar-valued, that's what I mean by τ. So a tuple $(a1, \ldots, a_k)$ are in $\mathcal{A}_1$ and $(b_1, \ldots, b_k) \in \mathcal{A}_2$ have the same distribution if, let me say what I'm doing.. I want to extract everything from probability theory in terms of moments. No we say they have the same distribution if $\tau_1(a_{i_1} \cdots a_{i(m)})$. So if X is distributed as a Bernoulli variable, then X has the same distribution as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The third example tells me that I can consider tensor products, I get $(A_1 \otimes A_2, B_1 \otimes B_2, \mathbb{F}_1 \otimes \mathbb{F}_2)$. The distribution of the matrices with respect to this, if I can make [unintelligible], we get matrices with entries now in our probability space, and the functional is the normalized trace, and so I get $\frac{1}{N}\operatorname{Tr}\otimes\mathbb{F}$. This puts everything in context, tells us why we can consider [unintelligible]. We can considr free inedependence in this case.

There were some works, started by Roland on how to classify independence. There's a nice description, well, this gives us a way to compute mixed moments. Say that A_i are all distinct subalgebras. I say that these are freely independent if $\langle a_1^k \cdots a_k\rangle$ is $\mathbb{F}(a_1^m)\cdots\mathbb{F}(a_k^{n_k})$, I want to assume that $\langle A_1\, ldtos A_k$ is a comumutative algebra.

To discuss independence consider $\cdot a = a - \mathbb{F}(a)$, and if I want $\mathbb{F}(\dot{a}_1\cdots\dot{a}_m) = 0$ whenever a_j and a_{j+1} never match.

So for Boolean independence, this is the same as $\mathbb{F}(a_1\cdots a_m) = \mathbb{F}(a_1)\cdots\mathbb{F}(a_m))$.

Now I want to present some Segal realizations in terms of graphs. So classical independence can be realized using the tensor product. In the context of the second example of matrices and the normalized trace. If a matrix A is normal, then we can associate the sums of,

$$\frac{1}{N}\operatorname{Tr}(A^k A^{*\ell}) = \frac{1}{N}\sum_{i=1}^{N} \lambda_i^k \bar{\lambda}_i^{\ell}.$$

Say that $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$, then we can consider $A \otimes I_m$ and $I_n \otimes B$. If these are $\tilde{A}$ and $\tilde{B}$, tensoring with the identity doesn't affect the spectral measure. Then $\tilde{A}$ and $\tilde{B}$ are realizations of independent random variables.

One can also see that if UAU^* is diagonal, and likewise for V and B, then $U \otimes V$ simultaneously diagonalizes $\tilde{A}$ and $\tilde{B}$. If we are now in the, [unintelligible]Muraki invented monotone independence. Think now that A and B are adjacent. The tensor products of two adjacency matrices is again an adjacency matrix, so this realizes $\otimes$ in terms of graphs.

Now we consider $M_N(\mathbb{C})$ and the functional is instead $\tau_{1,1}$, I consider the first vertex to be the root. This τ_{11} is just the first entry. So $\tau_{11}(A_k)$ is the moments, a weighted version of [unintelligible]. I'll associate a probability measure to each matrix and then [missed some]

The interesting thing is that it is possible to realize other kinds of independence in terms of graphs like this. Maybe I'll just tell you what to do with the Boolean case.

The Boolean realization, the realization of Boolean convolution, this is, if I consider A_{G_1} and A_{G_2} with a root, say G_1 is this [picture] and G_2 is this [picture] then the star product $G_1 \star G_2$ is to identify the two graphs by their roots. This formula that I displayed for the moments is satisfied by this construction, I have no time to introduce the cumulants but the combinatorial picture is very important, there are cumulants which are important because they are polynomials in the moments that linearize the sum of independent random variables. Let me just say it in words. Boolean cumulants count irreducible cycles within the graph and what I mean is that, there is a path within the vertices of the graph going through the edges. A cycle is irreducible if the root is only visited once. Then the irreducible cycles of size k is the sum of the number for each of the graphs because if I want to make a cycle that goes through both components, then I have to go through the root.

I was thinking, then, what other kind of functional can I compute for graphs. If you compute Betti numbers and paste them, you get additivity. There are also versions to give monotone and free independence.

Maybe the last minute, the idea is how to work with simplicial complexes? The idea is that all our analysis was in the [unintelligible], we now want to use the faces. So the idea is to use the incidence matrix instead. One can recover the adjacency matrix from the incidence matrix. For instance, let me see, we have an incidence matrix of a simplicial complex and consider I^*I, this gives the degree matrix of X, plus the adjacency matrix of X. Then for dimension 2 (a graph) all the moments, we're getting nice information [missed something]. The question is what you get for higher dimensions, for higher dimension you no longer has $I^2 = 0$ and you can look instead at the boundary, and then the differential squaring to zero turns into the square of this matrix being zero, and if you consider $J^*J + JJ^*$, and the eigenvalues satisfy [unintelligible]. I have no time to say why but we can discuss.

3. Claus Koestler: Algebraic central limit theorems from distributional symmetries

So thanks for the invitation, I think I'll give a hint about what I'm *not* talking about today, which are semi-cosimplicial objects and spreadability, joint with Gohm and Evans. One knows that in physics, whenever you have systems and you look at joint distributions, expectations of monomials in several variables, this gives you a whole lot of data and when you have symmetry that incorporating that makes things simpler. So we have some general abstract nonsense approach for when symmetry comes from some data on a semi-cosimplicial objects. Today I'll talk about how distributional symmetries imply central limit theorems. This is a credo, basically. Say these are an infinite series of random variables that are exchangeable, and then you get conditional independence, and that leads to a central limit theorem. I'll do things only by example today in making this work in a non-commutative setting. So the task is to make the credo rigorous. And we want to identify the possible central limits. What I'm looking at today is the probability space which will become more explicit and explained, is $(\mathbb{C}[S_\infty], \mathrm{tr}_{1/d})$. So I'll take S_∞, the group of all bijections of the natural numbers such that there exists some N such that $\sigma(k) = k$ for all $k > N$. This is the inductive limit over n of S_n. I'll consider $\mathcal{T}$ the set of all transpositions. For non-identity σ in S_∞ we define this as

$$||\sigma|| = \min\{k \in \mathbb{N} | \tau_1 \cdots \tau_k = \sigma, \tau_i \in \mathcal{T}\}.$$

For the identity we give length 0.

Then, well, how do we go, we observe, put $\chi_q(\sigma) = q^{||\sigma||}$ for some $q \in \mathbb{R}$, and this is constant on conjugacy classes and normalized $X_q(\mathrm{id}) = 1$, and so what we need to produce a character is that it is positive definite. This is a condition we want to have, which just means that if we look at $\chi_q(\sigma_i^{-1}\sigma_j)_{i,j=1}^n$ is positive definite. This is in general not the case. There is a very nice argument by [unintelligible](2012) where this q should satisfy, this is equivalent to q being either $\pm\frac{1}{d}$ or 0 for d a natural number. So $\chi_{1/d}$ is a character, which complex linearly extends to a trace on the group algebra $\mathbb{C}[S_\infty]$. I'm considering the positive case $\frac{1}{d}$, one can reduce by taking the sign permutation, so it's really enough to consider that one. So we fix $d \in \mathbb{N}$ and work over it. For those with a little more background, in the infinite symmetric group, the so-called "block characters" of the infinite symmetric group, these have a few very important features, [unintelligible].

Okay, so this is our probability space. I have to talk a little bit about exchangeable sequences. Given $(\mathcal{A}, \varphi)$ a $*$-algebraic probability space (a unital $*$-algebra over the complex numbers and a unit preserving positive definite functional), we say that what we need, we say that given some sequence in $\mathcal{A}$, then $\varphi(a_{i_1} \cdots a_{i_k})$, these are the joint moments, and what we do is that, say two sequences (a_n) and (b_n) have the same distribution if whenever we compare the joint moments $\varphi(a_{i_1} \cdots a_{i_k}) = \varphi(b_{i_1} \cdots b_{i_k})$ for all $i_1, \ldots, i_k$. Then exchangeability compares with a permutation. So if we have $\varphi(a_{i_1} \cdots a_{i_k}) = \varphi(a_{\sigma(i_1)} \cdots a_{\sigma(i_k)})$ for all $i_1, \ldots i_k$ and all $\sigma \in S_\infty$.

Let's give an example. If you have $\varphi(a_i a_j a_i a_k)$ with i, j, and k pairwise disjoint, then this is the same as $\varphi(a_1 a_2 a_1 a_3)$. This uses quite a lot the complexity and that this only depends on the type of the underlying partition. So $P(k)$ is now the partitions with are blocks $V_1, \ldots, V_m$ which are nonempty and disjoint, and the union is $\{1, \ldots, k\}$. For example we could have $\{1, 4\}, \{2\}, \{3, 5\}, \{6\}$ is a partition in $P(6)$. Then one sees that by exchangeability, the joint moments depend only on the partition.

Then we can define $\alpha_\pi \coloneqq \varphi(a_{i_1} \cdots a_{i_k})$ for som $\pi \in P(k)$.

So potentially under the presence of exchangeablity, it's sufficient to look at the etalon collection of joint moments α_π for $\pi \in \sqcup_k P(k)$. Maybe what one sees immediately, if one wants to look, some simple observations are, $\varphi(a_i^k) = \varphi(a_1^k)$ so all of the random variables are identically distributed. One also sees the following. For notation later on, put $\varphi_\bullet = \varphi(a_1)$. What one sees from this etalon collection, we'd like to study those that go along with [unintelligible]partitions. So we define $P_2(k) = \{\pi \in P(k) | \text{each block has size } 2\}$, and examples are, well, let $\hat{\alpha}_k$ be the sum $\sum_{\pi \in P_2(k)} \alpha_\pi$. So $\hat{\alpha}_2 = \alpha_{\{1,2\}}$. When k is odd there is nothing. If α is four, we see three of them.

So let me come next to, what I'm marching toward is a central limit theorem in an algebraic setting, and there one knows that only pair partitions should survive in the central limit. So now what we're doing is, working in $*$-algebraic probability spaces. To make the credo rigorous. The symmetry I described is not strong enough. One can't control [unintelligible]in the algebraic point of view. So what I introduce next is the singleton-factorization property. I'll just give an example. Whenever we have φ_π, and π has one or more singletons, that means that you can put in $\varphi_\bullet^n \varphi'_\pi$ where you pull n singletons out of π to make π'. A simple remark,

Proposition 3.1. *Suppose (a_n) is exchangeable. Then when you pass over to $b_n = a_n + \lambda$, this is also exchangeable.*

So

Proposition 3.2. *For*

$$f(z) = 1 + \sum_{k=1}^{\infty} \frac{\hat{\alpha}_k}{k!} z^k$$

and $g(z)$ the same for β, where $\hat{\alpha}_k$ is as before for (a_n) and β for (b_n). Then the relation is

$$g(z) = e^{\lambda(\lambda + 2\alpha_\bullet)z^2/2} f(z).$$

[missed a little] Now the next part is the central limit theorem for an exchangeable sequence with the singleton factorization property.

Theorem 3.1. *If (a_n) is exchangeable with the singleton factorization property, then put $S_n = \frac{1}{\sqrt{n}}(a_1 + \cdots + a_n) - n\alpha_\bullet$. Then $(S_n)_{n=1}^\infty$ convergence in distribution , which means that there exists a probability measure on the reals such that*

$$\varphi(S_n^p) \xrightarrow{n\to\infty} \int_{\mathbb{R}} t^p \mu dt.$$

Just a hint on the proof: you extend and only the pair partitions remain in the limit. Now a remark. Essentially one could say, I could have said $\mu : \mathbb{C}[x] \to \mathbb{C}$ has certain properties, but what one has to say essentially is that $\mu(x^k) = \hat{\beta}_k$ for all $k \in \mathbb{N}$.

What I'm doing next is returning in the last step to the model which we had there, for $\varphi_{1/d}(u_\sigma)$, this is now $\chi_{\frac{1}{d}}(\sigma) = \frac{1}{d}^{||\sigma||}$ and now what one has to do is compute in pair partitions if one wants to see the limit.

First one has to figure out the exchangeable sequence. If one looks at the Coxeter generators $(i, i+1)$, these are not exchangeable, they don't commute if they're next to each other but they do if they're far. So instead take $(1, i)$. So $a_i = (1, i+1)$, these are self-adjoint and it's easy to see that these only have spectrum ± 1. These do not commute, so one has to have many ± 1 signs. If they all commute then it's just a sequence of [unintelligible], but they don't commute.

