HIM TRIMESTER SEMINAR

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1. June 23: David Gepner

I'd like to first say that Stephan should know the title because he made it up. I was asked to give this talk to do something about equivariant forms and elliptic cohomology. That's what I'll try to do. This will be expositional. The main reference here is Lurie's *Survey of elliptic cohomology.* Let me start by giving a little outline of global homotopy theory, and then some things that we'll need to know about derived algebraic geometry. Time permitting we'll then get to equivariant forms in elliptic cohomology.

Global homotopy theory is supposed to be equivariant homotopy theory for all groups at the same time. Another way to think about it is that global homotopy theory is the homotopy theory of stacks. One way to make this precise is, well, "stacks" is extremely flexible. We'll focus on topological stacks since we want to talk about elliptic cohomology. Even with elliptic cohomology, you have smooth stacks, not smooth stacks, I'll fix an indexing category, which I'll call Top, it could be paracompact spaces or topological manifolds or smooth manifolds. I want it small so you might have to choose a cardinality bound in some sense.

Top should be equipped with a Grothiendieck topology and a functor down to the infinity category of spaces, $\mathcal{S}$ which is Gpd_∞. We basically just need one condition, which is that it takes covers to covers.

If $\{U_\alpha\}$ is a cover of X, set $U = \coprod U_\alpha$, then in $\mathcal{S}$ I need that X is equivalent to the realization of the Cech nerve of the cover

$$|U \leftleftarrows U \times_X U \Lleftarrow U \times_X U \times_X U \cdots|.$$

So let $F : Top^{op} \to (\tau_{\le n})\mathcal{S} \cong Gpd_n$. Then F is a *stack* if $F(X) = \lim\{F(U) \rightrightarrows F(U \times_X U) \cdots\}$.

There's a realization functor from stacks to $\mathcal{S}$ which commutes with homotopy colimits and sends X to X in $\mathcal{S}$. If $F \leftarrow X_0 \leftleftarrows X_1 \cdots$ then $|F| = |X_\cdot|$. For example, I could take F to be a G-space, which is a stack by thinking of it as a topological groupoid, that gives a functor to groupoids, and then you stackify it. If X is a point then $F = \mathbb{B}G$, which is bold because it's not the space BG, it's the stack, and then $|F| = X \times_G EG = X//G$, and so $|\mathbb{B}G| = BG$.

We don't want to apply this objectwise to our stacks, but we want to apply it on the morphisms.

If $E \to F$ is a map to stacks, I have the mapping stack $\underline{Map}(E, F)$, and I can take the realization of the mapping stack to get what I'll call the space of maps $map(E, F)$.

If I take $E = \mathbb{B}H$ and $F = \mathbb{B}G$, then you can easily compute that $\underline{Map}(E, F)$ is a stack version of $Hom(H, G)$ with G acting. Then $map(\mathbb{B}H, \mathbb{B}G) = \underline{Hom}(H, G)//G$. I can make these smooth stacks if I choose and have these be maps between Lie groups. Now Orb is the category whose objects are compact Lie groups $\mathbb{B}G$, and maps $map_{Orb}(\mathbb{B}H, \mathbb{B}G) = Hom(H,G)//G$. You should think of these as points. The definition here is, a map $f : E \to F$ is a weak homotopy equivalence if $map(\mathbb{B}G, E) \xrightarrow{\sim} map(\mathbb{B}G, F)$ for all $\mathbb{B}G$ in Orb.

This gives a localization of the ∞-category of stacks equivalent to presheaves on Orb, that is, $Fun(Orb^{op}, \mathcal{S})$. This fits in the general framework of Elmendorf's theorem, identifying G-spaces as presheaves on Orb_G.

I think this is amenable to dealing with elliptic cohomology, which I will explain.

Within this orbit category I want to single out some subcategories.

So you also have a notion of a representable morphism of stacks. This roughly corresponds to inducing an injection on stabilizer groups. I don't want to be more precise for time purposes.

Lemma 1.1. *Stacks over $\mathbb{B}G$ (take G a point to get all stacks) are equivalent to presheaves on $Orb/\mathbb{B}G$.*

Also, $Spaces/\mathbb{B}G$, which are representable stacks over $\mathbb{B}G$, is equivalent to presheaves on representable orbits over $\mathbb{B}G$. In fact, $\mathbb{B}H \xrightarrow{f} \mathbb{B}G$ is representable if and only if f is an open embedding. So this is the same as presheaves on Orb_G or just G-spaces.