Now is the bad thing, one has to compute or prove something. So the proposition or the lemma or something, we only have to look at pair partitions, so let $k = 2\ell$ and we look at $\hat{\alpha}_{2\ell}$ which is

$$d^{\ell-1} \sum_{\pi \in P_2(2\ell)} (\frac{1}{d})^{||P_\pi(\sigma_{2\ell})||}.$$

Where $\sigma_{2\ell}$ is $(1, 2, \ldots, 2\ell)$ and $\pi = \{(p_1, q_1), \ldots (p_\ell, q_\ell)\}$ and $P_\pi = (p_1, q_1) \cdots (p_\ell, q_\ell)$. The good thing about collaborations, my collaborator knew what this formula is, this is essentially the exponential moment generating function of Gaussian unitary ensembles, self-adjoint $d \times d$ matrices with independent entries. One has seen these guys. This is the sequence α, the unshifted ones, but for the unshifted ones, we can't carry this out. One knows what distribution belongs to that. For those we can't carry out the central limit law. We had to do the shifting to center them, which produces the Gaussian distribution. We can't work with a_i directly, they are not centered, so we have to center them, and what we have to understand is that, we're interested only in central moments. If one plugs in, choose $\lambda = -\alpha_\bullet$, then you get $e^{-\alpha^2 z^2/2}$, and this is $g(z)e^{z^2/2d^2} = f(z)$. We know, then, that ν (this is the measure of a GUE normalized $d \times d$ matrix) and this is $\mu * N(0, \frac{1}{d^2})$ where μ is the central limit law for an exchangeable sequence of star generators with respect to this given state $\frac{1}{d}$. So we started with this noncommutative exchangeable sequence and carried out a central limit argument and could identify that the central limit law of that sequence is the law of a GUE. That, we were a little bit surprised.

This is where I can stop but as a remark, there are several remarks about generalizations.

- One could do tuples of Gaussian unitary ensembles by taking the first n blocks, then the next and so on.

- The Taylor algebra is [unintelligible]and we'd want to pass to an operator-valued version. One nice thing that we don't know whether it's a nice observation or we can capitalize further is that we started with exchangeable sequences and a character on them, and now what we did was somehow did an $N \to \infty$ thing and came to $d \times d$ GUEs, but we can also do, sending $d \to \infty$ and we have $\mathbb{C}S_\infty$ with the left regular representation, and there you find the semicircle law as your central limit and now you can take the $d \to \infty$ for the GUE. We have somehow in distribution a commuting diagram. Maybe this is only the most simple example where one can do this. We have general results identifying the [unintelligible]algebra. Just to control, to identify this, then one can make this rigorous, [unintelligible], [unintelligible], and of course this is all capital letters, we wanted to see new examples, and this is the most simple example and one already meets an interesting connection between block characters on the infinite symmetric group and Gaussian unitary ensembles.

4. September 5: Jae-Suk Park: Homotopy Theory of Probability Spaces I

My goal was to arrange a kind of marriage between homotopy theory and probability theory in the algebraic world. I'm not a specialist in either so I thought my role could be the *arrangement* of this marriage.

I'd like to give a very simple talk, a plain talk, and start with an example of what I want to achieve here. As a former physicist my favorite distribution is Gaussian, but many people here are non-commutative so theirs might be the semicircular distribution. So let me start with that. Let $A = \mathbb{R}[x]$ and consider a map $\iota : A \to \mathbb{R}$ which is

$$\iota(o) = \int_{-2}^{2} o\sqrt{4-x^2}dx / \int_{-2}^{2} \sqrt{4-x^2}dx$$

and the goal is to compute $\iota(e^{tX}) = Z_t$, the moment generating function. I'm not good at integrals so I won't do any integration. I'll indicate three ways of doing this. Let me pick a Lie algebra $\mathfrak{g} = \mathbb{R}e$, and a representation of this Lie algebra $\rho : \mathfrak{g} \to L\operatorname{Diff}_{\mathbb{R}}(A)$, this is part of the endomorphism Lie algebra, $L\operatorname{End}_{\mathbb{R}}(A, A)$, and this is one dimensional, so the representation is completely specified by its action on the generator e, which is $(4-x^2)\frac{d}{dx} - 3x\cdot$, this is a very simple linear representation. Then I claim (easy to check) that the image of ρ, that $\iota \circ \rho = 0$. This is really taking a total derivative out of this guy. This, I'll say, is because

$$\iota \circ \rho(o) \propto \int_{-2}^{2} d(o \cdot (4-x^2)^{3/2}) = 0.$$

A simple computation shows that $\rho(x^{n-1}) = 4(n-1)x^{n-2} - (n+2)x^n$. This means that $\iota(x^n) = \frac{4(n-1)}{(n+2)}\iota(X^{n-2})$, and we know that $\iota(1) = 1$ and the conclusion is that

$$\iota(x^n) = \begin{cases} 0 & n \text{ odd} \\ 2^{n+1}\frac{(n-1)!!}{(n+2)n!} & n \text{ even.} \end{cases}$$

So we see the Catalan numbers already, and we can write down $Z_t = \sum_{n\geq 0} \frac{t^{2n}}{(2n)!}C_n$ and we've solved the problem without doing any integration.

Another method, take a family $\rho_t = e^{-tx}\rho e^{tx}$, and let ι_t, instead of measuring with $\sqrt{4-x^2}dx$, measure with $e^{tx}\sqrt{4-x^2}dx$, and we can check that $\iota_t \circ \rho_t = 0$ and $\iota_t(1) = Z_t$, and you can work out Z_t using this formula, $\rho_t(1) = -tx^2 - 3x + 4t$, which gives us a relation which can easily be translated into the differential equation $\left(t\frac{d^2}{dt^2} + 3\frac{d}{dt} - 4t\right) Z_t = 0$ and this is the same as a recursion relation satisfied by the Catalan numbers: $(n+2)C_{n+1} = 2(2n+1)C_n$.

The third method, a little more complicated, gives a way to give the cumulant generating function directly, using L_∞ homotopy theory, a more advanced language.

So what I did here, let me give a variant of the first method, remember the original representation matrix acting on e, is $(4-x^2)\frac{d}{dx} - 3x$, and we originally had $A = \mathbb{R}[x]$. The variant would use some cohomology or homology theory. Let's change, extend this algebra A to $\mathbb{R}[x, \eta]$, I've added a variable η, and let me introduce a grading (ghost number) where x has degree 0 and η has degree -1. Then $\eta^2 = 0$ and $x\eta = \eta x$. So this breaks into $A^{-1} \oplus A^0$ and this is $\mathbb{R}[x]\eta \oplus \mathbb{R}[x]$. The commutative product is extended into a supercommutative product and from ρ let me create some differential which is written as $\rho(e)\frac{\partial}{\partial\eta} = -3x\frac{\partial}{\partial\eta} + (4-x^2)\frac{\partial^2}{\partial x\partial\eta}$. If you apply to A^{-1} I'll land in A^0. This operator then satisfies $K^2 = 0$. This is a very simple cochain complex.

Now we have A and K, and note that I just made up K out of the representation. Consider the image of K in A^0. Any element Σ in A^{-1} can be written as $O(x)\eta$. If I apply K to the Σ, this is the same as applying ρ to $O(x)$, so the image of the operator ρ is in the image of K. Then we can construct a map c from A to our ground field $\mathbb{R}$, of ghost number 0. It has two pieces, $c = \iota$ on A^0 and the zero map on degree -1. Then we know that $c(1_A) = 1$ and $c \circ K = 0$ (this latter is a translation of the fact that $\iota \circ \rho = 0$). So this is just a unit-preserving cochain map.

So I'll call this a homotopical realization of the original probability space. Then what is the relation from the ρ action on x^{n-1}? it now says that $4(n-1)x^{n-2} \sim (n+2)x^n$. This is a homotopical method to achieve the calculation, but it's more than that. Let's turn to cumulants. What are the cumulants? They are maps K_n from $S^n\mathcal{A} \to \mathbb{R}$. These are defined by the formula, using the sum over all partitons:

$$\iota(x_1 \cdots x_n) = \sum_{\pi \in P(n)} K(B_1) \cdots K(B_{|\pi}).$$

For me the striking property of probability theory is the following. In algebraic topology or geometry, we study algebras with structure-preserving morphisms. We study Lie algebras with Lie morphisms or associative algebras with algebra maps. But in this case ι does not preserve structure. Everyone knows that $K_2(x_{1,2}) = \iota(x_1 \cdot x_2) - \iota(x_1)\iota(x_2)$. This way of writing classical cumulants gives a particular way of telling how the morphism deviates from preserving structure. This is the heart of probability, telling you how the variables are correlated.

We can of course generalize this, applying the same set of arguments for the graded one too. The grading gives some sign factor, we can derive some formula for the cumulants and c, now an expectation, let me derive $\underline{\phi}^c$, so this is $\phi_1^c, \phi_2^c, \ldots$ all maps $S^nA \to \mathbb{R}$, these are the "descendents" and this gives cumulants.

There's one more thing that probability theorists usually didn't consider. This differential has a striking property, which is that it does not preserve the algebra structure. A is an associative and graded commutative unital algebra. But the differential K, if I apply $K(o_1o_2) - K(o_1)(o_2) - (-1)^{|o_1|}o_1K(o_2)$, this is not zero, and let me define this as $\ell_2^K(o_1, o_2)$. You can easily compute that ℓ_2 is a derivation of this guy, that $K\ell_2^K(o_1, o_2) = \ell_2^K(Ko_1, o_2) \pm \ell_2^K(o_1, Ko_2)$. If you compute further, then you can measure whether ℓ_2 is a derivation of the product, then if I do $\ell_2^K(o_1, o_2o_3) - \ell_2^K(o_1, o_2)o_3 - o_2\ell_2^k(o_1, o_3)$ this turns out to be zero. This happens to be true because K is a second order differential operator, and (A, K, ℓ_2^K), this is a differential graded Lie algebra, in particular something shifted because the degrees are wrong. This should have a degree 1 differential and a degree 1 bracket.

We can also do the same thing with our target $\mathbb{R}$, and there we have another shifted differential graded Lie algebra with no differential or bracket.

In general you could have third or higher order operators. Then you'd end up with an sL_∞ algebra, a shifted L_∞ algebra.

Okay, this might not be obvious to you, but actually you see that I'm using exactly the same methodology as how you define cumulants. I'm going to use classical partitions, we have this expectation.

So out of an ordinary probability space I created something with a differential and some version of cumulants, and also because I have K, I introduced some version of cumulants with respect to them here. Then we see that $\underline{\phi}^c$ is an sL_∞ morphism.

In topological spaces we have singular chains or cochains, we have some algebraic structure there, sometimes it wants to be commutative but it's not strictly commutative. [some questions]

Let me emphasize why K can't be a derivation of the product in doing correlations. Because $c \circ K = 0$, if I have o which is $K\Lambda$, and this says, oh, $c(o) = 0$. Then consider $c(o^n)$, this cannot be zero in general. If K is a derivation of the product, then o^n is K of something so the nth moment is zero. The failure of being a derivation of the product tells you something about the correlation between random variables here, using the same kind of argument.

This is an L_∞ algebra, and an L_∞ morphism, and in this case this comes with a notion of homotopy. There's some other L_∞ morphism which looks completely different but they're the same in the L_∞ world. The generating function turns out only to depend on the homotopy class of the L_∞ morphism. All these sort of things are invariant of the homotopy type. Algebraic topologists want some kinds of homotopy functor. The law of random variables has to be regarded as, [some questions].

What is the use of the notion of homotopy. I'll explain my third method of computation. You can always find another L_∞ morphism which is just the actual cumulant generating function. We can, we have some cumulants corresponding to this generating function, there is some very simple L_∞ morphism homotopic to this, where the map is given by a number times the unit. I won't tell you the details of how to get this guy here.

This is my example, and I've already spent the entire lecture explaining the entire example, but let me give just a little more remarks about physics that Owen was talking about. Owen had a differential that looks like something called classical and something that was called Δ, the classical part was a first order operator and Δ was second order. He had a measure that looked like $dxe^{-S(x)}$. The semicircular is $dx\sqrt{1-x^2}$. It doesn't really matter. He showed how to compute the Gaussian integral, this was the same kind of computation I did for this one, right?

So, now, another way of measuring the failure of structure-preserving ones, we could use interval partitions, instead of classical partitions, and throw away the assumption you're dealing with a commutative thing, and you end in the sA_∞ world, homotopy associative. You change the way of measuring here. Classical cumulants lead to L_∞ and Boolean lead to A_∞ world.

These are two famous examples coming from algebraic topology.

Up to this part was kind of easy to do.

I'm very familiar with A_∞ and L_∞, but what about free independence? There must be some kind of homotopy theory. It turns out that there's no corresponding example, Gabriel will talk about this part. So for me, homotopy probability theory can be thought of in two ways. You can either think you're using homotopy theory to do ordinary probability theory. In that sense every probability theory is a homotopy probability theory. There you have a unital algebra and a unit-preserving map to the ground field. The target is one dimensional, so the kernel is a huge thing, always codimension 1, you can do $x - \iota(x)1_A$, this is in the kernel. This argument, familiar to you, says this is codimension 1. So two elements differing by a kernel, give the same expectation.

So you use this symmetry to drive everything to the known situation. We can kind of use that thing to figure out some kind of symmetry. The usual symmetry should be a representation of a group to a representation of A, so that $\iota \circ \rho = \iota$. But you can also consider an infinitesimal symmetry $\rho \circ \iota = 0$. This is the version I'm using here. For a discrete symmetry the infinitesimal version can't see it.

Then the original map $A \to \mathbf{k}$ should factor through the coinvariants $A/\mathfrak{g} \cdot A$. Then there's a unique map $A/\mathfrak{g} \cdot A \xrightarrow{\iota_{\mathfrak{g}}} \mathbf{k}$, but this causes a huge problem because when you push to the coinvariants, it will almost never have an algebra structure. There's not a natural algebra structure unless the kernel of ι belongs to an ideal of the algebra. When it's an ideal, ι must be an algebra homomorphism. We are not interested in that case. The combined idea here is to create some larger space, we upgrade, and this does not preserve the structure of this guy, and then you measure the failure together and using the homotopy theory you can induce a probability space which knows all about the original information, but the induced thing has not only a binary product but also a triple product and so on, but can be nicely organized using some gemoetric data like flat connections and describe the law of random variables from here. That's the basic idea of how, of the computational method. I think tomorrow I'll give a more abstract setting without examples, and how we can solve this problem in this way.

Thank you for listening.

5. Patrizio Frosini: An approach to topological data analysis via persistent topology and invariant operators.

[I do not take notes at slide talks.]

6. Frédéric Patras: Probability and shuffle products I: shuffle bialgebras

Thank you. This is the first talk of the series. Kurush Ebrahami-Fard is going to give two lectures coordinated with this one and explain how shuffle bialgebras can be used in the context of free probability. This is an introduction to the general setting. This, I'll try to argue, it's natural to use shuffles in thinking about non-commutative probability. I won't speak about free probability. I'll discuss shuffles in the commutative setting first and discuss the relations with classical probability. In particular I'll state how some results can be stated or translated into this context of shuffle algebras. In the next talks you'll see that there's a way to make this non-commutative in a natural way. A shuffle is an operation you perform to mix two decks of cards, so that the two decks of cards are kept in the same relative order. For instance if you mix 123 with 45 you get all the configurations where 1, 2, and 3 stay in the same order and likewise with 4 and 5 so for instance 14235. So this is called a $(3, 2)$-shuffle. You see immediately that there's a bijection between shuffles and partitions. If you take the example, and you look at the positions of the cards 1 through 3, then this corresponds to the ordered set partition of 5 as $\{1, 3, 4\} \sqcup \{2, 5\}$. In a certain sense this is why you can encode computations in terms of shuffles. They are a natural framework in which to encode set partitions. Dually, there is unshuffling, which is an operation that you perform when you want to split a deck of cards in two decks, where you split the first card either to the right or to the left and similiarly for the next card, in order. So you could separate 12345 into 134 and 25, for instance, creating two decks.

What you get with the unshuffling operation is, unshufflings are encoded by all ordered partitions if you have $[n] = S \sqcup T$. You can decide to divide the first deck into two new decks or you could do the same thing by splitting the second deck of cards. When you do so, moving from the initial deck and doing either of these, you get all partitions of $[n]$ into three subsets. If you iterate unshuffling, you'll get all partitions of your set. This is the basic idea, which goes back, essentially, to Poincaré in something like the 1910s or around these dates, when he used this concept to study probability. I should mention in view of the talks we already had in this conference that shuffles play an important role in algebraic topology and also in homological algebra. In a certain sense this makes a connection with what we've heard about in other areas. They encode Cartesian products of spaces or topological objects like simplices, and in homological algebra they show up in Hochschild and cyclic homology of commutative algebras. That's an interesting connection to what we've seen. The first axiomatization I'll discuss was done by Eilenberg and MacLane in the 1950s. They really want to understand how to construct a product in homology. This won't really appear in what I'm going to say later on.

If we want to go further, the next step is to construct algebraic structures controlling what's going on. So construct the tensor algebra and equip it with a Hopf algebra structure. I don't want to enter into details but a Hopf algebra H is an algebra and a coalgebra, where you can split the space $H \to H \otimes H$, and you ask this map to be coassociative in the sense that the diagram

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"] & H \otimes H \arrow[d, "\Delta\otimes\mathrm{id}"] \\ H \otimes H \arrow[r, "\mathrm{id}\otimes\Delta"'] & H \otimes H \otimes H \end{tikzcd}$$

commutes. There's a compatibility when you request that the product $H \otimes H \to H$ is a morphism of coalgebras and the coproduct is a morphism of algebras. There is also a technical condition, the existence of an antipode, but this will be automatic and not very important in my case.

It's important to start from this systematic picture in the classical case. The tensor algebra of X is the linear span of words, sequences of letters in X, and I'll write X^* for the set of words including the empty set.

This space is equipped with a Hopf algebra structure, by defining the product and the coproduct. There are two ways, dual to each other, to equip this with a product and coproduct. One is to use the shuffle product, by the process I've indicated, but it's useful to do this recursively. You can define

$$y_1 \cdots y_n \sqcup\!\sqcup\, z_1 \cdots z_m = y_1(y_2 \cdots y_n \sqcup\!\sqcup\, z_1 \cdots z_m) + z_1(y_1 \cdots y_n) \sqcup\!\sqcup\, z_2 \cdots z_m).$$

and I'll break this apart into the sum of two operations which I'll write $y_1 \cdots y_n \prec z_1 \cdots z_m + y_1 \cdots y_n \succ z_1 \cdots z_m$. The coproduct is deconcatenation, cutting into two words in all possible ways.

That's the first Hopf algebr astructure and you can dualize. The product is concatenation, gluing words together, and the coproduct is unshuffling, essentially the same definition dealing with words. The coproduct δ splits into a left and right piece $\delta^{\prec}$ and $\delta^{\succ}$.

We want to relate this to probability. When you have two linear forms, something you can do, given f and g in $T(X)^*$, you can make a new linear form using the convolution product, by first using the coproduct and then using the product. So you go $T(X) \xrightarrow{\delta} TX \otimes TX \xrightarrow{f \cdot g} \mathbb{R}$ and $*$ is commutative and associative.

Let's look at a concrete example. Look at $X = \{x_1, \dots, x_n\}$ and $\mathcal{X}_1, \dots, \mathcal{X}_n$ random variables with all moments, then I can write $\mu \in T(X)^*$ by sending $x_{i_1} \otimes \cdots \otimes x_{i_k}$ to $\mathbb{E}(\mathcal{X}_{i_1} \cdots \mathcal{X}_{i_k})$. Now you are in an algebra so you can solve equations like $\mu \exp^*(\kappa)$ and it's easy to see that there's a solution if you require $\kappa(1) = 0$. What happens if you do so? You want to write $\mu = \sum_{n \geq 1} \frac{\kappa^{\otimes n}}{n!} \delta^{(n)}$ where $\delta^{(n)}$ goes from $T(X)$ to $T(X)^{\otimes n}$. Then what happens is when you apply δ to a word, for example the trivial word $x_1, \dots, x_n$, you'll get the sum over all ordered set partitions $I_1 \sqcup \cdots \sqcup I_k = [n]$ as follows:

$$\sum x_{I_1} \otimes = cdots \otimes x_{I_k}.$$

So by doing this I get all ordered set partitions of n. I'm averaging over the corresponding factorial, and I get the sum over all non-ordered partitions

$$\mu(x_1, \dots, x_n) = \sum_k \sum_{P_k(n)} \kappa_{p_1} \cdots \kappa_{p_k}$$

and when I solve for κ I get the classical cumulants. But even the case where you take just one variable $X = \{x\}$ it's interesting.

Then you have $\mathcal{X}_1$ and $\mathcal{X}_2$, say, as an exercise, if $\mathcal{X}_1$ and $\mathcal{X}_2$ are independent then $\mu_1 * \mu_2$, the moment map for $\mathcal{X}_1 + \mathcal{X}_2$, is $\exp^*(\kappa)$ which because it's commutative is the exponential of $\kappa_1 + \kappa_2$, the sum of the corresponding cumulants. This is one way to state that cumulants linearize.

The next step is to understand shuffles formally. If you remember the recursive definition of shuffle products, we can summarize the construction that way. The shuffle product is commutative and associative and decomoposes as the sum of a left and a right product, and $a \prec b = b \succ a$.

This operation tells me that if I know one of the two products I know everything. In the end the key relation is the one that is satisfied by one of the two products, which is

$$(x \prec y) \prec z = x \prec (y \sqcup\!\sqcup z) = x \prec (y \prec z + z \prec y)$$

which gives me the abstract definition of a shuffle algebra. This is a vector space V equipped with a product map $\prec$ satisfying the relation given above. It's useful to go back to Schützenberg. He says that if you start with a set X and look at the algebra freely generated by X using a product generated by X and $\prec$ satisfying this, then this is the free algebra.

The next point is to move from algebra to bialgebra in this context. I gave the definition of a bialgebra actually, and the Hopf algebras I want are just bialgebras with antipodes.

So what are bialgebras in relation to shuffles. When you have a notion of algebra, you can associate to it a notion of bialgebras. Now what happens is if A and B are algebras, then I can construct a shuffle algebra on the tensor product and the definition is simple, you have to define $(a \otimes b) \prec (a' \otimes b')$ and you define this to be $(a \prec a') \otimes b \sqcup\!\sqcup b'$. Categories of algebras that satisfy this property are called algebras over Hopf operads, whatever it means, so you can define a consistent notion of algebra in this way. So you can define a notion of a shuffle bialgebra, and namely this is a shuffle algebra equipped with a coassociative coproduct such that Δ is a morphism of shuffle algebras. This is exactly the definition we have for classical algebra and bialgebra. There's a way to define, a framework to deal with coproducts in this world. In particular, if you look at TX with the shuffle product and concatenation coproduct you get a shuffle bialgebra. In a way we capture all the shuffle structure here.

I'll change all these things moving from commutative to noncommutative. So to make this theory non-commutative, what I've done with set partitions, suppose I start not from partitions but from non-crossing partitions. I want to construct them recursively. What I did previously was to extract a component and then put to the right hand side the remaining part. [picture]. Then I'd like to construct this by iterating. But I have to take into account that the partition needs to be noncrossing. So I have to remember the process of extraction I've used. So I should keep track of some information. This will be the key of the definition I make for the Hopf algebra involivng non-crossing partitions. There is something that is going to make impossible the use of the same tools when I move from the classical to the free world.

This tells us something about moving to non-commutative. The other argument is more formal. What happens if you want to make shuffle algebras non-commutative? The simplest way to make an algebra non-commutative is to move from a commutative algebra A to its algebra of matrices. We can try to do this with shuffle algebras to see what happens. Start with a shuffle algebra B and see what happens when I look at matrices over it. Take $M_2(B)$, that's enough. I take a product

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

and this is

$$\begin{pmatrix} a_{11} \prec b_{11} + a_{12} \prec b_{21} & \cdots \\ \cdots & \cdots \end{pmatrix}$$

and this gives the matrices the structure of an associative algebra, but then $M \succ N$ is not $N \prec M$. I have

$$b_{11} \prec a_{11} + b_{12} \prec a_{21} \neq a_{11} \succ b_{11} + a_{12} \succ b_{21}.$$

For the rest everything is kept and so I end up with a definition, again, a classical process to study algebraic theories, these are particularly meaningful for shuffle algebras. If I collect everything I say together, what I get is a notion of a non-commutative shuffle algebra, also called a dendriform algebra in the literature. For various reasons I don't like this name, which is not informative or historical. This is a vector space equipped with two operations $\prec$ and $\succ$, we don't have the identity so we have to take both of them in the definition, that add up to an associative product $\sqcup$, and satsifying

$$(x \prec y) \prec z = x \prec (y \sqcup z) = x \prec (y \prec z + y \succ z) \tag{1}$$

$$x \succ (y \succ z) = (x \sqcup y) \succ z \tag{2}$$

$$(x \succ y) \prec z = x \succ (y \prec z) \tag{3}$$

and this is the definition we used to move to the non-commutative setting. Now it will be essential that we have three products, and the study of each of these products will give one of the probability theories. This will be important, to have all three of these products together. These will give access to different pieces of information.

Now one last comment and then I'll stop here. We'll need a suitable notion of bialgebras. What is a bialgebra in this context? The answer is essentially the same, the axioms when you write them down, are a little different, but it's the same theoretically as in the classical or commutative case. If you take two non-commutative shuffle algebras, then the tensor product is again a non-commutative shuffle algebra, with $(a \otimes b) \prec (a' \otimes b') = (a \prec a') \otimes (b \sqcup\!\sqcup b')$ and similarly $(a \otimes b) \succ (a' \otimes b') = (a \succ a') \otimes (b \sqcup\!\sqcup b')$. Then a non commutative shuffle bialgebra is a shuffle algebra equipped with a coassociative coproduct and such that Δ is a morphism of non-commutative shuffle algebras. What's going to happen is that the algebraic structures we're going to capture will be associated to this kind of object (actually the dual).

7. Franz Lehner: Spreadability, Cumulants and Hausdorff Series

Let me thank the organizers for the invitation. There's nothing new in this talk except part of the audience, which hopefully justifies things. For me, this is joint work with Hasebe, for me, cumulants are functionals κ_n, so these are functionals on random variables, with some axioms,

(1) they are additive, $\kappa_n(X+Y) = \kappa_n(X)+\kappa_n(Y)$ if X and Y are independent
(2) they are homogeneous, $\kappa_n(\lambda X) = \lambda^n \kappa_n(X)$, and
(3) they are polynomial, $m_n(X) = \kappa_n(X) + Q_n(\kappa_1, \dots, \kappa_{n-1})$

So missing from this is monotone cumulants, because if X and Y are independent, then Y and X need not be. One replaces the first axiom with the axiom that this holds for things with the same distribution: $\kappa_n(X^(1) + \cdots + X^{(n)}) = N\kappa_n(X)$.