We have Orb^{tori} inside Orb^{ab} inside Orb, where you restrict your groups to being tori or Abelian. If G is Abelian, then $map(\mathbb{B}H, \mathbb{B}G) = hom(H,G) \times BG$, because an Abelian group acts trivially by conjugation.

I should have said that $Hom(H,G)//G$ is always the same as $\coprod_{\rho \in [H,G]} BZ(\rho)$ but if G is Abelian you have this simplification.

So $map(\mathbb{B}T(m), \mathbb{B}T(n)) \cong hom(T(m), T(n)) \times BT(n) \cong (hom(T(m), \mathbb{T}) \times \mathbb{B}\mathbb{T})^n$, so you can say Orb^{tori} is an ∞-category with objects $\mathbb{B}T(n)$ where $\mathbb{B}T(n) \cong \mathbb{B}T(1))^n$.

What is this? Well, you can check that the homotopy category of this is theory of Abelian groups. An Abelian group, as Charles was explaining, an Abelian group object in an ∞-category $\mathcal{C}$ is exactly a product-preserving functor from $\mathcal{T}^{ab}$ to $\mathcal{C}$. If $\mathcal{C}$ is presentable, then this is $\mathcal{C} \otimes Ab$. Here I'm thinking of the homotopy category really, as an infinity category via the nerve.

You're taking the homotopy category, well, maps from an m-torus to the n-torus by Pontrjagin duality is the same as a map of their dual groups. [missed]. You can show easily that to give a product preserving functor from the category of tori into an ∞-category $\mathcal{C}$ is to give an Abelian group plus some extra structure. That extra data is exactly what is called a preorientation in Lurie's world.

From this point of view you can start to create global homotopy theories valued in various ∞-categories from pre-oriented Abelian group objects.

Let me be precisely more precise about what this means. An Abelian group object A in an ∞-category $\mathcal{C}$ with finite products equipped with a map of Abelian group spaces $B\mathbb{T} \to maps(*, A)$ in $Ab(\mathcal{S})$ which is equivalently the ∞-category of simplicial Abelian groups. This is not the naive notion of what you might expect, not an E_∞ thing but something much more rigid than that. This is called a preorientation. Using the fact that $B\mathbb{T}$ is $K(\mathbb{Z}, 2)$, this is the same thing as an element of $\pi_2 map(*, A)$.

This is the assertion that in simplicial Abelian groups, $B\mathbb{T}$ is the double delooping of $\mathbb{Z}$.

If $\mathcal{C}$ has finite limits, then you can extend it to $Orb^{ab} \to \mathcal{C}$. Let me back up. I said that a map from tori was an Abelian group object plus a preorientation. I extend because we know the structure theory of compact Abelian Lie groups. They are tori crossed with finite Abelian groups. So this functor, $A(\mathbb{B}T(n))$ is n copies of $\mathbb{BT}$. I need to tell you the n-torsion in the circle, and you extend by taking fibers, so you get the n-torsion of the circle by multiplying by n from $\mathbb{BT}$ to itself.

Later we'll be able to extend to all orbits. The way you construct equivariant versions of elliptic cohomology is by starting off with the Abelian ones.

Now I want to turn to the derived algebraic geometry part of the story. Lurie classifies (pre)oriented derived elliptic curves, and those will give us equivariant cohomology theories. I need to present some definitions, tell you what a preoriented derived elliptic curve is in a somewhat rigorous way.

I want to consider étale sheaves on commutative S-algebras. For technical reasons, I might want to assume connective S-algebras. So these are functors $CAlg_S \to \mathcal{S}$ which satisfy an étale descent condition. I'll just call this $\mathcal{X}$, it's an ∞-topos but we don't need to know what that means for the purposes of this lecture. In other words F is an étale sheaf if for every étale cover $A \to B$, $F(A)$ is the limit of $F(B) \rightrightarrows F(B \otimes_A B) \cdots$.

Let me tell you what an étale cover is very quickly. Does anyone have any questions at the moment?

There's this strong notion of flatness, if A is a commutative S-algebra and M an A-module, then M is (faithfully) flat over A if $\pi_0(M)$ is (faithfully) flat over $\pi_0(A)$ and the other homotopy groups $\pi_*(M) = \pi_0(M) \otimes_{\pi_0(A)} \pi_*(A)$. A map $A \to B$ is étale if B is flat over A and $\pi_0 A \to \pi_0 B$ is étale. This map f is an étale cover if B is faithfully flat over A.