So existence follows from general nonsense, we'll hear about this in Roland Friedrich's talk, we've seen a logarithm and exponential in previous talks. In the classical case, you have $\sum \kappa_n \frac{z^n}{n!} = \log \sum m_n \frac{z^n}{n!}$, but there is another idea from discrete Fourier transform, found by [unintelligible]and [unintelligible]around 1975. Let $\omega = e^{2\pi i/n}$ and $X^{(i)}$ independent copies of X. Then $\kappa_n(X) = \frac{1}{n}\mathbb{E}(S_n^n)$ where $S_n = \omega X^{(1)} + \omega^2 X^{(2)} + \cdots + \omega^n X^{(n)}$. So $\mathbb{E}(S_n^n)$ has the sum of independent variables so by the first axiom, you have $\kappa_r(S_n) = \kappa_1(\omega X^{(1)} + \cdots + \omega^n X^{(n)})$ and so this is $\kappa_r(\omega X^{(1)}) + \cdots + \kappa_r(\omega^n X^{(n)})$ and then this is eventually $(\omega^r + \omega^{2r} + \cdots + \omega^{nr})\kappa_r(X)$ whire this sum is 0 if $n \nmid r$ and is n if $n|r$. So then you have $m_n(S_n) = \kappa_n(S_n) = n\kappa_n(X)$. Similarly, $\kappa_n(X_1, \dots, X_n) = \frac{1}{n}\mathbb{E}$, where $S_i = \omega X_i^{(1)} + \cdots + \omega^n X_i^{(n)}$.

This is the background for the following definition. The basic and trivial observation is the following. If you have a random vector (X, Y) of two variables and you take $(X^{(1)}, Y^{(1)})$ and $(X^{(2)}, Y^{(2)})$ of this random variables, then X and Y are independent if and only if the joint distribution of X and Y is the same as the joint distribution of the vector $(X^{(1)}, Y^{(2)})$. What do you actually need? This is what I call

Definition 7.1. Let (A, φ) be a noncommutative probability space. An *exchangeability system* is a triple $(U, \tilde{\varphi}, J)$ where $(U, \tilde{\varphi})$ is another noncommutative probability space and J is a collection ι_k of embeddings $A \to A^{(k)} \subset U$ preserving the expectation. The notation is that $X \mapsto X^{(k)}$. We demand that $\tilde{\varphi}(X^{(k)}) = \varphi(X)$. Moreover we require that ι_k are exchangeable, that $\tilde{\varphi}(X_1^{(i)} X_2^{(i_2)} \cdots X_n^{(i_n)}) = \tilde{\varphi}(X_1^{(h(i_1))} \cdots X_n^{(h(i_n))})$.

To give an example,

$$\tilde{\varphi}(X^{(1)}Y^{(2)}Z^{(1)}W^{(3)}U^{(2)}) = \tilde{\varphi}(X^{(3)}Y^{(1)}Z^{(3)}W^{(5)}U^{(1)})$$

so this now is called $\varphi_\pi(X, Y, Z, W, U)$ where π is the partition encoding this pattern of indices. So in this example $U = A^{\otimes\infty}$ and $\tilde{\varphi}$ is $\varphi^{\otimes\infty}$. This is classical. For free, you take $U = *^\infty(A, \varphi)$, the reduced free product.

Then you can define cumulants just the same way as you did here. Then this depends on which exchangeability you take. So instead of $\mathbb{E}$ you put $\tilde{\varphi}$ where $S_i = \omega X_i^{(1)} + \cdots + \omega^n X_i^{(n)}$. So what is, for example, the second cumulant? The root of unity of order two is -1, so you get

$$\begin{aligned}
&\frac{1}{2}\tilde{\varphi}((-X^{(1)} + X^{(2)})(-Y^{(1)} + Y^{(2)}))\\
&\qquad = \frac{1}{2}\tilde{\varphi}(X^{(1)}Y^{(1)} - X^{(1)}Y^{(2)} - X^{(2)}Y^{(1)} + X^{(2)}Y^{(2)})\\
&\qquad = \frac{1}{2}(\varphi(XY) - \varphi_{11}(X, Y) - \varphi_{11}(X, Y) + \varphi(XY))\\
&\qquad = \varphi(XY) - \varphi_{11}(X, Y).
\end{aligned}$$

So in general, you get

Proposition 7.1.

$$\kappa_n(X_1, \ldots, X_n) = \sum_{\pi\in P(n)} \varphi_\pi(X_1, \ldots, X_n)\mu(\pi, \hat{1}_n)$$

where $\mu(\pi, \hat{1}_n) = (-1)^{|\pi|-1}(|\pi| - 1)!$ *And so*

$$\kappa_\pi(X_1, \ldots, X_n) = \sum_{\sigma\leq\pi} \varphi_\sigma(X_1, \ldots, X_n)\mu(\sigma, \pi)$$

where in the free case this vanishes unless the partition is non-crossing.

Okay let's move on to independence.

So let's do an example. We'll say that two variables are independent if the distribution of the pair is the same as taking X from the first copy and Y from the second copy in some exchangeability system. So we have

$$\varphi_{\{1,2\}}(X, Y) = \varphi(XY) = \varphi(X^{(1)}Y^{(2)}) = \varphi_{11}(X, Y)$$

or

$$\varphi_{\{1,2,3,4\}}(XYXY) = \varphi(X^{(1)}Y^{(2)}X^{(1)}Y^{(2)}) = \varphi\{\{1, 3\}, \{2, 4\}\}(X, Y, X, Y)$$

so you're separating out the variables.

Definition 7.2. Subalgebras A_1 and A_2 of A are called $\mathcal{E}$ independent if for all $X_1, \ldots, X_n$ in $A_1 \cup A_2$ and $I_1 \cup I_2 = \{1, \ldots, n\}$ so that $(X_i)_{i\in I_1}$ are in the first algebra and $(X_i)_{i\in I_2}$ are in the second algebra, and that $\varphi_\pi(X_1, \ldots, X_n) = \varphi(\pi \wedge \rho)(X_1, \ldots, x_n)$ where $\pi \wedge \rho$ is what you get when you intersect the blocks of π with the blocks of ρ, where ρ is the partition into I_1 and I_2.

So for $\varphi_{\{1,3,4\},\{2\}}$ this is

$$\begin{aligned}
\varphi_{\{1,3,4\},\{2\}}(X, Y, X, Y) &= \tilde{\varphi}(X^{(1)}Y^{(2)}X^{(1)}Y^{(1)})\\
&= \tilde{\varphi}(X^{(1)}Y^{(2)}X^{(1)}Y^{(3)})\\
&= \varphi_{(\{1,3\},\{2\},\{4\})}(X, Y, X, Y)
\end{aligned}$$

How does this translate to cumulants? As we expect. Namely:

Proposition 7.2. *Subalgebras A_1 and A_2 of A are $\mathcal{E}$ independent if and only if mixed cumulants vanish, i.e., if $X_1, \dots, X_n$ in $A_1 \cup A_2$ with $\rho = I_1 \cup I_2$ as before, and $\pi \not\leq \rho$ then $K_\pi(X_1, \dots X_n)$ vanishes.*

Here $\pi \leq \rho$ means that any block of π lies either in I_1 or I_2. So if there's one block with elements from both subalgebra, then it vanishes.

Proof. One direction is easy. If mixed cumulants vanish, then we have this φ_π equation, this is

$$\varphi_\pi(X_1 \dots, X_n) = \sum_{\sigma \leq \pi} K_\sigma(X_1, \dots, X_n) = \sum_{\sigma \leq \pi, \rho} K_\sigma = \sum_{\sigma \leq \pi \wedge \rho} K_\sigma = \varphi_{\pi \wedge \rho}$$

and this is the definition of independence. For the ether direction we use Weisner's lemma. Let $(P, \leq)$ be a lattice containing a, b, and c, then

$$\sum_{x \in P, x \wedge a = c} \mu(x, b) = \begin{cases} \mu(c, b) & a \geq b \\ 0 & \text{otherwise.} \end{cases}$$

So this turns out to be exactly the thing we need. So we compute these cumulants. $K_\pi(X_1, \dots, X_n) = \sum_\sigma \varphi_\sigma(X_1, \dots, X_n)\mu(\sigma, \pi)$ and we acn replace φ_σ with $\varphi_{\sigma\wedge\rho}$ and this is

$$\sum_\tau \sum_\sigma \varphi_\tau \mu(\sigma, \pi)$$

and we can move φ_τ out and then we use Weisner with $x = \sigma$, $a = \rho$, $c\tau$ and $b = \pi$ and get that this is

$$\sum_\tau \varphi_\tau = \begin{cases} \mu(\tau, \pi) & \rho \geq \pi \\ 0 & \text{otherwise,} \end{cases}$$

but then this is zero because $\pi \not\leq \rho$. ∎

I gave this proof because I need to generalize it, what about monotone independence? Well, monotone independence is not exchangeable. So $\tilde{\varphi}(X^{(1)}Y^{(2)}Z^{(1)}) = \varphi(Y)\varphi(XZ)$. But if I permute 1 and 2, I get $\tilde{\varphi}(X^{(2)}Y^{(1)}Z^{(2)})$, then this gives $\varphi(X)\varphi(Y)\varphi(Z)$ which is not the same in general so we do not have exchangeability. But we have *spreadability* when we keep the relative order. We do have $\tilde{\varphi}(X^{(3)}Y^{(7)}X^{(3)})$. This I call a *spreadibility system.*

So S is $(U, \tilde{\varphi}, J)$ so that $\tilde{\varphi}(X_1^{(i_1)}, \dots X_n^{(i_n)}) = \tilde{\varphi}(X_1^{(h(i_1))} \cdots X_n^{(h(i_n))})$ where h is now a monotone function. So this is $\varphi_\pi(X_1, \dots, X_n)$ where now we have to keep track of the *order* of the partition. So for $\tilde{\varphi}(X^{(1)}Y^{(2)}Z^{(1)}U^{(3)}V^{(2)})$ we get the partition $\{1,3\}, \{2,5\}, \{4\}$ where now the order is important. So for $\tilde{\varphi}(S_1^{(N)}, \dots, S_n^{(N)})$, and you calculate this thing, with $S_K^{(N)}$ as $X_k^{(1)} + \dots + X_k^{(N)}$ and this is polynomial in N with no constant term and we say this is $K_n(X_1 ldots, X_n)N + O(N^2)$.

So for instance, for $K_2(X, y)$, we compute

$$\begin{aligned} &\varphi((X^{(1)} + \dots + X^{(N)})(Y^{(1)} + \dots + Y^{(N)})) \\ &\qquad = \sum \tilde{\varphi}(X^{(i)})Y^{(i)}) + \sum_{i<j} \tilde{\varphi}(X^{(i)}Y^{(j)}) + \sum_{i>j} \tilde{\varphi}(X^{(i)}Y^{(j)}) \end{aligned}$$

and this is $N\varphi(XY)$ so we get

$$\frac{N(N-1)}{2}(\varphi_{1,2}(X,Y)+\varphi_{2,1}(X,Y))$$
$$= N(\underbrace{\varphi(XY)-\frac{1}{2}(\varphi_{1,2}(X,Y)+\varphi_{2,1}(X,Y))}_{K_2(X,Y)})+\frac{N^2}{2}\cdots$$

and in general

$$K_n(X_1,\ldots,X_n)=\sum_{\pi}\in OP(n)\varphi_\pi(X_1,\ldots,X_n)\tilde{\mu}(\pi,\hat{1}_n)$$

where

$$\tilde{\mu}(\pi,\hat{1}_n)=\frac{(-1)^{|\pi|-1}}{|\pi|}=\frac{\mu(\bar{\pi},1n)}{\pi!}.$$

This is not a lattice. How do we define independence? You have $\pi \curlywedge \rho=(P_1\cap R_1,P_1\cap R_2,\ldots)$ deleting empty blocks, so for example, $(3/12/4)\curlywedge(13/24)$ you get

$$(3/\emptyset/1/2/\emptyset/4)\to(3/1/2/4)$$

and then we define independence as before using this operation. So now mixed cumulants do *not* vanish because that would imply commutativity. We need an analogue of Weisner's lemma and in the end what comes out is, for $\rho=(I_1,I_2)$ as before

$$K_n(X_1,\ldots,X_n)=\sum_\tau \varphi_\tau(X_1,\ldots,X_n)w(\tau,\rho)$$

where the weight

$$w(\tau,\rho)=\sum_{\sigma\curlywedge\rho=\tau}\tilde{\mu}(\sigma,\hat{1}_n)=\begin{cases}\frac{(-1)^{(|\tau|-\operatorname{asc}\lambda(\pi,\rho)-1)}}{|\tau|\binom{|\tau|-1}{\operatorname{asc}\lambda(\pi,\rho)}} & \bar{\tau}\le\bar{\rho}\\ 0 & \text{otherwise.}\end{cases}$$

What is λ? So for $\lambda(\tau,\rho)$, I look at each block of τ and ask which block of ρ it's in. Now I count how many rises I have in the word I make out of this, and so this is $\operatorname{asc}\lambda(\tau,\rho)$

Now if I express the moments in terms of cumulants, I get

$$\sum_\tau K_\tau(X_1,\ldots,X_n)g_{\lambda(\tau,\rho)}$$

where g_w is the *Goldberg coefficient* from the Baker–Campbell–Hausdorff series. If a and b commute then $e^ae^b=e^{a+b}$. If a and b do not commute, then this has a replacement $e^ae^b=e^c$ where c is the Hausdorff series $c=\sum_{w\in\{a,b\}^*}g_w w$ and these g_w are what appear here.

There's another way that the Goldberg coefficients arise. Namely if you look at the following, I'll take two more minutes, the following spreadability system, you can replace everything with operator-valued maps, so you take the following, $\mathcal{A}=\mathbb{C}\langle X\rangle$, the algebra of non-commutative polynomials, and φ is the identity map $A\to A$. the big algebra U is again $A^{\otimes\infty}$. The embedding is again $\iota_k(X)$ is the kth tensor, and $\tilde{\varphi}$ is the concatenation. You see immediately that this preserve the identity, since $\tilde{\varphi}(\iota_k(A))=$ [*unintelligible*]. And it's spreadable. But what is the cumulant in this case? Here $e^ae^b=\sum_{p_1,p_2}\frac{1}{p_1!p_2!}K_{p_1+p_2}(\underbrace{\qquad}_{a\ p_1\text{ times; }b\ p_2\text{times}})$. I'll stop here.

8. September 6: Patrizio Frosini: Some advances in the application of group-invariant persistent homology to topological data analysis.

[I do not take notes at slide talks.]

9. Jae-Suk Park: Homotopy Theory of Probability Spaces II

Okay, I'll begin my second talk, I'll draw some diagram. Say, I'll define a certain category called *probability data* $PDat_C(\mathbf{k})$, where $\mathbf{k}$ is a field of characteristic zero, I'll do this category, which is very simple, and design a functor to the category of shifted L_∞ algebras over $\mathbf{k}$, and do some kind of commutative probability, and the functor will be the (commutative) descendant functor $\mathcal{D}_C$, which will organize classical cumulants and so on. I'll explain the domain category and then explain sL_∞, objects, morphisms, and homotopies.

So a probability datum $(A, 1_A, M, K)$ is a tuple where A is a $\mathbb{Z}$-graded $\mathbf{k}$-vector space, $\bigoplus_{z\in\mathbb{Z}} A^z$, the unit 1_A is a distinguished element in A^0, and $M : \bar{S}(A) \to A$, where $\bar{S}(A) = A \oplus S^2A \oplus S^3A \oplus \cdots$ This M should satisfy something, we can project this to each summand, $M = M_1, M_2, \ldots$, with $M_n : S^n(A) \to A$. So M_1 is the identity from $A \to A$ and $|M|$, the degree, is zero, and I require that $M_{n+1}(X_1, \ldots, X_n, 1_A) = M_n(X_1, \ldots, X_n)$, so this plays the role of the unit. So suppose that $(A, 1_A, \cdot)$ is a unital graded commutative associative algebra over $\mathbf{k}$, then $M_n(x_1, \ldots, x_n) = x_1 \cdots x_n$. So K is a differential, a degree 1 operation $A^\bullet \to A^{\bullet+1}$ which annihilates the unit and squares to zero, $K1_A = K^2 = 0$. What is the idea behind this one? We usually assume that random variables have an algebra structure and an expectation, a linear functional, so that higher moments are determined by iterated products applied to a single expectation. I think that's too much assumption, I want to get rid of that. Instead of that, I want to assume in the space of random variables, I have some expectation $A \to \mathbf{k}$, I can get a map $\bar{S}(A) \to \mathbf{k}$ by composing c with M and that gives the moments, and then we do some analysis of the data. If there's an underlying binary product we can proceed as usual but that doesn't need to be, so we should do some sort of obstruction theory for this.

Now what is the morphism of these guys, if I have $(A, 1_A, M, K)$ and another one $(A', 1_{A'}, M', K')$, what is a morphism? It's a linear map of degree zero. We want $f : A \to A'$ to preserve the unit, $f(1_A) = 1_A$ and we want it to commute with the differential, $K' \circ f = f \circ K$. If you forget from the probability data, you have pointed cochains and pointed cochain maps. I assume no compatibility between M and the differential. This obviously forms a category.

Let's now think about homotopy. We say f and $\tilde{f}$ are homotopic to each other if $\tilde{f} = f + K' \circ \lambda + \lambda \circ K$ where λ is a degree -1 map that annihilates the unit. It's elementary to check that if f is a morphism then f' is a morphism for any λ. This is the usual definition of homotopy between (pointed) cochain maps. Now we can say that this probability data, we can change it to something called the homotopy category, and what is the homotopy category, we're doealing with the same objects, but instead of morphisms, we regard them as homotopy types of morphisms. This defines the homotopy category.

So you now can write A_C as an object here and cook up a very simple example of an object, the ground field $\mathbf{k}$ is an object in the category, you can see that this is the initial object in the category, and then a homotopy probability space of type

C is an object A_C and a morphism $A_C \xrightarrow{c} \mathbf{k}_C$. This is a diagram in the category. If I translate this one, $c : A \to \mathbf{k}$, and the degree of c is zero, and $c(1_A) = 1$, and $c \circ K = 0$. Here the target has no diffrential, so that's why you have $c \circ K = 0$.

What is the example of homotopy probability space? Every classical probability space is an example. It's ungraded so there's no differential, the M is given by the product, and then you get a probability datum in this sense.

Now I'll explain what is this category, this L_∞ algebra category over $\mathbf{k}$. To define this category, I have to give objects, morphisms, and composition of morphisms. Here composition was obvious. So an sL_∞ algebra is the data $(A, 1_A, \ell)$ where A is a graded vector space, 1_A is a distinguished element, and $\ell \in Hom(\bar{S}A, A)^1$ so $\ell = \ell_1, \ell_2, \ldots$, all of these of degree 1, and usually we write $K = \ell_1$, satisfying a bunch of relations.

Let me write the relations in a couple of ways. What is K? It's a map $A \to A$ which squares to zero. What is ℓ_2? It's a map $S^2A \to A$, with $K\ell_2(x, y) - \ell_2(Kx, y) \pm \ell_2(x, Ky) = 0$ and you can regard this as a bracket.

What about ℓ_3? It's like, your bracket doesn't quite satisfy Jacobi, but

$$\underbrace{\ell_2(x, \ell_2(y, z)) + \cdots}_{\text{Jacobi}} = K\ell_3(x, y, z) \pm \ell_3(Kx, y, z) \pm \ell_3(x, Ky, z) \pm \ell_3(x, y, Kz),$$

and so this is like a generalization of Jacobi that collapses to the Jacobi relation if $K = 0$. And there are a bunch of higher relations, let me write down the general form of the relations.

The relation is that

$$\sum_{\pi \in P(n), |B_i| = n - \pi + 1} \pm \ell(X_{B_1}, \ldots, X_{B_{i-1}}, \ell(X_{B_i}), X_{B_{i+1}}, \ldots, X_{B_{|\pi|}}) = 0.$$

Here you write $\pi = B_1 \sqcup \cdots \sqcup B_{|\pi|}$.

Usually this relation is written in terms of unshuffles. You'll maybe be happier to see the definition of L_∞ morphisms. There should also be a unit condition, but let me ignore it for now. So suppose I have $(A, 1_A, \ell)$ and $(A', 1_{A'}, \ell')$, and what is a morphism? It's a map in $\mathrm{Hom}(\bar{S}(A), A')$ where $\phi = \phi_1, \ldots, \phi_n, \ldots$ where $\phi_n : S^nA \to A'$. This also satisfies a bunch of relations. What are these relations? These relations can also be written

$$\sum_{\pi \in P(n)} \ell'(\phi(X_{B_1}), \ldots, \phi(X_{B_{|\pi|}})) + \sum_{\pi \in P(n), |B_i| = n - |\pi| + 1} \phi_{|\pi|}(X_{B_1}, \ldots, \ell(X_{B_i}), \ldots, X_{B_{|\pi|}}).$$

So what does this mean? So you have a map ϕ_1, which is not strictly a homomorphism of ℓ_2 and ℓ_2', but is so up to homotopy, so

$$\phi_1(\ell_2(x, y)) - \ell_2'(\phi_1(x), \phi_1(y)) = K\phi_2(x, y) \pm \phi_2(Kx, y) \pm \phi_2(x, Ky)$$

so it's not really structure preserving but is so up to some relations. This is an L_∞ morphism. If I have, I'll say something about how we compose these guys, If I have $(A, \ell) \xrightarrow{\phi} (A', \ell') \xrightarrow{\phi'} (A'', \ell'')$, then I have a composition $\phi' \bullet \phi$, where the nth component

$$(\phi' \bullet \phi)_n(x_1, \ldots, x_n) = \sum_{\pi \in P(n)} \pm \phi'_{|\pi|}(\phi(x_{B_1}), \ldots, \phi(x_{B_{|\pi|}}))$$

and you can check that this composition is associative so that these form a category.

Maybe this is too complicated to remember, so let me introduce a shorthand notation using coalgebra structures. If A is a graded vector space, you can consider the reduced symmetric coalgebra, $(\bar{S}A, \bar{\blacktriangle})$ where

$$\bar{\blacktriangle}(x_1 \odot \cdots \odot x_n) = \sum_{i=1}^{n-1} \sum_{\sigma \in sh(i,n-i)} x_{\sigma(1)} \odot \cdots \odot x_{\sigma(i)} \underline{\otimes} x_{\sigma(i+1)} \odot \cdots \odot x_{\sigma(n)}$$

and this coalgebra has a freeness property, it's cofree as a conilpotent symmetric coalgebra, if I have any map $\ell : \mathrm{Hom}(\bar{S}(A), A)$, I can always extend it to a coderivation $D : \mathrm{Hom}(\bar{S}(A), \bar{S}(A))$, so that $\bar{\blacktriangle} \circ D = (D \otimes I + I \otimes D) \circ \bar{\blacktriangle}$. If ℓ is an L_∞ algebra, then this coderivation D squares to zero. In fact, these are equivalent, that $D^2 = 0$ and that ℓ forms an L_∞ algebra. You'll see the explicit formula soon.

Perhaps, for better understanding there should be a separate talk purely about L_∞ and A_∞ algebra. But unfortunately, oh well. So what does this coderivation look like?

$$D(x_1 \odot \cdots \odot x_n) = \sum_{\pi \in P(n), |B_i| = n - |\pi| + 1} x_{B_1} \odot \cdots \odot X_{B_{i-1}} \odot \ell(X_{B_i}) \odot X_{B_{i+1}} \odot \cdots \odot X_{B_{|\pi|}}$$

So for example

$$\begin{aligned} D(X_1) &= KX_1 \\ D(X_1 \odot X_2) &= KX_1 \odot X_2 + X \odot KX_2 \pm \ell_2(X_1, X_2) \end{aligned}$$

Okay, some linear algebra. Now if you have this L_∞ algebra and L_∞ morphism between them, you obtain this coderivation $(\bar{S}(A), \bar{\blacktriangle}, D)$ and likewise $(\bar{S}(A'), \bar{\blacktriangle}, D')$, and the basic analysis tells you that for any $\phi : \mathrm{Hom}(\bar{S}(A), A')$ there exists a unique extension to a coalgebra map $\mathrm{Hom}(\bar{S}A, \bar{S}A')$, so that $\bar{\blacktriangle} \circ F = (F \otimes F) \circ \blacktriangle$. If $F \circ D = D' \circ F$, this is equivalent to saying that ϕ is an sL_∞ morphism. How does this extension look like? What is the extension of ϕ as a coalgebra map. It's easy to work out that $F(x_1 \bullet \cdots \bullet x_n) = \sum_{\pi \in P(n)} \phi(X_{B_1}) \odot \cdots \odot \phi(X_{B_{|\pi|}})$, this is how you translate individual ϕ to a coalgebra map. This already looks like the definition of the classical cumulants. This is related to the presentation yesterday. So then there's a notion of homotopy between morphisms, but it's a little too complicated to say. They're homotopic when there is a one-parameter family connecting the two. So there's some flow equation and time zero is one of the morphisms and time one is the other morphism then they're homotopic. Then these can be composed and reversed, so this forms an equivalence relation. I'm not going to write down the formulas.

So from the probability data, you can define a functor $\mathcal{D}_C$ to $sL_\infty(\mathbf{k})$, but you have homotopy categories on both sides, you have equivalence classes, I'm going to define this functor, it induces a well-defined functor at the level of homotopy categories. Most of the statistics will be defined in this language. From the data $A_C = (A, 1_A, M, K)$, I have to produce an L_∞ algebra $(A, 1_A, \ell^K)$. If I have another one A'_C, then I have a simple morphism, and another L_∞ algebra $(A', 1_A, \ell^{K'})$ and I should obtain an L_∞ morphism ϕ^f. This should also induce a well-defined functor in the homotopy category. Okay so we have $K : A \to A$ and we want to make a coderivation $\bar{S}(A) \to \bar{S}(A)$ that squares to zero. So you have the following commutative diagram

$$\begin{tikzcd} \bar{S}(A) \arrow[r, "D"] \arrow[d, "M"] & \bar{S}(A) \arrow[d, "M"] \\ A \arrow[r, "K"'] & A \end{tikzcd}$$

and this defines D, you define $K \circ M = M \circ D$. The fact that $K^2 = 0$ implies that $D^2 = 0$. So what is the formula? Let me rewrite this in terms of formulas. Let's assume that $M_n(X_1, \ldots, X_n)$ is an iterated binary product, so that this is

$$K(X_1 \cdots X_n) = \sum_{\pi \in P(n)} x_{B_1} \cdots \ell(X_{B_i}) \cdots x_{B_\pi},$$

this is the definition of ℓ out of K. So you can find out

$$\begin{aligned} Kx_1 &= \ell_1(x_1) \\ K(x_1 x_2) &= \ell_2(x_1, x_2) \pm KX_1 \cdot X_2 \pm X_1 K X_2 \end{aligned}$$

so for instance ℓ_2 is measuring the failure of K of being a derivation of the product. And the claim is that this is automatically an L_∞ algebra.