If you're given f which is faithfully flat, and form this [unintelligible]complex, the one that appears in the limit above, there's a map from A to the limit, and using the notion of flatness and the Bousfield–Kan spectral sequence, you can see that A is equivalent to this limit.

This means that representable functors are étale sheaves, even sheaves for the flat (and even étale) homology. Then I should say a little bit about what a derived scheme is.

Definition 1.1. A *derived scheme* $X, \mathcal{O}_X$ is locally an E_∞-ringed space which is locally $(\operatorname{spec} \pi_0 A, \mathcal{O}_{\operatorname{spec} A})$. And $\mathcal{O}_{\operatorname{spec} A}$ is $\operatorname{spec} \pi_0(A)[\frac{1}{f}] = A[\frac{1}{f}]$.

Definition 1.2. A *derived Deligne–Muford stack* is a locally E_∞ ringed space (I should really say topos) which is locally of the form $\operatorname{spec} A$.

The somewhat surprising, difficult theorem of Lurie, is that you can classify elliptic curves. I have everything I need to tell you about that.

A *derived elliptic curve* $C \xrightarrow{p} S$ is a map of derived schemes such that p is flat, C is an Abelian group object in, here $\mathcal{X}$ is all étale sheaves on commutative S-algebras, C is an Abelian group object in S, and the map of underlying schemes $\underline{C} \to \underline{S}$ is an elliptic curve. The underlying scheme of $\underline{(X, \mathcal{O}_X)}$, is $(X, \pi_0(\mathcal{O}_X))$. So for example if X is $\operatorname{spec} A$ then $\underline{X}$ is $\pi_0(A)$.

Theorem 1.1. *The functor F from connective commutative S-algebras to $\mathcal{S}$ which assigns to A the ∞-groupoid of oriented derived elliptic curves over A is represented by a derived Deligne–Mumford stack whose underlying stack is $\mathcal{M}_{1,1}$, genus 1 curves with a single marked point.*

I should have given $\mathcal{M}$ a structure sheaf, and there is a Bott element locally defined in $\pi_2(\mathcal{O}_{\mathcal{M}})$ such that, now I can invert this class β, and $(\mathcal{M}, \mathcal{O}_{\mathcal{M}}[\beta^{-1}])$ representds oriented derived elliptic curves. The preorientation, if C is living over $\operatorname{spec} A$, then I can realize it as happening in formal schemes, and the orientation condition says that the formal completion along the identity is the same as this derived formal schemee.

I don't have time to explain this, but this is saying that when you have the orientation, then A is an E_∞ elliptic spectrum whose formal group law is the group law of the underlying elliptic curve of the derived elliptic curve.

Now how do you extract elliptic cohomology? It's easy to describe this now. The recipe we sort of already know how it goes, we have the orbit $\mathbb{B}G$ where G is an Abelian compact Lie group. Then the elliptic cohomology of C over S, an oriented derived elliptic curve, $Ell(\mathbb{B}G) = \Gamma(C_{\mathbb{B}G}, \mathcal{O}_{\mathcal{X}})$, the E_∞-ring spectrum of functions on this derived geometric object $C_{\mathbb{B}G}$. For example, what did we have? If we plug in the point, we just get $\Gamma(S)$, which, if, let's say, how should I say this? At the moment this is a functor from Abelian orbits to $\mathcal{X}_{/S}$, but then I can compose with global sections to go down to E_∞ ring spectra, which reverse the variance. So then this can be extended before taking *ops* to presheaves on the various categories

$$\begin{tikzcd}
(Orb^{ab})^{ob} \arrow[r] \arrow[d] & \mathcal{X}^{op}_{/S} \arrow[r, "\Gamma"] & E_\infty \text{ ring spectra} . \\
(Pre(Orb^{ab}))^{ob} \arrow[ur, dotted] & &
\end{tikzcd}$$

So $Ell(*) = \Gamma(S) = A$ if $S = \operatorname{spec} A$ and if $S = \mathcal{M}$ then $Ell(*)$ is TMF. So $Ell(\mathbb{B}\mathbb{T}) = \Gamma(\mathcal{O}_C)$ and you can use a spectral sequence to see that $H^p(\underline{C}, \pi_q \mathcal{O}_C) \Rightarrow \pi_{q+p}(\Gamma(\mathcal{O}_C))$.

[some silly back and forth.]

I think I'll stop here. If people are interested in how this works more generally and how you can get genuine equivariant cohomology theories from it, I'm happy to tell you about that either in another lecture or in person.