So what about morphisms? You make a unique lifting as before as a coalgebra map:

$$\begin{tikzcd} \bar{S}(A) \arrow[r, dotted, "\phi^f"] \arrow[d, "M"] & \bar{S}(A') \arrow[d, "M'"] \\ A \arrow[r, "f"'] & A' \end{tikzcd}$$

and so what sort of formula do you get out of $f \circ M = M' \circ \phi^f$? You get

$$f(x_1 \cdots x_n) = \sum_{\pi \in P(n)} \phi^f(x_{B_1}) \cdots \phi(x_{B_{|\pi|}}).$$

Then you further prove that this is functorial and respects homotopy, so that, for instance, if $A \xrightarrow{f} A' \xrightarrow{f'} A''$ then $\phi^{f' \circ f} = \phi^{f'} \bullet \phi^f$.

This is the end of my L_∞ exposition? What about A_∞? You replace $\bar{S}(A)$ with the tensor coalgebra with the deconcatenation, and you replace classical partitions with interval partitions.

I'm already over time. Let me draw this diagram, I have

$$\begin{tikzcd} \mathcal{A}_C \arrow[r, "c"] \arrow[d, dotted] & \mathbf{k} \\ (A, \ell^K) \arrow[ur, "\phi^c"'] & \end{tikzcd}$$

and so this interpolates between expectation and cumulants. What about random variables? Let me distinguish between homological and homotopical random variables. For the former, $x \in A$ and $Kx = 0$, that's a homological random variable. If I have no K then any element is a random variable. I say that $\tilde{x} \sim x$ if $\tilde{x} = x + K\lambda$. I have expectation $c(x) = c(\tilde{x})$, and the expectation of a homological random variable depends only on the homology class of the random variable and the homotopy class of the expectation c. But what if I take products? I have $K(x \cdot y)$ (let me assume these are random variables so $Kx = Ky = 0$), then this is $\ell_2(x, y)$ and it's not a random variable. More seriously, even if this is a random variable, if $x \sim \tilde{x}$ and $y \sim \tilde{y}$ then we do *not* necessarily have $xy \sim \tilde{x}\tilde{y}$. So I can't define the moment generating function with this definition.

A *homotopy random variable*, take a graded vector space with a trivial (0) structure of an L_∞ algebra, and then suppose we have an L_∞ morphism $(V, 0) \xrightarrow{\phi} (A, \ell^K)$, that's my definition of a homotopy random variable. This may look mysterious but if I compose with ϕ^c I get a map, an L_∞ map $(V, 0) \to \mathbf{k}$, and this has components $\bar{S}V \to \mathbf{k}$, and these are joint cumulants of the random variable which only depend on the homotopy type of φ and of ϕ^c (thus c). So this is the statistics that only depend on homotopy type. Let me finish, emphasizing what this means. Assume we have such a situation, and that V has a basis $\{e_\alpha\}$. Then we can form the series (choose a dual basis t^α) depending on φ as

$$\Gamma^\varphi = \sum_{n=1}^{\infty} \frac{1}{n!} t^{\alpha_1} t^{\alpha_n} \varphi_n(e_{\alpha_1}, \ldots, e_{\alpha_n}) = t^\alpha \phi_1(e_\alpha) + \frac{1}{2} t^\alpha t^\beta \phi_2(e_\alpha, e_\beta) + \cdots$$

and the two claims are

$$Ke^{\Gamma^\varphi} = 0$$

if this is an L_∞ morphism, and second that

$$c(e^{\Gamma^{\tilde{\varphi}}}) - c(e^{\Gamma^\varphi}) = c(K(\quad)) = 0$$

if φ and $\tilde{\varphi}$ are homotopic.

Finally I want to talk about "hidden" integrability. So suppose we can find $\tilde{\varphi}_n(e_{\alpha_1}, \ldots, e_{\alpha_n}) = K_n^\vee(e_{\alpha_1}, \ldots, e_{\alpha_n}) 1_A$. Then if you do $c(e^{\Gamma^{\tilde{\varphi}}})$, this is just

$$e^{\sum t^{\alpha_1} \cdots t^{\alpha_n} K_n^\vee(e_{\alpha_1}, \ldots, e_{\alpha_n})}$$

and your question is basically homotopic to your answer, you don't need to do any calculation.

10. September 7: Gabriel C. Drummond-Cole: A homotopy-algebraic point of view on free probability

[I do not take notes at my own talks]

11. Antonio Rieser: An introduction to algebraic topology, stochastic topology, and topological data analysis I

[I do not take notes at slide talks; I didn't realize until too late that this was mostly a chalk talk.]

12. Kurusch Ebrahimi-Fard: Probability and shuffle products II: moment-cumulant relations and shuffle-exponentials

Thank you and thank you also for this nice workshop and the opportunity to present our work here. This is jonit work with Frédéric Patras relating shuffle products with moment cumulant relations. Let me briefly recall some of what Frédéric said last time. We start with random variables $X = \{X_1, \ldots, X_n\}$ and look at the tensor algebra over this set, and look at a linear map $\mu(X_{i_1}, \ldots, X_{i_n})$ that thakes this to the expectation of the product $\mathbb{E}(X_{i_1} \cdots X_{i_n})$, and $\mu(\mathbf{1}) = 1$. So now we introduce another map κ, with $\kappa(\mathbf{1}) = 0$ and try to solve the equation $\mu = \exp^*(\kappa)$, and here $\exp^*(\kappa) = \sum_{n\geq 0} \frac{\kappa^{*n}}{n!}$, where this is convolution using the unshuffle coproduct. We know already from the cocommutativity of the shuffle coproduct, that the convolution product is commutative. If we take $\mu(x_{i_1}, \cdots, x_{i_n})$, and when you extend this, working out that $\kappa(\mathbf{1}) = 0$, using this principle multiple times, you find out that you get

$$\sum_{\pi \in \mathcal{P}_k} \prod_{p_i \in \pi} K_{|P_i|}(X_{P_i})$$

and this provides the classical moment cumulant relation through this exponential representation.

Now Frédéric briefly mentioned the following statement, that the space of linear maps $(T(X)^*, *)$, is a commutative shuffle algebra. What does that mean? In particular, it means that $f * g$ can be written $f \prec g + f \succ g$, and by commutativity, this is $f \prec g + g \prec f$. With this information, we can have another look at this exponential, and that's what I'll try to indicate by doing some calculations. Let's look at $\kappa * \kappa$. Then following this rule, we get $2\kappa \prec \kappa$, so we get a 2 here. If we look at $\kappa * \kappa * \kappa$, it's somewhat of a lengthy calculation, you get

$$\begin{aligned}\kappa * \kappa * \kappa &= \kappa \prec (2\kappa \prec \kappa) + \kappa \succ (2\kappa \prec \kappa)\\ &= \kappa \prec (2\kappa \prec \kappa) + (2\kappa \prec \kappa) \prec \kappa\end{aligned}$$

and then I use one of the shuffle rules and get

$$\begin{aligned}&= 2\kappa \prec (\kappa \prec \kappa) + 2\kappa \prec (\kappa * \kappa)\\ &= 6\kappa \prec (\kappa \prec \kappa)\end{aligned}$$

and what you can show in general is that

$$\kappa^{*n} = n!\kappa \prec (\kappa \prec (\cdots (\kappa \prec \kappa)))$$

and so

$$\mu = \exp^*(\kappa) = \sum_{n\geq 0} \frac{\kappa^{*n}}{n!} = \sum_{n\geq 0} \kappa^{\prec n}$$

and so now I can motivate this with the fixed point equation

$$\mu = \mathbf{1} + \kappa \prec \mu$$

and this will be carried over to the non-commutative world. In the commutative case, this would coincide with the fixed point equation $\mu = \mathbf{1} + \mu \succ \kappa$. So now we get two equations in the non-commutative world and you can wonder what is the right one and how to deal with this.

So then to make this precise I should tell you what is $*$ and $\prec$ at the level of the tensor algebra? So let me give the splitting of the shuffle in explicit form.

$$\Delta(x_{i_1} \otimes \cdots \otimes x_{i_n}) = \sum_{\mathcal{J} \in [n]} x_{\mathcal{J}} \otimes x_{[n] \setminus \mathcal{J}}$$

where $x_{\mathcal{J}} = x_{j_1} \otimes \cdots \otimes x_{j_k}$ where $\mathcal{J} = \{j_1, \dots, j_k\}$.

What we observed, what we want to see is a splitting, that the splitting is extremely simple, if you look into the work of Neu–Speicher where they introduce non-crossing cumulants, you can see that what we do now at the classical level amounts to this. So what we want is that $x_{\mathcal{J}}$ must contain the first letter, all the time,

$$\Delta(x_{i_1} \otimes x_{i_n}) = \sum_{1 \in \mathcal{J} \subset [n]} x_{\mathcal{J}} \otimes x_{[n] \setminus \mathcal{J}} + \sum_{1 \notin \mathcal{J} \subset [n]} x_{\mathcal{J}} \otimes x_{[n] \setminus \mathcal{J}}.$$

Let me make some observations, that $\Delta_{\prec} = \tau \Delta_{\succ}$ which is equivalent to this coproduct being commutative. What you see then here is that $\mu = \mathbf{1} + \kappa \prec \mu$, you see that if you now apply $(\kappa \otimes \mu)\Delta_{\prec}$ to a word, you get a word, and if this is a word with one letter, you get something with Bell's numbers that you use to do moments cumulants in the classical setting.

So now let me do a calculation at order 4, let me do $\mu(x^4)$, which I mean a word with four letters, I want to not put in the tensor symbol. We want to see the part with division into two blocks of size two. So within the expansion you get $(\kappa \prec \kappa)(x^4)$, and this is the same as $(\kappa \otimes \kappa) \sum_{1 \in S \subset [4]} X_S \otimes X_{[4] \setminus S}$ and if you work this out, you get

$$(\kappa \otimes \kappa)(xx \otimes xx + xx \otimes xx + xx \otimes xx)$$

where in the three cases I've got three different partitions, 12|34 and 14|23 and 13|24 and this term makes the crossing partition that we'd like to get rid of in the moment cumulant expansion.

The idea here is to add structure, when we do this extraction operation, I put a bar there, write $x_1 x_3 \otimes x_2 \mid x_4$. Then it's clear that in doing this, I add this information and I'd like to require (this is ad hoc and I'll provide definitions in the next minute) that $\kappa(x \mid x) = 0$. The compatibility of my κ is whenever I have a bar, I get zero, and then we get the moment cumulant relation as I expect in the free case. In this identification I get

$$\mu(x^4) = \sum_{\pi \in NC_4} \kappa_\pi(x, x, x, x)$$

but for that we need more sturcture on TX (and in fact need to go to a larger space). For this I take (A, φ) and what I'm interested in is working with $T(\bar{T}A)$, words made out of words by a new product, and an element here I write $w = w_1 \mid w_2 \mid \cdots$ and

$w_i \in \bar{T}A$. Now the coproduct, this is of course non-commutative, the coproduct is

$$\Delta(w) = \Delta(a_1, \ldots, a_n) = \sum_{S \in [n]} a_S \otimes a_{\mathcal{J}_1} \mid \cdots \mid a_{\mathcal{J}_\ell},$$

where $a_{\mathcal{J}_i}$ is the sequence of letters broken by bars where something in S was taken out. So this lands in $T(A) \otimes T(T(A))$.

Let me make an example, with $S = \{2, 4, 5\}$, ancting on $a_1a_2a_3a_4a_5a_6a_7$, I get the term

$$a_2a_4a_5 \otimes a_1 \mid a_3 \mid a_6a_7$$

and I extend this multiplicatively $\Delta(w_1 \mid w_2) = \Delta(w_1) \mid \Delta(w_2)$.

The theorem is that $H = T(\bar{T}A)$ is a non-commutative, non-cocommutative graded connected Hopf algebra.

Now H^* with the convolution product is non-commutative, and let me briefly remind you

(1) a *character* is a map $\phi \in H^*$ such that $\phi(\mathbf{1}) = 1$ and also $\phi(w_1|w_2) = \phi(w_1)\phi(w_2)$.
(2) an *infinitesimal character* α is a linear map on H such that $\alpha(\mathbf{1}) = 0$ and $\alpha(w_1|w_2) = 0$.

Between these, G and $\mathfrak{g}$ a group and a Lie algebra, I have an exponential and a logarithm. All of this is kind of hinting at finding this analogous picture for moments and cumulants in a non-commutative setting.

Definition 12.1. $\Phi(a_1 \cdots a_n) = \varphi(a_i \cdot_A \cdots \cdot_A a_n)$, this defines a character,

$$\Phi(w_1|w_2) = \phi(w_1)\phi(w_2).$$

So this defines Φ in G.

Let's simply again do some calculation, considering $\rho \in \mathfrak{g}$, I want to find ρ so that $\Phi = \exp^*(\rho)$. Let's calculate. So simply look at $\Phi(aa) = m_2 = \rho(aa) + \frac{1}{2}(\rho * \rho)(aa)$, this is unshufflings of aa, and if you extract a letter from between two you put a bar. So you find $\rho(aa) + \rho(a)\rho(a) = h_2 + h_1h_1$. Now I'm interested in $\Phi(aaa) = m_3 = (\rho + \frac{1}{2}\rho * \rho + \frac{1}{6}\rho * \rho * \rho(aaa)$ and I get

$$\rho(aaa) + \frac{1}{2}(\rho * \rho)(aaa) + \frac{1}{6}\rho^{*3}(aaa).$$

Remember that $\Delta(aaa) = aaa \otimes \mathbf{1} + \mathbf{1} \otimes aaa + 2aa \otimes a + a \otimes aa + a \otimes a \mid a + aa \otimes a + a \otimes aa$ and you get $\frac{1}{2}(\rho * \rho)(aaa) = \frac{5}{2}\rho(aa)\rho(a) = \frac{5}{2}h_2h_1$. So at the end I get $m_3 = h_3 + \frac{5}{2}h_2h_1 + h_1^3$.

Without doing the complete calculation, if I do the fourth moment, I get

$$\phi(a^4) = m_4 = h_4 + 3h_1h_3 + \frac{3}{2}h_2h_2 + \frac{13}{3}h_1h_1h_2 + h_1^4$$

and the theorem is that for $h_n = \rho(a^n)$, $\Phi(a^n) = m_n$ and $\Phi = \exp^*(\rho)$, that

$$m_n = \sum_{s=1}^{n} \sum_{1=i_0 < \cdots < i_s = n} \frac{1}{s!} \prod_{j=1}^{s} i_{j-1} h_{i_j - i_{j-1}}.$$

Let me remark:

(1) $\rho = \log^*(\Phi) = \sum \frac{(-1)^n}{n}(\Phi - \epsilon)^{*n}$,
(2) $\Phi * \Psi = \exp^*(\mathrm{BCH}(\alpha, \beta))$,
(3) and I have an inverse $\Phi^{-1} \in G$ which is $\Phi \circ S = \sum_{n>0}(\Phi - \epsilon)^(* n)$.

What does inverse really mean? I will come to it.

We have understood monotone moment cumulants in terms of this Hopf algebra, but there's a motivation. Recall the splitting,

$$\Delta(a_1, \dots, a_n) = \sum_{S \subset [n]} a_S \otimes a_S \otimes a_{J_1} \mid \cdots \mid a_{\mathcal{J}_s}$$

which is

$$\underbrace{\sum_{1 \in S \subset [n]} \cdots}_{\Delta_\prec} + \underbrace{\sum_{1 \notin S \subset [n]} \cdots}_{\Delta_\succ}$$

Theorem 12.1. *$(H^*, \succ, \prec)$ is a unital shuffle algebra (with $f * g = f \prec g + f \succ g$).*

The noncommutative relations are

$$\begin{aligned}(f \prec g) \prec h &= f \prec (g * h)\\ (f \succ g) \prec h &= f \succ (g \prec h)\\ f \succ (g \succ h) &= (f * g) \succ h.\end{aligned}$$

Theorem 12.2. *For $\Phi \in G$ and κ, β in $\mathfrak{g}$,*

(1) *$\Phi = \mathbf{1} + \kappa \prec \Phi$ which looks like*

$$\Phi(a_1 \dots a_n) = \sum \kappa(a_S)\Phi(a_{J_1}) \cdots \Phi(a_{J_\ell}) = \sum_{\pi \in NC_n} K_\pi(a_1 \cdots a_n)$$

gives the free moment cumulant relation.

(2) *If I look instead at $\mathbf{1} + \Phi \succ \beta$, which looks like*

$$\Phi(a_1, \dots, a_n) = \sum_{k=1}^{n} \Phi(a_{k+1} \cdots a_n)\beta(a_1 \cdots a_k) = \sum_{\pi \in IP_n} \beta_\pi(a_1 \cdots a_n)$$

which is the inversion formula for Boolean cumulants.

So this is a rewriting of Φ in terms of two sets of cumulants. I can say that $\Phi \succ \beta = \kappa \prec \Phi$ and if I now define $L_{f\succ} g = f \succ g$ and $R_{\prec f} g = g \prec f$, then

$$L_{\Phi^{-1}\succ}(\Phi \succ \beta) = \Phi^{-1} \succ (\Phi \succ \beta) = (\Phi^{-1} * \Phi) \succ \beta = \beta = \Phi^{-1} \succ \kappa \prec \Phi$$

and similarly

$$\kappa = \Phi \succ \beta \prec \Phi^{-1}$$

and this is something, we were trying to understand the paper of Arizmendi et al. in Adv. Math., 2015, and then you get the theorem

Theorem 12.3.

$$\beta(a_1, \dots, a_n) = \sum_{1, n \in S \subset [n]} \kappa(a_S)\Phi(a_{\mathcal{J}_1}) \cdots \Phi(a_{\mathcal{J}_\ell})$$

and so you get

$$b_n(a_1, \dots, a_n) = \sum_{\pi \in NC_n^{irr}} r_\pi(a_1, \dots, a_n).$$

The point is that I want both the first and the last letter which leads to irreducibility. If you invert this, you work with inverses and involve a little more shuffling, and get a formula for free cumulants in terms of Boolean cumulants, but you get something like (-1) to the number of blocks minus one.

My very last statement, the link between the monotone infinitesimal cumulants and the free cumulants, here it's a little more complicated, we don't have a fixed point formula here, at least not until the next lecture, when we'll have three exponentials, at which time we'll have a relationship among the three cumulants using them as infinitesimal characters, and this will use the preLie Magnus expansion. This is another angle of shuffle algebras, and this has an inverse, and for those of you who know, this is a refined version of the classical Magnus expansion and this then links the monotone cumulants with the others and you can get formulas involving the preLie expansion. But I think I will do this tomorrow in the last lecture.

13. Roland Friedrich: Types, Structures and (Free) Harmonic Analysis

[I do not take notes at slide talks.]

14. September 8: Kurusch Ebrahimi-Fard: Probability and shuffle products III: additive convolutions

Let me first recall the picture, so to say. The starting point is a noncommutative probability space with an expectation φ, that is, (A, φ) with $\varphi : A \to \mathbf{k}$. Then the first step was to define a Hopf algebra $(H = T(\bar{T}A), \Delta)$, with the $|$ product. This is a non-commutative non-cocommutative graded Hopf algebra. The coproduct, remember, is $\Delta(a_1 \cdots a_n) = \sum_{S \subset [n]} a_S \otimes a_{\mathcal{J}_1} \mid \cdots \mid a_{\mathcal{J}_\ell}$. There is a splitting of this coproduct

$$\Delta = \underbrace{\sum_{1 \in S \subset [n]}}_{\Delta_{\prec}} + \underbrace{1 \notin S \subset [n]}_{\Delta_{\succ}}.$$

The splitting dualizes and we get $(H^*, \succ, \prec)$ a unital shuffle algebra. Here the product is $f * g = (f \otimes g)\Delta = f \prec g + f \succ g$. Inside here we have G the group of characters and the Lie algebra $\mathfrak{g}$ of infinitesimal characters, and the natural relation between the two is given in terms of the exponential and the logarithm: $\mathfrak{g} \xrightarrow{\exp^*} G$.

Coming back to the probability situation, we have $\Phi \in G$, then trying to calculate the expectation, well, we can look at $\Phi = \exp^*(\rho)$, this is one way of looking at Φ, as a (proper) exponential of a Lie algebra element. I also, due to the splitting, I can also write Φ as a solution to a fixed point equation

$$\Phi = \mathbf{1} + \kappa \prec \Phi$$

for a Lie algebra element $\kappa \in \mathfrak{g}$, or I could write

$$\Phi = \mathbf{1} + \Phi \succ \beta$$

and this gives three different answers: ρ is the monotone case, κ the free case and β the Boolean case.

This is the most exciting situation, this gives three ways of writing a grouplike element in terms of these characters.

Okay, so one thing that comes naturally with these two fixed point equations is that we immediately see that β can be written in terms of κ: $\beta = \Phi^{-1} \succ \kappa \prec \Phi$. So this gives some linear combination of moments with signs, this inversion, in principle I can just calculate what is $\Phi^{-1} = \Phi \circ s$, this is $-\Phi(w)$ plus many terms. This adjoint operation, this group acting on the Lie algebra in terms of half-shuffles, that contains all the interesting structure. The other question where we stopped yesterday is, what is the relationship between free, monotone, and Boolean cumulants. Let me introduce something you'd do immediately if you see the fixed point formula, you get a Dyson expansion, so I define the object $\mathcal{E}_{\prec}(\alpha)$ for $\alpha \in \mathfrak{g}$ and I define this to be

$$\sum_{n \geq 0} \alpha^{\prec n}$$

where $\alpha^n = \alpha \prec (\alpha^{\prec(n-1)})$ and $\alpha^{\prec 0} = \mathbf{1}$. Then I similarly have $\mathcal{E}_{\succ}(\alpha) = \sum_{n \geq 0} \alpha^{\succ 0}$.

So you have $1 + \alpha + \alpha \succ \alpha$, and so this expansion order by order gives you all the parts with one cumulant, with two cumulants, et cetera. So these αs aren't just maps but send $\mathbf{1}$ to zero and combining this you get some cancellation and it adds up to the right moment cumulant formula, in this case the Boolean case.

The next results will be about abstract shuffle algebras but I'll state them in this context.

Lemma 14.1.
$$\mathcal{E}_{\prec}(\alpha) * \mathcal{E}_{\succ}(-\alpha) = \mathbf{1}$$
so that $\mathcal{E}_{\prec}^{-1}(\alpha) = \mathcal{E}_{\succ}(\alpha)$.

The central property you use is the associativity $f \succ (g \prec h) = (f \succ g) \prec h$. This is most of what you use.

Now that we have left and right half-shuffle exponentials, let me state the following. If you're interested in the infinitesimal character of cumulants:

Lemma 14.2. *For* $\Phi = \mathcal{E}_{\prec}(\alpha)$*, you have* $\alpha = (\Phi - \mathbf{1}) \prec \Phi^{-1}$ *and for* $\Phi = \mathcal{E}_{\succ}(\beta)$ *you have* $\beta = \Phi^{-1} \succ (\Phi - \mathbf{1})$.

Then you can write this as $\mathcal{L}_{\prec}(\mathcal{E}_{\prec}(\alpha)) = \alpha$ and $\mathcal{L}_{\succ}(\mathcal{E}_{\succ}(\beta)) = \beta$, defining half-shuffle logarithms. Here $\Phi^{-1} = \Phi \circ s = \sum_{n\geq 0}(-1)^n(\Phi - 1)^{*n}$.

Okay, so we have written $\Phi = \exp^{\prec}(\kappa) = \exp^*(\rho)$, writing this in two different ways, and now we can calculate the left half-shuffle logarithm, which should be
$$\kappa = \mathcal{L}_{\prec}(\exp^*(\rho)).$$
You do a little calculation,
$$\frac{d}{dt}\exp^*(t\rho) \succ (\exp^*((1-t)\rho) - \mathbf{1}) = -\exp^*(t\rho) \succ \rho \prec \exp^*((1-t)rho).$$
Integrate this from zero to one and you get
$$\exp^*(\rho - \mathbf{1}) = \int_0^1 \exp^*(s\rho) \succ \rho \prec \exp^*((1-s)\rho)ds$$
and I rewrite this in terms of left and right half-shuffle operators, $L_{a\succ}(b) := a \succ b$ and $R_{\prec a}(b) = b \prec a$, and so I can write my integral as
$$\int_0^1 e^{sL_{\rho\succ}}e^{-sR_{\prec\rho}}(\rho)ds \prec \exp^*(s) = \int_0^1 e^{s(L_{\rho\succ} - R_{\prec\rho})}(\rho)ds \prec \exp^*(\rho)$$
and this operation $L_{\rho\succ} - R_{\prec\rho}$ I call $L_{\rho\rhd}$ and then by a standard step this is
$$\frac{e^{L_{\rho\rhd}} - \mathrm{id}}{L_{\rho\rhd}}(\rho) \prec \exp^*(\rho)$$
the inverse of the generating function for the Bernoulli numbers. So let me finish the calculation:
$$\exp^*(\rho) = \mathbf{1} + \underbrace{\left(\frac{e^{L_{\rho\rhd}} - \mathrm{id}}{L_{\rho\rhd}}(\rho)\right)}_{W'(\rho)} \prec \exp^*(\rho)$$
and so I can say this is $\exp^*(\rho) = \mathcal{E}_{\prec}(W'(\rho))$ so that $\kappa = W'(\rho)$ and I can invert this as so I can write $\rho = \Omega'(\kappa)$ which is the same as $\frac{L_{\rho\rhd}}{e^{L_{\rho\rhd}} - \mathrm{id}}(\kappa)$ and this is the (preLie) Magnus expansion
$$\sum_{n\geq 0}\frac{B_n}{n!}L^{(n)}_{\Omega'(\kappa)\blacktriangleright}(\kappa)$$
This preLie product is another binary product on the shuffle algebra.

Proposition 14.1. *The product* $a \rhd b = a \succ b - b \prec a$ *is left preLie:*
$$(a \rhd b) \rhd c - a \rhd (b \rhd c) = (b \rhd a) \rhd c - b \rhd (a \rhd c)$$

and the interesting thing is that the Lie bracket $[a,b] = a \triangleright b - b \triangleright a$ is a Lie bracket, in fact the one from $*$ itself.

Now $W'(\rho) = \rho + \frac{1}{2}\rho \triangleright \rho + \frac{1}{6}\rho \triangleright (\rho \triangleright \rho) + \cdots$ and now we can go between all of these cumulants using the preLie expansions:

$$\begin{aligned} \Phi = \mathcal{E}_{\prec}(\kappa) \qquad & = \exp^*(\rho) \qquad & = \mathcal{E} \succ (\beta) \\ & = \mathcal{E}_{\prec}(W'(\rho)) & \\ & = \mathcal{E}_{\succ}(-W'(-\rho)) & \end{aligned}$$

and so at the end I get $\kappa = W'(\rho)$, $\rho = \Omega'(\kappa)$, $\beta = -W'(-\rho)$, and $\rho = -\Omega'(-\beta)$. Then I can forget the half-shuffle exponentials, so $\mathcal{E}_{\prec}(\kappa) = \exp^*(\Omega'(\kappa))$ and $\mathcal{E}_{\succ}(\beta) = \exp^*(-\Omega'(-\beta))$. This can be further extended, writing something we already know from a preLie algebraic point of view. I can write $\kappa = W'(-\Omega'(-\beta))$, and of course I know that this is the same as $\Phi \succ \beta \prec \Phi^{-1}$ and same for the Boolean cumulants, $\beta = -W'(-\Omega'(K)) = \Phi^{-1} \succ \kappa \prec \Phi$.

You've here gotten a slightly more conceptual point of view, that the transformations involve the preLie structure of the shuffle algebra.

A natural question to ask is, what is the group law for the half-shuffle exponentials. Let's simply calculate the shuffle product $\mathcal{E}_{\prec}(\alpha) * \mathcal{E}_{\prec}(\beta) = \mathcal{E}_{\prec}(\alpha \# \beta)$, and the product is just

$$\alpha \# \beta = W'(\mathrm{BCH}(\Omega'(\alpha), \Omega'(\beta))).$$

In fact, one can be more precise here, this is

$$\alpha + \mathcal{E}_{\prec}(\alpha) \succ \beta \prec \mathcal{E}_{\prec}^{-1}(\alpha)$$

and the same (with more minus signs) for the other half-shuffle exponential. To calculate this, there is a beautiful intertwining between Baker–Campbell–Hausdorff and the preLie expansion first noticed by [unintelligible]in control theory.

Let me now introduce some notation. It's clear that to get to this, I first write $\mathcal{E}_{\prec}(\alpha)$ in terms of the proper exponential, and then multiply the exponentials to get the Baker–Campbell–Hausdorff part, and then I use the intertwining to get the formal group law.

So now for notation, if I have α and β in $\mathfrak{g}$, then I'll write

$$\alpha^{\beta} = \mathcal{E}_{\prec}^{-1}(\beta) \succ \alpha \prec \mathcal{E}_{\prec}(\beta).$$

I'll use this a few times to simplify notation. Let me define

$$\Phi_1 = \mathcal{E}_{\prec}(\gamma_1), \qquad \Phi_2 = \mathcal{E}_{\prec}(\gamma_2)$$

and now I'll define

$$\Phi_2 \boxplus \Phi_2 := \mathcal{E}_{\prec}(\gamma_2^{\gamma_1})$$

and when I translate this, writing $\psi_1 = \mathcal{E}_{\succ}(\alpha_1)$ and $\psi_2 = \mathcal{E}_{\succ}(\alpha_2)$, then $\psi_1 \boxplus \psi_2 = \mathcal{E}_{\succ}(\alpha_2^{-\alpha_1})$

Lemma 14.3. *For ϕ_1 and ϕ_2 in G, and $\phi_i = \mathcal{E}_{\prec}(\gamma_i)$, we have*

$$\phi_1 * (\phi_2 \boxplus \phi_1) = \phi_2 * (\phi_1 \boxplus \phi_2) = \mathbf{e}_{\prec}(\gamma_1 + \gamma_2)$$

and analagously for $\boxplus$.

This now, I turn this into a definition and define

$$\phi_1 \boxminus \phi_2 = \mathcal{E}_{\succ}(\alpha_1 + \alpha_2)$$

and

$$\phi_1 \boxslash \phi_2 = \mathcal{E}_{\prec}(\gamma_1 + \gamma_2)$$

and this gives a shuffle-additive convolution for free and Boolean cumulants. One, at least, for myself, benefit, is that this has a very precise description showing how the Baker–Campbell–Hausdorff description fits into this definition. This should amount to the fact that I can always write additive convolution in terms of [unintelligible].

As a remark, you see quickly a distributivity

Lemma 14.4.

$$(\mathcal{E}_{\prec}(\gamma_1) \boxslash \mathcal{E}_{\prec}(\gamma_2)) \boxplus \mathcal{E}_{\prec}(\gamma_3) = (\mathcal{E}_{\prec}(\gamma_1) \boxplus \mathcal{E}_{\prec}(\gamma_3)) \boxslash (\mathcal{E}_{\prec}(\gamma_2) \boxplus \mathcal{E}_{\prec}(\gamma_3))$$

And a final observation, for $\phi = \mathcal{E}_{\prec}(\gamma)$ for $\gamma \in \mathfrak{g}$, we have

$$\mathcal{L}_{\succ}(\phi) = \phi^{-1} \succ (\phi - 1) = \phi^{-1} \succ \gamma \prec \phi = \gamma^{\gamma}$$

so that

$$\phi \boxplus \phi = \mathcal{E}_{\prec}(\gamma^{\gamma}) = \mathcal{E}_{\prec} \circ \mathcal{L}_{\succ}(\phi)$$

and this is something called $\mathbb{B}(\phi)$, the Berkovichi [unintelligible], and this is using the subordination product, using this half-shuffle exponential and the other half-shuffle logarithm in this formalism.

15. Antonio Rieser: An introduction to algebraic topology, stochastic topology, and topological data analysis II

I want to start by talking about the origin of A_∞ algebras. So Stasheff in 1963 wanted to study the loop space ΩX, this is the space of all loops $\{f : ([0,1]/0 \sim 1, *) \to (X, *)\}$ and you can define a concatenation

$$f * g(t) = \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

and this is great but it has a problem, $(f * g) * h \neq f * (g * h)$. You can see this by seeing which portion is when you do f, g, and h. In the first, the first quarter is f, the second quarter is g, and the second half is h; in the second, the first half is f, the third quarter is g, and the fourth quarter is h. But these are homotopic. There is a homotopy

$$H : [0,1] \times \Omega X \times \Omega X \to \Omega X[\text{sic}]$$

where you get $f * g$, but what if you do three things? Then you have five ways of multiplying here, and you get a pentagon when you put things together, and you want to fill this with a polygon K_4. Stasheff has a beautiful picture of this, at least in words, of how to construct these. For the next case, I'll draw it [picture].

There is an infinite sequence of these where you construct the next one from the ones before. For each one of these, note that $M_3[0,1] \times (\Omega X)^3 \to \Omega X$, and my chain complex $C_*(\Omega X)$ now has a differential, and I'm going to define a bunch of $m_i : C_*^{\otimes i}(\Omega X) \to C_*(\Omega X)$. Our m_1 will be the differential and all the rest of them will be $m_i(a_1 \otimes \cdots \otimes a_i) = M_{i,*}(K_i \otimes a_1 \otimes \cdots a_n)$ [sic].

Theorem 15.1. *$C_*(X, d)$ and m_i for $i \geq 1$ form an A_∞ algebra.*

There's one theorem I'm going to state that came a little later. These are the reason these are called homotopy associative algebras.

The theorem I'm going to state is by Kadeishvili, around the early seventies. If I take C_* an A_∞ algebra and a chain complex then $H_*(C_*)$ has an A_∞ structure such that $m_1 = 0$, the differential, you already took homology, and m_2 is induced by the the product m_2 on C_*, and the others are something different. The extraordinary thing is that there is a quasi-isomorphism between as A_∞ algebras between these two objects $H_*(C_*)$ and C_*. So this is definitely not true if C_* is just a chain complex. If you take homology you lose information but here you can recover everything up to homotopy.

So that was part one. Now I'll talk about a number of models of generating random simplicial complexes and then say some things known about them as objects. These are, with the Laplacian I mentioned yesterday, this has a non-commutative thing sitting around somewhere.

Let me start with the notion of an Erdös–Rényi random graph $G(n, p)$. This is a graph on n vertices, and edges exist chosen independently with probability p. The game is that, let $p = f(n)$ and describe asymptotic structure of the graph. So the kind of thing that we'll concentrate on are the phase transitions that are topological in nature. There are plenty that are combinatorial in nature but the ones I know are topological.

Theorem 15.2 (Erdös–Rényi 1959)**.** *For $p \geq \frac{\log n}{n}$, then with high probability (which means probability approaching 1 as $n \to \infty$) then $g \in G(n, p)$ is connected.*

This is the most basic topological feature of a graph. You could also ask about π_1, but the main thing that's interesting is whether it's connected. This is sharp, and so if p is less than this, then with high probability it's not connected.

Many many years later, Linial–Meshulam in 2006 generalized this to random 2-complexes. What is a random 2-complex? You have $Y(n, p)$, a simplicial complex, with 2 dimensions, so X^0, X^1, and X^2. My X^0 will be my n vertices. My X^1 will be all possible edges, all $\binom{n}{2}$ of them, and then I add faces with probability p. Here you see an analagous theorem.

Theorem 15.3 (Linial–Meshulam, 2006)**.** *For $p \geq \frac{2 \log n}{n}$, with high probability $s \in Y(n, p)$ has $H_1(S, \mathbb{R}) = 0$.*

This is the analogous result to saying that the graph is connected.

A little later, this is peculiar, by a bunch of people,

Theorem 15.4. *(Koslov, 2010; Lineal–Peled; Aronshtam–Lineal 2015) If $p \geq \frac{2.753}{n}$ implies that $H_2(S, \mathbb{R}) \neq 0$.*

This is not sharp. The rest of this talk will be basically writing down theorems of this nature.

There was another group studying 2-complexes at the same time, looking for counterexamples (or the lack of them) to the Whitehead conjecture. What's the Whitehead conjecture? This says that for X an aspherical simplicial 2-complex, every subcomplex of X is aspherical. This is an old conjecture, either from [unintelligible]or slightly before, that $\pi_n(X) = 0$ for $n \geq 2$. Often creating examples by hand is difficult but you can sometimes show with positive probability that graphs with some property exist. They didn't resolve the conjecture but learned a lot of things about 2-complexes. The theorems have a slightly different flavor. You can say things like

Theorem 15.5 (Costa–Farber, 2015 (among others))**.** *If $p \ll n^{-\frac{3}{5}}$ then the fundamental group $\pi_1(S)$ of a random simplicial 2-complex is torsion-free with high probability.*

This is a bit surprising, because you somehow know that any group can appear so the fact that you don't get torsion when p is low, that's a bit surprising.

Theorem 15.6. *If $0 < \epsilon < 0.1$ and $n^{-\frac{3}{5}} \ll p \ll n^{-\frac{1}{2}-\epsilon}$ then $\pi_1(S)$ has nontrivial elements of order 2.*

They've pushed this farther but I don't have this in my notes. You can say a little more in the regime. As a kind of partial response to their initial set of questions, suppose that $p \ll n^{-\frac{1}{2}-\epsilon}$ for a fixed $\epsilon > 0$, then with high probability, $Y \in Y(n,p)$ has the following property: any subcomplex X contained in $X' \subset Y$ of an aspherical subcomplex X' is aspherical.

What happened? I let $n \to \infty$, all of these statements are with high probability, and then any large number of these things have the property that subcomplexes of aspherical complexes are aspherical.

How do we know this exists? Oh, I have the wrong things written down.

Theorem 15.7. *Let $p \gg n^{-\frac{1}{2}-\epsilon}$ and S an arbitrary simplicial complex. Then with high probability there exists a* topological *embedding $S \subset Y$ in $Y(n,p)$.*

Let me give one more kind of result. Let $X(n,p)$, well, let $G(n,p)$ be an Erdös–Rényi random graph. For a set of $k+1$ vertices, if all the edges exist, then fill in a k-simplex. The randomness still happens at the level of the edges, and I look at all my collections of $k+1$ points and if all of them are there you fill in a simplex. For a long time we had theorems about real homology.

Theorem 15.8 (Kahle 2014)**.** *Fix $m \geq 1$, denote, let, $X \in X(n,p)$ and $\omega(1)$ is a function that satisfies $\frac{\omega(1)}{n} \to 0$. Then if*

$$p \geq \frac{(\frac{m}{2}+1)\log n + \frac{k}{2}\log\log n - \omega(1)}{n})^{\frac{1}{k+1}}$$

then $H_m(X,\mathbb{R}) = 0$ with high probability.

So this is interesting, the phase changes happen at different times for different m. So if I look at β_1, I get a graph like this [picture], for the second homology I get [picture], a little lower, and so forth. All of the results so far are for characteristic zero homology.

I won't exactly tell you how this works, but this already relies critically on information about spectra of random graphs. This is also the only way to get anything at all about homology with integral coefficients.

Theorem 15.9 (Hoffman, Kahle, Paquette)**.** *Let $d \geq 2$ and $Y \in Y_d(n,p)$, I fill in all the skeleta up to $d-1$ and then take d-simplices with probability p. If $p \geq \frac{40d\log n}{n}$ then with high probability, the integral homology $H_{d-1}(Y,\mathbb{Z})$ vanishes.*

They conjecture that the phase transition happens at $\frac{d\log n}{n}$. This has been proved for $d = 2$ by Łutzak–Peled in 2016.